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# EQUIVARIANT BUNDLES ON TORAL VARIETIES

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ABSTRACT. Equivariant bundles on toral varieties are described in terms of filtrations which arise canonically in the fiber over a fixed point. The cohomology groups and characteristic classes are computed in terms of these filtrations, and problems of linear algebra which arise from them are discussed.

Bibliography: 20 titles.

## §0. Introduction

**0.1.** Let $X$ be a nonsingular toral variety. This means that an action of an algebraic torus $T$ is defined on $X$ and $X$ contains an open orbit on which this action is free. We call a vector bundle $p: \mathscr{E} \to X$ an *equivariant* or *toral bundle on* $X$ if it has an equivariant $T$-structure, i.e. an action of the torus $T: \mathscr{E}$ which is linear on the fibers and makes the following diagram commute:

$$\begin{array}{ccc} \mathscr{E} & \xrightarrow{t} & \mathscr{E} \\ p\downarrow & & p\downarrow \\ X & \xrightarrow{t} & X \end{array} \qquad \forall t \in T.$$

The starting point of this paper is the following description of equivariant bundles given in the language of linear algebra. We recall (see §1) that a toral variety $X$ is determined by a fan $\Sigma = \Sigma(X)$ in a lattice $\widehat{T}^0$ dual to the character lattice $\widehat{T} = \operatorname{Hom}(T, G_m)$. The cones $\sigma \in \Sigma$ correspond bijectively to orbits $O_\sigma$ of the torus $T$ in $X$, and $\tau \subset \sigma \Leftrightarrow O_\sigma \subset \overline{O}_\tau$; $\dim \sigma = \operatorname{codim} O_\sigma$. We denote by $|\Sigma|$ the set of primitive vectors of the lattice $\widehat{T}^0$ which generate one-dimensional cones in $\Sigma$. For a cone $\sigma \in \Sigma$ we shall put $|\sigma| = \sigma \cap |\Sigma|$.

0.1.1. THEOREM. *The category of toral bundles on a variety $X = X(\Sigma)$ is equivalent to the category of vector spaces $E$ with a family of decreasing $\mathbf{Z}$-filtrations $E^\alpha(i)$ $(\alpha \in |\Sigma|,\ i \in \mathbf{Z})$ which satisfy the following compatibility condition:*

(C) *For any $\sigma \in \Sigma$ the filtrations $E^\alpha(i)$, $\alpha \in |\sigma|$, consist of coordinate subspaces of some basis of the space $E$.*

EQUIVALENT FORMULATION. *The subspaces $E^\alpha(i)$, $\alpha \in |\sigma|$, $i \in \mathbf{Z}$, generate a distributive lattice.*

In this theorem, and henceforth, the filtrations are assumed to be full: $E^\alpha(i) = 0$, $i \gg 0$, and $E^\alpha(i) = E$, $i \ll 0$.

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The equivalence of categories is established by assigning to a bundle $\mathscr{E}$ the fiber $E = \mathscr{E}(x_0)$ over a fixed point $x_0$ of the open orbit. The filtrations on $E$ arise in the following way. For each orbit $O_\alpha$, $\alpha \in |\Sigma|$, of codimension one we choose a point $x_\alpha \in O_\alpha$ and put

$$E^\alpha(\chi) = \{e \in E = \mathscr{E}(x_0) | \exists \lim_{tx_0 \to x_\alpha} \chi^{-1}(t)(te) ;\, t \in T\}, \qquad \chi \in \hat{T}.$$

It turns out that the spaces $E^\alpha(\chi)$ depend only on the number $i = \langle \chi, \alpha\rangle$ and determine a family of filtrations $E^\alpha(i)$ which satisfy the compatibility condition of Theorem 0.1.1.

A major part of this paper is of a linguistic character; its aim is to translate geometric notions related to bundles, cohomology groups, and characteristic classes into the language of filtrations. In preparing this paper we were helped by the hope that a new ecological niche would appear in which the geometry of bundles could exist and which would serve as a source of deep examples.

**0.2. A short summary of the results.** In §1 we give an account of basic facts about the construction of toral varieties and bundles. Note Theorem 1.2.3, which says that an equivariant bundle $\mathscr{E}$ is torally indecomposable if and only if it is indecomposable in the usual sense; if indecomposable toral bundles $\mathscr{E}$ and $\mathscr{F}$ are isomorphic as ordinary bundles, then $\mathscr{E}$ is torally isomorphic to $\mathscr{F} \otimes \chi$ for some character $\chi \in \hat{T}$.

In §2 a modified version of the above mentioned theorem which is suitable for singular varieties (Theorem 2.2.1) is proved. This theorem is applied to describe bundles on varieties of dimension $\leq 2$, and bundles of rank $\leq 2$ on arbitrary varieties.

In §3 for an equivariant bundle $\mathscr{E}$ on a complete nonsingular toral variety $X$, $\dim X = n$, a canonical resolution

$$0 \to \mathscr{E} \to \mathscr{F}_0 \to \mathscr{F}_1 \to \cdots \to \mathscr{F}_n \to 0,$$

is constructed, which consists of splittable bundles (Theorem 3.1.1). It is applied to compute the total Chern class of the bundle (Theorem 3.2.1):

$$c(\mathscr{E}) = \prod_{\sigma \in \Sigma} \det\left(1 + \sum_{\alpha \in |\sigma|} \alpha X_\alpha | \mathscr{E}(x_\sigma)\right)^{(-1)^{\operatorname{codim} \sigma}}, \tag{0.1}$$

where $x_\sigma \in O_\sigma$, $X_\alpha = \overline{O}_\alpha$ is the closure of the orbit of codimension 1, and the vector $\alpha \in |\sigma| \subset \hat{T}^0_\sigma$ is considered as an element of the Lie algebra $\mathscr{L}ie\, T_\sigma = \hat{T}^0_\sigma \otimes \mathbf{C}$ of the stabilizer $T_\sigma$ of the point $x_\sigma \in \overline{O}_\sigma$.

In §4 the cohomology groups of toral bundles $\mathscr{E}$ are computed in terms of the filtrations $E^\alpha$, $\alpha \in |\Sigma|$. We will denote by $H^p(X, \mathscr{E})_\chi$ the isotypical component of the cohomology group which corresponds to the character $\chi \in \hat{T}$. It can be computed with the help of some complex $C^*(\mathscr{E}, \chi)$ constructed from the spaces

$$E^\sigma(\chi) = \bigcap_{\alpha \in |\sigma|} E^\alpha(\chi), \qquad E^\alpha(\chi) = E^\alpha(\langle \chi, \alpha\rangle), \quad \alpha \in |\Sigma|$$

(Theorem 4.1.1). The result for the projective space takes the simplest form:

$$H^{n-p}(\mathbf{P}^n, \mathscr{E})_\chi = E^0(\chi) \cap \cdots \cap E^{p-1}(\chi) \cap \sum_{k \geq p} E^k(\chi) \Big/ \sum_{k \geq p} E^0(\chi) \cap \cdots \cap E^{p-1}(\chi) \cap E^k(\chi),$$

where the $E^\alpha$, $\alpha = 0, \dots, n$, are the filtrations which determine the bundle $\mathscr{E}$, and $0 < p < n$.

In the general case we have the formulas

1) $H^0(X, \mathscr{E})_\chi = \bigcap_{\alpha\in|\Sigma|} E^\alpha(\chi)$;

2) $H^n(X, \mathscr{E})_\chi = E \Big/ \sum_{\alpha\in|\Sigma|} E^\alpha(\chi)$, $n = \dim X$, $X$ complete;

3) $\sum_p (-1)^p \dim H^p(X, \mathscr{E})_\chi = \sum_{\sigma\in\Sigma} (-1)^{\operatorname{codim}\sigma} \dim E^\sigma(\chi)$.

The last formula for the Euler characteristic has a useful interpretation in the form of a trace formula (Theorem 4.2.1):

$$\sum_p (-1)^p \operatorname{Tr}(t|H^p(X, \mathscr{E})) = \sum_\Delta \operatorname{Tr}(t|\mathscr{E}(x_\Delta)) \Big/ \prod_{\chi\in\Delta^*} (1 - \chi^{-1}(t)),$$

where $\Delta \in \Sigma^{(n)}$ runs over cones of maximal dimension $n = \dim X$, $x_\Delta \in X^T$ is the fixed point corresponding to $\Delta$, and $\Delta^*$ is the basis of the group of characters $\widehat{T}$ dual to the basis $|\Delta|$ of the lattice $\widehat{T}^0$.

As an application of the trace formula, in §5 the intersection index of cycles is calculated (Theorem 5.1.1):

$$(X_\sigma \cdot X_\tau) = \sum_{\Delta\supset\sigma,\tau} \prod_{\alpha\in|\sigma|} \alpha_\Delta^* \cdot \prod_{\alpha\in|\tau|} \alpha_\Delta^* \Big/ \prod_{\alpha\in|\Delta|} \alpha_\Delta^*, \tag{0.2}$$

where $X_\sigma = \overline{O}_\sigma$, $\dim X_\sigma + \dim X_\tau = \dim X$, and $\alpha_\Delta^*$, $\alpha \in |\Delta|$, denotes elements of the dual basis $\Delta^*$. The right-hand side of this formula is viewed as a rational function on $\widehat{T}^0 \otimes \mathbf{C}$ which is in fact constant. Therefore formulas of type (0.2) may serve as a source of various algebraic identities including many classical ones.

A combination of (0.1) and (0.2) leads to explicit formulas for Chern numbers (Theorem 5.2.1). In the same section we construct canonical bases of the additive groups of the Chow ring $\mathrm{CH}(X)$ and the Grothendieck ring $K(X)$ (Theorem 5.3.1).

The final section, §6, is devoted to the analysis of the compatibility condition of Theorem 0.1.1. We call the family of filtrations $E^\alpha(i)$, $\alpha \in A$, $i \in \mathbf{Z}$, of the space $E$ *splittable* if the subspaces $E^\alpha(I)$ generate a distributive lattice. Splittable families of filtrations can be represented in the form of a direct sum of filtered spaces of dimension one. It is convenient to write the splittability condition of filtrations in the language of parabolic subgroups $P^\alpha = \{g \in \mathrm{GL}(E) | g(E^\alpha(i)) = E^\alpha(i)\}$:

$$\{E^\alpha | \alpha \in A\} \text{ splittable} \Leftrightarrow \bigcap_{\alpha\in A} P^\alpha \text{ contains a maximal torus} \tag{0.3}$$

Formula (0.3) together with Theorem 0.1.1 shows that the study of toral bundles with fiber $E$ on nonsingular varieties $X = X(\Sigma)$ is essentially equivalent to describing simplicial maps of the fan $\Sigma$ to the complex $\mathscr{P}(E)$, the vertices of which are parabolic subgroups $P \subset \mathrm{GL}(E)$, and the simplices form families $P^\alpha \subset \mathrm{GL}(E)$, $\alpha \in A$, which contain a common maximal torus. The complex $\mathscr{P}(E)$ plays the role of a classifying space for toral bundles.

We prove for $\mathscr{P}(E)$ the following analog of Helly's theorem concerning convex sets (Theorem 6.1.2): for a family of parabolic subgroups $P^\alpha \subset \mathrm{GL}(m)$, to have a common maximal torus it is necessary and sufficient that any $m+1$ subgroups of this family contain a common maximal torus. Asd a corollary we get that toral bundles on $\mathbf{P}^n$ of rank $r < n$ split. A more subtle criterion for splittability is proved in Proposition 6.3.2.

For a family of Borel subgroups $B^\alpha \subset \mathrm{GL}(m)$ more can be proved (Theorem 6.2.1): the intersection $\bigcap_\alpha B^\alpha$ contains a maximal torus if any triple of subgroups contains a common maximal torus. This condition can also be expressed in the language of permutations $\pi_{\alpha|\beta} \in S_m$ of subgroups $B^\alpha$ and $B^\beta$: $\pi_{\alpha|\beta} \cdot \pi_{\beta|\gamma} \cdot \pi_{\gamma|\alpha} = 1$, $\forall \alpha, \beta, \gamma$.

In the same section the restrictions of toral bundles to the closures of orbits $X_\sigma \subset X$ (Theorem 6.3.1) are studied, and the following classification result is proved. We call a filtration $E^\alpha(i)$, $i \in \mathbf{Z}$, *short* if it contains at most one nontrivial subspace $E^\alpha(i) \neq 0, E$. It turns out (Theorem 6.4.1) that a toral bundle $\mathscr{E}$ on $\mathbf{P}^n$ defined by a family of short filtrations decomposes into a direct sum of line bundles and twisted bundles of $p$-forms $\Omega^p \otimes O(f)$. This result leads to restrictions on the ranks of the cohomology groups of an arbitrary toral bundle on $\mathbf{P}^n$:

$$\sum_p \binom{n}{p} \dim H^p(\mathbf{P}^n, \mathscr{E})_\chi \leq \operatorname{rk} \mathscr{E},$$

from which it follows, for example, that $H^p(\mathbf{P}^n, \mathscr{E}) = 0$ for toral bundles $\mathscr{E}$ of rank less than $\binom{n}{p}$.

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**0.4. Permanent notation.** $X = X(\Sigma)$ is a toral variety defined by a fan $\Sigma$ together with the action of a torus $T$; $\widehat{T} = \mathrm{Hom}(T, G_m)$ is the character lattice; $\widehat{T}^0 = \mathrm{Hom}(\widehat{T}, \mathbf{Z})$ is its dual lattice of one-dimensional subtori; $\widehat{T}^0_{\mathbf{R}} = \widehat{T}^0 \otimes \mathbf{R}$; and $|\Sigma|$ is the set of primitive vectors of the lattice $\widehat{T}^0$, which generate one-dimensional cones in $\Sigma$. If $\sigma \in \Sigma$, then $|\sigma| = \sigma \cap |\Sigma|$.

Captial greek letters $\Delta, \Gamma \in \Sigma$ will denote cones of maximal dimension $n = \dim X$ of the fan $\Sigma$. $O_\sigma$ is the orbit corresponding to the cone $\sigma \in \Sigma$, $\dim \sigma = \operatorname{codim} O_\sigma$; $T_\sigma \subset T$ is the stabilizer of the point $x_0 \in O_\sigma$; and $X_\sigma = \overline{O}_\sigma$. In particular, $X_\Delta$ or $x_\Delta$ is the fixed point corresponding to the maximal cone $\Delta \in \Sigma$.

In the paper the decreasing $\mathbf{Z}$-filtrations $E^\alpha(i)$, $i \in \mathbf{Z}$, parametrized by vectors $\alpha \in |\Sigma|$ will often appear. In this case we put

$$E^\alpha(\chi) = E^\alpha(\langle \chi, \alpha \rangle) \quad \text{and} \quad E^\sigma(\chi) = \bigcap_{\alpha \in |\sigma|} E^\alpha(\chi), \qquad \sigma \in \Sigma, \chi \in \widehat{T}.$$

## §1. General facts about toral varieties and bundles

**1.1.** Let $X$ be a toral variety. This means that $X$ is normal, an action of an algebraic torus $T$ is defined on it, and $X$ contains an open orbit on which this action is free.

For example the projective space $\mathbf{P}(V)$ is a toral variety with respect to the action of the maximal torus $T \subset \mathrm{PGL}(V)$.

We will recall some facts about the construction of toral varieties (see [1]–[3]). Let $X = \coprod_\sigma O_\sigma$ be an orbit decomposition (the number of orbits is always finite). We shall identify the open orbit $O_0$ with the torus $T$ and we shall consider characters $\chi \in \widehat{T}$ as rational functions on $X$. Then for each orbit $O_\sigma$ a subgroup of characters $\hat{\sigma} \subset \widehat{T}$ regular on $O_\sigma$ is defined and also a cone dual to it

$$\sigma = \{\alpha \in \widehat{T}^0_{\mathbf{R}} | \langle \chi, \alpha \rangle \geq 0, \forall_\chi \in \hat{\sigma}\}.$$

The set of cones $\sigma \subset \widehat{T}^0_{\mathbf{R}}$ is called the *fan associated to* $X$ and is denoted by $\Sigma = \Sigma(X)$. There is a one-to-one correspondence between cones $\sigma \in \Sigma$ and orbits $O_\sigma \subset X$; moreover, $\sigma \subset \tau \Leftrightarrow \overline{O}_\sigma \supset O_\tau$ and $\dim O_\sigma = \operatorname{codim} \sigma$.

A variety $X = X(\Sigma)$ is uniquely determined by its fan $\Sigma$. In fact, $X$ is constructed from affine pieces $U_\sigma = \operatorname{Spec} k[\hat{\sigma}]$ by identifying $U_\sigma \cap U_\tau$ with $U_{\sigma\cap\tau}$. Here $\hat{\sigma} = \{\chi \in \widehat{T} | \langle \chi, \alpha \rangle \geq 0, \ \forall \alpha \in \sigma\}$. A fan can be an arbitrary finite collection of convex cones in the space $\widehat{T}^0_{\mathbf{R}}$ which satisfy the following conditions:

i) The cones in $\Sigma$ are generated by a finite number of vectors in the lattice $\widehat{T}^0$ and do not contain straight lines.

ii) The faces of a cone contained in a fan are also contained in the fan.

iii) Any two cones in $\Sigma$ intersect along a common face.

All the geometric properties of the variety $X = X(\Sigma)$ can be expressed in terms of the fan $\Sigma$. For example:

$X$ is nonsingular $\Leftrightarrow$ every cone $\sigma \in \Sigma$ is generated by part of the basis of the lattice $\widehat{T}^0$; in particular, a fan of a nonsingular variety is simplicial.

$X$ is complete $\Leftrightarrow$ the union of the cones $\sigma \in \Sigma$ coincides with the whole space $\widehat{T}^0_{\mathbf{R}}$.

1.1.1. Example. We shall consider the projective space $\mathbf{P}^n$. Let $(x_0 : x_1 : \ldots : x_n)$ be homogeneous coordinates on which the action of the maximal torus $T \subset \mathrm{PGL}(n+1)$ is diagonal. We shall identify the torus $T$ with the orbit of the point $(1 : 1 : \ldots : 1)$. Then the ratios $x_i/x_j = \chi_{ij}$ can be viewed as characters of the torus $T$. For a given $j$ these form a basis of the semigroup of characters $\hat{\sigma}_j$ which are regular at the fixed point $p_j$ with coordinates $x_i = \delta_{ij}$. The fan $\Sigma(\mathbf{P}^n)$ consists of dual cones $\sigma_j$, $j = 0, \ldots, n$, and their faces. Each cone $\sigma_j$ is generated by the basis $(\alpha_0, \alpha_1, \ldots, \hat{\alpha}_j, \ldots, \alpha_n)$ of the lattice $\widehat{T}^0$, where the $\alpha_k$ are defined by the conditions $\langle \alpha_k, \chi_{ij} \rangle = \delta_{ki}$ for $k \neq j$. Moreover, $\alpha_0 + \alpha_1 + \cdots + \alpha_n = 0$.

**1.2.** Let $p : \mathscr{E} \to X$ be a vector bundle over a toral variety $X$. We shall say that $\mathscr{E}$ is an equivariant or toral bundle if an action, linear on the fibers, of the torus $T : \mathscr{E}$ is given, making the following diagram commute for all $t \in T$:

$$\begin{array}{ccc} \mathscr{E} & \xrightarrow{t} & \mathscr{E} \\ p\downarrow & & p\downarrow \\ X & \xrightarrow{t} & X. \end{array} \tag{1.0}$$

For example, all the canonical bundles over $X$ are toral (they depend on $X$ functorially and therefore diagram (1.0) is defined for an arbitrary morphism $X \to X$). As another example consider rigid bundles $\mathscr{E}$, i.e. bundles for which

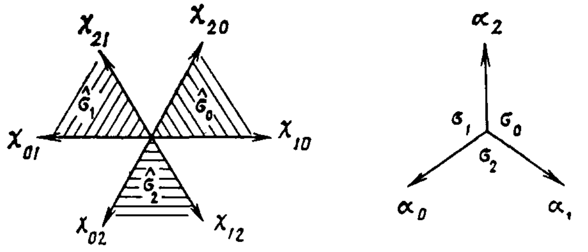


Figure 1

$\mathrm{Ext}^1(\mathscr{E}, \mathscr{E}) = H^1(X, \mathscr{E}nd\,\mathscr{E}) = 0$. This follows from the following characterization of toral bundles in the moduli space of all bundles.

1.2.1. PROPOSITION. *A bundle $\mathscr{E}$ on a complete toral variety $X$ can be endowed with a toral structure if and only if $\mathscr{E} \simeq t^*\mathscr{E}$, $\forall t \in T$ (i.e. $\mathscr{E}$ is a fixed point in the moduli space for the action of the torus $T$).*

PROOF. Let the bundle $\mathscr{E}$ satisfy the assumption of the proposition. Then for each $t \in T$ there exists an automorphism $g_t : \mathscr{E} \to \mathscr{E}$ of the variety $\mathscr{E}$ which maps fibers to fibers, is linear on the fibers, and induces translations by $t$ on the base. We denote the group of such automorphisms by $G$. The bundle $\mathscr{E}$ is consequently endowed with an equivariant $G$-structure. Moreover, there is a short exact sequence

$$1 \to \mathrm{Aut}_X \mathscr{E} \to G \to T \to 1, \tag{1.1}$$

where $\mathrm{Aut}_X \mathscr{E}$ is the group of automorphisms of the bundle $\mathscr{E}$ (it is finite dimensional since $X$ is complete). Let $S \subset G$ be a maximal torus. Since for surjective homomorphisms of linear groups maximal tori map to maximal tori, (1.1) induces an exact sequence

$$1 \to S_0 \to S \to T \to 1.$$

By construction, the bundle $\mathscr{E}$ is endowed with an equivariant $S$-structure. Moreover, $S_0$ acts trivially on the base $X$ and therefore we have a decomposition of the bundle $\mathscr{E}$ into isotypical components $\mathscr{E} = \bigoplus_{\chi \in \widehat{S}_0} \mathscr{E}_\chi$. Every character $\chi$ of the diagonalizable group $S_0 \subset S$ can be prolonged to a character $\tilde{\chi}$ of the torus $S$. Then the group $S_0$ acts trivially on the $S$-bundle $\bigoplus_\chi \mathscr{E}_\chi \otimes \tilde{\chi}^{-1}$ which, consequently, admits a toral structure (and is isomorphic to $\mathscr{E}$ as an ordinary bundle).

1.2.2. COROLLARY. *Rigid bundles on a complete toral variety can be endowed with a toral structure.*

Indeed, a rigid bundle $\mathscr{E}$ cannot be included in a continuous family, and therefore $t^*\mathscr{E} \simeq \mathscr{E}$ for all $t \in T$.

One would like to extend this corollary to noncomplete varieties. In the affine case the results of Gubeladze [20] concerning the freeness of projective modules over rings generated by monomials would easily follow.

A toral structure on a bundle $\mathscr{E}$ is, generally speaking, not uniquely defined. The following theorem and its corollary allow us to examine this arbitrariness.

1.2.3. THEOREM. *Let $\mathscr{E}$ and $\mathscr{F}$ be equivariant bundles on a complete toral variety $X$. Then:*

i) *If $\mathscr{E}$ is torally indecomposable then it is indecomposable in the usual sense.*

ii) *If $\mathscr{E}$ is indecomposable and is a direct summand of $\mathscr{F}$ as an ordinary bundle, then $\mathscr{E} \otimes \chi$ is a toral direct summand of the bundle $\mathscr{F}$ for some character $\chi \in \widehat{T}$.*

1.2.4. COROLLARY. *If indecomposable toral bundles $\mathscr{E}$ and $\mathscr{F}$ are isomorphic as ordinary bundles, then $\mathscr{E}$ is torally isomorphic to $\mathscr{F} \otimes \chi$ for some character $\chi$.*

PROOF OF THE THEOREM. i) The indecomposability of the bundle $\mathscr{E}$ is equivalent to the absence of nontrivial idempotents in the algebra $\mathrm{End}\,\mathscr{E}$. This means that the semisimple rank of the algebra $\mathrm{End}\,\mathscr{E}$ equals one. In the case of a toral bundle we have a grading of this algebra

$$\mathrm{End}\,\mathscr{E} = \bigoplus_{\chi \in \widehat{T}} (\mathrm{End}\,\mathscr{E})_\chi,$$

where $(\ )_\chi$ is an isotypical component corresponding to the character $\chi \in \widehat{T}$. The toral indecomposability implies the absence of idempotents in the component of the neutral element $(\operatorname{End}\mathscr{E})_0 = \operatorname{End}_T \mathscr{E}$. The result we want now follows from the following fact.

1.2.5. PROPOSITION. *Let $A = \bigoplus_i A_i$ be a finite-dimensional $\mathbf{Z}^n$-graded algebra over a field $k$. Then the semisimple rank of the algebra $A$ equals the semisimple rank of the algebra $A_0$.*

PROOF. It suffices to analyze the case of $\mathbf{Z}$-graded algebras.

STEP 1. *Reduction to the case of a semisimple algebra.*

The radical $R$ of the algebra $A$ coincides with the kernel of the trace form $\operatorname{Tr}_A xy$ and therefore is a homogeneous ideal. It is enough to prove the theorem for the semisimple algebra $A/R$.

STEP 2. *If $A$ is semisimple, so is $A_0$.*

Since $A_i$ and $A_j$ are orthogonal with respect to the trace form for $i+j \neq 0$, the trace form induces a nondegenerate pairing between $A_i$ and $A_{-i}$. In particular, its restriction to $A_0$ is nondegenerate.

STEP 3. *Reduction to the case* $\dim A_0 = 1$.

Let $C_0 \subset A_0$ be a maximal commutative semisimple subalgebra in $A_0$ and $C = \bigoplus_i C_i$ its centralizer in $A$. It suffices to prove the theorem for the algebra $C$, the center of which coincides with $C_0$. Moreover, using idempotents from $C_0$, one can decompose $C$ into a sum of graded algebras with one-dimensional center, i.e. to a sum of matrix algebras with a grading in which the center has degree zero (here the field $k$ is assumed to be algebraically closed).

STEP 4. *The full matrix algebra $A = M_n(k)$ does not admit a grading with $A_0 = k$ for $n > 1$.*

Indeed, $N = \bigoplus_{i>0} A_i$ is a nilpotent subalgebra of $A$ of dimension $\frac{1}{2}(\dim A - 1) = (n^2-1)/2$. But every nilpotent subalgebra of $M_n(k)$ is conjugate to a subalgebra of triangular matrices, which have dimension $n(n-1)/2$. Consequently $n = 1$.

Proposition 1.2.5 together with the first assertion of Theorem 1.2.3. are proved. We now prove the second assertion of the theorem. As in case i) the action of the torus on $\operatorname{Hom}(\mathscr{E}, \mathscr{F})$ induces gradings

$$\operatorname{Hom}(\mathscr{E}, \mathscr{F}) = \bigoplus_\chi \operatorname{Hom}_T(\mathscr{E}, \mathscr{F} \otimes \chi) \tag{1.2}$$

$$\operatorname{Hom}(\mathscr{F}, \mathscr{E}) = \bigoplus_\chi \operatorname{Hom}_T(\mathscr{F} \otimes \chi, \mathscr{E}) \tag{1.3}$$

Let $\mathscr{E}$ be a direct summand of $\mathscr{F}$, with $\varphi\colon \mathscr{E} \to \mathscr{F}$, $\psi\colon \mathscr{F} \to \mathscr{E}$, and $\psi \circ \varphi = 1_{\mathscr{E}}$. We shall decompose $\varphi = \bigoplus_\chi \varphi_\chi$ and $\psi = \bigoplus_\chi \psi_\chi$ corresponding to (1.2) and (1.3). Then the identity map decomposes into $1_{\mathscr{E}} = \psi \circ \varphi\colon \mathscr{E} \to \bigoplus_\chi \mathscr{F} \otimes \chi \to \mathscr{E}$, where the sum is taken over the characters $\chi$ for which $\varphi_\chi \neq 0$. Thus $\mathscr{E}$ is a toral component of the bundle $\bigoplus_\chi \mathscr{F} \otimes \chi$. Our assertion which says that $\mathscr{E}$ is a component of some summand $\mathscr{F} \otimes \chi$ follows now from the Remak-Schmidt theorem for toral bundles:

1.2.6. PROPOSITION. *Indecomposable components $\mathscr{E}_i$ of the toral bundle $\mathscr{F} = \bigoplus_i \mathscr{E}_i$ on a variety $X$ are uniquely determined up to an isomorphism and up to order.*

PROOF. The proof follows from the finite dimensionality of the algebra of toral endomorphisms $\mathrm{End}_T \mathcal{F}$ in which, by well-known structure theorems, all decompositions of unity into orthogonal sums of minimal idempotents are conjugate.

## §2. Moduli of toral bundles

**2.1.** First we shall describe the construction of toral bundles over affine varieties $X(\sigma) = \mathrm{Speck}[\hat{\sigma}]$, where $\hat{\sigma}$ is a semigroup of characters dual to the rational cone $\sigma \subset \widehat{T}^0_{\mathbf{R}}$ (see §1.1). All affine toral varieties take this form. Each of them contains a unique closed orbit $O_\sigma \subset X$. We denote by $T_\sigma$ the stabilizer of an arbitrary point $x_\sigma \in O_\sigma$.

2.1.1. PROPOSITION. i) *All toral bundles on an affine variety $X = X(\sigma)$ take the form $\mathcal{E} = E \times X$, where $E$ is a linear representation space of the torus $T$.*

ii) *Two toral bundles $\mathcal{E} = E \times X$ and $\mathcal{F} = F \times X$ are isomorphic if and only if the restrictions of the representations are isomorphic: $E|_{T_\sigma} \simeq F|_{T_\sigma}$.*

iii) *Define decreasing $\mathbf{Z}$-filtrations on the space $E$*

$$E^\alpha(i) = \bigoplus_{\langle \chi , \alpha\rangle \geq i} E_\chi, \qquad \alpha \in |\sigma|,$$

*where $E_\chi \subset E$ is the isotypical component of the character $\chi \in \widehat{T}$. Then the space of toral homomorphisms $\mathrm{Hom}_T(\mathcal{E}, \mathcal{F})$, $\mathcal{E} = E \times X$, $\mathcal{F} = F \times X$, is canonically isomorphic to the space of the linear operators $\varphi\colon E \to F$, compatible with the filtrations: $\varphi(E^\alpha(i)) \subset F^\alpha(i)$, $\forall \alpha \in |\sigma|$, $i \in \mathbf{Z}$.*

Note that every representation $T_\sigma : E$ of a subgroup $T_\sigma \subset T$ can be extended to a representation of the torus $T$. This follows from the complete reducibility of $E$ and the surjectivity of the character map $\widehat{T} \to \widehat{T}_\sigma$. The restriction $E|_{T_\sigma}$, as oposed to the whole representation $T : E$, has a simple geometrical meaning: $E|_{T_\sigma} \simeq \mathcal{E}(x_\sigma)$.

PROOF. i) Consider the canonical projection

$$p\colon \Gamma(X, \mathcal{E}) \to \mathcal{E}(x_\sigma),$$

which maps a section $s$ to its value $s(x_\sigma)$.

STEP 1. *There exists a $T$-invariant subspace $E \subset \Gamma(X, \mathcal{E})$ on which $p$ induces an isomorphism $p\colon E \simeq \mathcal{E}(x_\sigma)$.*

Indeed, let $E \subset \Gamma(X, \mathcal{E})$ be a maximal $T$-module on which $p$ is injective. If $p(E) \neq \mathcal{E}(x_\sigma)$, then there exists an eigenvector $\gamma \in \Gamma(X, \mathcal{E})$, $t\gamma = \chi(t)\gamma$, $t \in T$, for which $p(\gamma) \notin p(E)$. Then $p$ is injective on $E + \langle\gamma\rangle$. This contradicts the maximality of $E$. Consequently $p|_E$ is surjective.

STEP 2. *Let $s_i \in E$, $i \in I$, be a basis of eigenvectors of the torus $T$. Then the sections $s_i$ are linearly independent at every point $x \in X$.*

Indeed, if the sections $s_i$ were not independent at $x \in X$, then they would not be independent at every point of the orbit $O_x$ and also at every point of the closure $\overline{O}_x$, due to the semicontinuity of rank. But the closure of every orbit contains a closed orbit on which sections are independent because of Step 1. From Step 2 we get that $\mathcal{E} \simeq E \times X$, which proves i).

iii) It follows from i) that toral bundles on affine varieties decompose into a sum of line bundles. Therefore, without loss of generality we can assume that $\dim E = \dim F = 1$.

A bundle morphism $f\colon E \times X \to F \times X$ is a family of linear maps $\varphi_x\colon E \to F$, $x \in X$. The equivariance condition means that $\varphi_{tx}(te) = t\varphi_x(e)$, i.e. $\chi_E(t)\varphi_{tx}(e) = \chi_F(t)\varphi_x(e)$.

Now fix a point $x_0$ in an open orbit, and let $\varphi = \varphi_{x_0}$. Then the formula

$$f(e \times tx_0) = (\chi_E^{-1}\chi_F(t)\varphi(e) \times tx_0) \tag{2.1}$$

determines a rational equivariant map $f\colon E \times X \to F \times X$. It is regular everywhere provided either $\varphi = 0$ or the character $\chi_E^{-1}\chi_F$, considered as a rational function, is regular everywhere on $X = \operatorname{Spec} k[\hat{\sigma}]$; here $\hat{\sigma} = \{\chi | \langle \chi, \alpha \rangle \geq 0, \forall \alpha \in \sigma\}$. The latter condition means that

$$\langle \chi_F, \alpha \rangle \geq \langle \chi_E, \alpha \rangle, \qquad \forall \alpha \in |\sigma|.$$

In both cases the map $\varphi\colon E \to F$ respects the filtrations $E^\alpha$ and $F^\alpha$.

Conversely, if $\varphi\colon E \to F$, $\varphi \neq 0$, is compatible with the filtrations $E^\alpha$ and $F^\alpha$, then the character $\chi_E^{-1}\chi_F$ extends to $X$, and (2.1) determines a bundle morphism $f\colon E \times X \to F \times X$.

ii) If the bundles $\mathscr{E}$ and $\mathscr{F}$ are isomorphic then $E|_{T_\sigma} = \mathscr{E}(x_\sigma) \simeq \mathscr{F}(x_\sigma) = F|_{T_\sigma}$. Conversely, if $E|_{T_\sigma} \simeq F|_{T_\sigma}$ then the filtered spaces $(E; E^\alpha, \alpha \in |\sigma|)$ and $(F; F^\alpha, \alpha \in |\sigma|)$ are isomorphic. Consequently, by iii), the bundles $\mathscr{E}$ and $\mathscr{F}$ are isomorphic.

2.1.2. Remark. In the case of the complex number field we can give the filtrations $E^\alpha(i)$, $\alpha \in |\sigma|$, the following geometric interpretation. We fix a point $x_0 \in X$ of the open orbit, and put $E = \mathscr{E}(x_0)$. Let $O_\alpha$, $\alpha \in |\sigma|$, be an orbit of codimension 1, and take an $x_\alpha \in O_\alpha$. For each character $\chi \in \widehat{T}$ we define subspaces

$$E^\alpha(\chi) = \left\{ e \in E | \exists \lim_{tx_0 \to x_\alpha} \chi^{-1}(t)(te);\ t \in T \right\}.$$

Here we consider $te$ as an element of the fiber $\mathscr{E}(tx_0)$.

It follows from Proposition 2.1.1. that the spaces $E^\alpha(\chi)$ depend only on the number $i = \langle \chi, \alpha \rangle$ and determine $\mathbf{Z}$-filtrations $E^\alpha(i) = E^\alpha(\chi)$, $i = \langle \chi, \alpha \rangle$.

**2.2.** We now describe the bundles on an arbitrary toral variety $X = X(\Sigma)$. We denote by $|\Sigma|$ the set of primitive vectors of the lattice $\widehat{T}^0$, which generate one-dimensional cones in $\Sigma$. Then an arbitrary cone $\sigma \in \Sigma$ is generated by the set $|\sigma| = \sigma \cap |\Sigma| = \{\alpha_1, \dots, \alpha_m\}$. In this case we write $\sigma = \langle \alpha_1, \dots, \alpha_m \rangle$. As usual, $T_\sigma$ is the stabilizer of the point $x_\sigma \in O_\sigma$.

2.2.1. Theorem. *The category of toral bundles over the variety $X = X(\Sigma)$ is equivalent to the category of vector spaces $E$ with a family of decreasing $\mathbf{Z}$-filtrations $E^\alpha(i)$, $\alpha \in |\Sigma|$, which satisfy the following compatibility condition:*

C) *For any $\sigma \in \Sigma$ there exists a $\widehat{T}_\sigma$-grading $E = \bigoplus_{\chi \in \widehat{T}_\sigma} E^{[\sigma]}(\chi)$ for which*

$$E^\alpha(i) = \sum_{\langle \chi, \alpha \rangle \geq i} E^{[\sigma]}(\chi), \qquad \forall \alpha \in |\sigma|.$$

In this theorem and in what follows the filtrations are assumed to be full: $E^\alpha(i) = 0$, $i \gg 0$, and $E^\alpha(i) = E$, $i \ll 0$.

The equivalence of categories is established by assigning to a bundle $\mathscr{E}$ its fiber $E = \mathscr{E}(x_0)$ at a fixed point $x_0$ of the open orbit. The filtrations on $E$ arise in the following way (see §2.1.2). For each orbit $O_\alpha$, $\alpha \in |\Sigma|$, of codimension one we choose a point $x_\alpha \in O_\alpha$ and put

$$E^\alpha(\chi) = \left\{ e \in E | \exists \lim_{tx_0 \to x_\alpha} \chi^{-1}(t)(te);\ t \in T \right\}.$$

The spaces $E^\alpha(\chi)$ depend only on the number $i = \langle \chi, \alpha \rangle$ and determine a compatible family $E^\alpha(i)$ of filtrations.

2.2.2. REMARK. If the variety $X$ is nonsingular, then every cone $\sigma \in X$ is generated by a basis of the lattice $\widehat{T}^0_\sigma$. In this case, the compatibility condition C) depends only on the combinatorial structure of the fan $\Sigma$, and is equivalent to the statement that one of the following equivalent assertions is satisfied for all $\sigma \in \Sigma$:

i) The filtrations $E^\alpha(i)$, $\alpha \in |\sigma|$, are composed of coordinate subspaces of some basis of the space $E$.

ii) The subspaces $E^\alpha(i)$, $\alpha \in |\sigma|$, $i \in \mathbf{Z}$, generate a distributive lattice.

iii) For arbitrary faces $\rho$, $\tau \subset \sigma$ and numbers $i_\alpha \in \mathbf{Z}$, $\alpha \in |\sigma|$,

$$\left( \sum_{\beta \equiv |\rho|} E^\beta(i_\beta) \right) \bigcap_{\alpha \in |\tau|} E^\alpha(i_\alpha) = \sum_{\beta \in |\rho|} \left( E^\beta(i_\beta) \bigcap_{\alpha \in |\tau|} E^\alpha(i_\alpha) \right).$$

The equivalence of conditions i) and ii) is well known. That they are equivalent to iii) follows from results of Johnson ([4], Chapter III, §7, Exercise 9).

2.2.3. COROLLARY. *A toral bundle on a nonsingular variety splits if and only if the filtrations $E^\alpha(i)$, $\alpha \in |\Sigma|$, generate a distributive lattice.*

The proof consists in constructing compatible filtrations from the bundle and, conversely, in reconstructing the bundle from the filtrations. Since these constructions are all canonical, they are functorial.

I. CONSTRUCTION OF THE FILTRATIONS. Each cone $\sigma \in \Sigma$ determines an affine toral variety $U_\sigma = \operatorname{Spec} k[\hat{\sigma}]$ and an equivariant open embedding $U_\sigma \circlearrowleft X$. They form an affine covering $X = \bigcup_\sigma U_\sigma$. Let $O_\sigma \subset U_\sigma$ be the unique closed orbit in $U_\sigma$, let $x_\sigma \in O_\sigma$, and let $T_\sigma$ be the stabilizer of the point $x_\sigma$.

According to Proposition 2.1.1, for an arbitrary toral bundle $\mathscr{E}$ we have

$$\mathscr{E}|_{U_\sigma} \simeq \mathscr{E}(x_\sigma) \times U_\sigma,$$

where the toral structure on $\mathscr{E}(x_\sigma) \times U_\sigma$ is determined by some extension of the action of the stabilizer $T_\sigma$ in the fiber $\mathscr{E}(x_\sigma)$ to a representation of the torus

$$\varphi_\sigma \colon T \to \operatorname{Aut} \mathscr{E}(x_\sigma). \tag{2.2}$$

The bundle $\mathscr{E}$ is uniquely determined by its restrictions $\mathscr{E}|_{U_\sigma}$ and its transition functions $f_{\sigma|\tau} \colon U_\sigma \cap U_\tau \to \operatorname{Hom}(\mathscr{E}(x_\tau), \mathscr{E}(x_\sigma))$, which satisfy the usual cocycle relations $f_{\sigma|\tau} f_{\tau|\rho} f_{\rho|\sigma} = 1$ and $f_{\sigma|\tau} f_{\tau|\sigma} = 1$, and ther equivariance condition

$$f_{\sigma|\tau}(tx) = \varphi_\sigma(t) f_{\sigma|\tau}(x) \varphi_\tau(t)^{-1}. \tag{2.3}$$

This shows that for given representations $\varphi_\sigma$, $\sigma \in \Sigma$, it suffices to define the cocycle $f_{\sigma|\tau}$ at an arbitrary point $x_0$ of the open orbit. Moreover, the isomorphisms $f_{\sigma|\tau}(x_0) \colon \mathscr{E}(x_\tau) \to \mathscr{E}(x_\sigma)$ allow one to identify all the spaces $\mathscr{E}(x_\sigma)$ with the fiber $E = \mathscr{E}(x_0)$. In other words, one can assume that all the representations $\varphi_\sigma$ act on the same space $E = \mathscr{E}(x_0)$, and $f_{\sigma|\tau}(x_0) = 1_E$. In this case $f_{\sigma|\tau}(tx_0) = \varphi_\sigma(t)\varphi_\tau(t)^{-1}$, and the map $\varphi_\sigma(t)\varphi_\tau(t)^{-1}$ can be extended from the torus $T$, which we identify with the orbit $Tx_0$, to the affine neighborhood

$$U_\sigma \cap U_\tau = U_{\sigma \cap \tau} = \operatorname{Spec} k[\chi \in \widehat{T} | \langle \chi, \alpha \rangle \geq 0, \forall \alpha \in \sigma \cap \tau].$$

2.2.4. ASSERTION. *Consider, for each* $\alpha \in |\sigma|$, *the following filtration of the space* $E$:

$$E^{\alpha,\sigma}(i) = \sum_{\langle \chi, \alpha\rangle \geq i} E^{[\sigma]}(\chi), \tag{2.4}$$

*where* $E^{[\sigma]}(\chi)$ *is an isotypical component corresponding to the character* $\chi$ *of the representation* $\varphi_\sigma$ (2.2). *In order for the function* $f_{\sigma|\tau}(tx_0) = \varphi_\sigma(t)\varphi_\tau(t)^{-1}$ *to extend from the open orbit to* $U_\sigma \cap U_\tau$ *it is necessary and sufficient that* $E^{\alpha,\tau}(i) \subset E^{\alpha,\sigma}(i)$ *for all* $\alpha \in |\sigma \cap \tau|$ *and* $i \in \mathbf{Z}$.

Indeed, let $e_i$, $i \in I$, be a diagonalizing basis of the representation $\varphi_\sigma$; $\varphi_\sigma(t)e_i = \chi_i(t)e_i$; $\chi_i \in \widehat{T}$; and let $f_j$, $j \in J$, be a diagonalizing basis of the representation $\varphi_\tau$; $\varphi_\tau(t)f_j = \psi_j(t)f_j$, $\psi_j \in \widehat{T}$. Let $f_j = \sum_i a_{ij}e_i$. Then

$$\varphi_\sigma\varphi_r^{-1}f_j = \sum_i a_{ij}\chi_i\psi_j^{-1}e_i.$$

In order for the character $\chi_i\psi_j^{-1}$ to extend to $U_\sigma \cap U_\tau = U_{\sigma\cap\tau}$, it is necessary and sufficient that $\chi_i\psi_j^{-1} \in (\widehat{\sigma \cap \tau})$, i.e. $\langle \chi_i, \alpha\rangle \geq \langle \psi_j, \alpha\rangle$ for all $\alpha \in |\sigma \cap \tau|$. In this way the extendibility of $f_{\sigma|\tau}$ to $U_\sigma \cap U_\tau$ means that for $a_{ij} \neq 0$ we have $\langle \chi_i, \alpha\rangle \geq \langle \psi_j, \alpha\rangle$ for all $\alpha \in |\sigma \cap \tau|$. In other words,

$$E^{[\tau]}(\psi) \subset \sum_{\langle \chi, \alpha\rangle \geq \langle \psi, \alpha\rangle} E^{[\sigma]}(\chi), \qquad \forall \alpha \in |\sigma \cap \tau|,$$

or, equivalently, $E^{\alpha,\tau}(i) \subset E^{\alpha,\sigma}(i)$, for all $\alpha \in |\sigma \cap \tau|$ and $i \in \mathbf{Z}$.

2.2.5. COROLLARY. *For a system of representations* $\varphi_\sigma$, $\sigma \in \Sigma$ (2.2), *associated to a toral bundle* $\mathcal{E}$ *the filtrations*

$$E^{\alpha}(i) := E^{\alpha,\sigma}(i), \qquad \alpha \in |\sigma|, \tag{2.5}$$

*do not depend on the choice of the cone* $\sigma$ *containing* $\alpha$ *and satisfy the compatibility condition of Theorem* 2.2.1.

Indeed, in this case both functions $f_{\sigma|\tau}$ and $f_{\tau|\sigma}$ have to be regular on $U_\sigma \cap U_\tau$, and therefore $E^{\alpha,\sigma}(i) = E^{\alpha,\tau}(i)$ for all $\alpha \in |\sigma \cap \tau|$. The compatibility condition is satisfied since the filtrations $E^{\alpha}(i) = E^{\alpha,\sigma}(i)$, $\alpha \in |\sigma|$, correspond to the grading $E = \bigoplus_\chi E^{[\sigma]}(\chi)$ associated to the isotypical decomposition of the representation $\varphi_\sigma$, or its restriction to $T_\sigma$.

II. THE CONSTRUCTION OF THE BUNDLES. We shall show that an arbitrary system of compatible filtrations $E^{\alpha}(i)$, $\alpha \in |\Sigma|$, of the space $E$ canonically determines a bundle $\mathcal{E}$ with fiber $E = \mathcal{E}(x_0)$. Indeed, let the filtrations $E^{\alpha}(i)$, $\alpha \in |\sigma|$, correspond to the grading

$$E = \bigoplus_\chi E^{[\sigma]}(\chi), \tag{2.6}$$

which determines a representation of the torus $T_\sigma$: $E$ with isotypical decomposition (2.6). We shall extend it to a representation $\varphi_\sigma$ of the torus $T$ and, in order not to complicate our notation, we shall assume that (2.6) is the isotypical decomposition of $\varphi_\sigma$. Then by Assertion 2.2.4 the function $f_{\sigma|\tau}(tx_0) = \varphi_\sigma(t)\varphi_\tau(t)^{-1}$ can be extended from the torus $T$ to $U_\sigma \cap U_\tau$. The cocycle $f_{\sigma|\tau}$ determines the required toral bundle $\mathscr{E}$ over $X$.

It remains to check that the bundle $\mathscr{E}$ does not depend on the choice of gradings $E^{[\sigma]}$. Let $E = \bigoplus_\chi \widetilde{E}^{[\sigma]}(\chi)$, $\sigma \in \Sigma$, be other gradings which induce the same filtrations $E^\alpha(i)$, and let $\tilde{\varphi}_\sigma$ be the representations of the torus $T$ constructed from them. Since the filtrations corresponding to the gradings $E^{[\sigma]}$ and $\widetilde{E}^{[\sigma]}$ coincide, by Assertion 2.2.4 the functions $\alpha_\sigma = \tilde{\varphi}_\sigma \varphi_\sigma^{-1}$ and $\alpha_\sigma^{-1} = \varphi_\sigma \tilde{\varphi}_\sigma^{-1}$ are regular on $U_\sigma$. Consequently, the cocyle $f_{\sigma|\tau} = \varphi_\sigma \varphi_\tau^{-1}$ is cohomologous to the cocycle $\tilde{f}_{\sigma|\tau} = \tilde{\varphi}_\sigma \tilde{\varphi}_\tau^{-1} = \alpha_\sigma(\varphi_\sigma \varphi_\tau^{-1})\alpha_\tau^{-1}$.

2.2.6 Remark. It can be seen from the proof that the gradings $E = \bigoplus_\chi E^{[\sigma]}(\chi)$ in the compatibility condition have an obvious geometric interpretation. Namely, if one identifies the grading $E^{[\sigma]}$ with the representation of the torus $T_\sigma$ for which it is an isotypical decomposition, then

$$E^{[\sigma]} \simeq \mathscr{E}(x_\sigma). \tag{2.7}$$

Note also that the representations in the fibers $\mathscr{E}(x_\sigma)$ cannot be arbitrary. They are related to each other by

$$\mathscr{E}(x_\sigma)|_\tau \simeq \mathscr{E}(x_\tau), \qquad \tau \subset \sigma. \tag{2.8}$$

Formally, this follows from (2.7) and Corollary 2.2.5. Here is a more direct geometric explanation: the character of the representation of the torus $T_\tau$ in the fiber $\mathscr{E}(x_\tau)$, because of continuity, does not depend on the choice of the point $x_\tau$ in the closure of the orbit $\overline{O}_\tau$. If $\tau \subset \sigma$ then $x_\sigma \in O_\sigma \subset \overline{O}_\tau$.

In the case of a complete variety $X$, (2.8) shows that all representations $T_\sigma : \mathscr{E}(x_\sigma)$ can be constructed from representations of the torus $T$ in the fixed fibers $\mathscr{E}(x_\Delta)$.

The compatibility conditions (2.8) do not guarantee the existence of a bundle $\mathscr{E}$ with the given representations in the fibers. Finding necessary and sufficient conditions for this is an interesting problem closely related to the question of which values of Chern classes of toral bundles determined by representations $\mathscr{E}(x_\sigma)$ (see Theorem 3.2.1) are possible.

When describing the moduli of bundles it is sensible to fix the representations $T_\sigma : \mathscr{E}(x_\sigma)$. We say that bundles $\mathscr{E}$ and $\widetilde{\mathscr{E}}$ *have the same spectrum* if the representations of the tori $T_\sigma$ in the fibers $\widetilde{\mathscr{E}}(x_\sigma)$ and $\mathscr{E}(x_\sigma)$ are isomorphic. If the variety $X$ is complete, then it suffices to check this condition for the representations of the torus $T$ in the fixed fibers $\mathscr{E}(x_\Delta)$ and $\widetilde{\mathscr{E}}(x_\Delta)$.

The following description of the moduli of toral bundles with a given spectrum will be useful for us.

2.2.7. Proposition. *Let $\mathscr{E}$ be an equivariant bundle on a nonsingular toral variety $X = X(\Sigma)$ with fiber $E = \mathscr{E}(x_0)$ defined by a system of filtrations $E^\alpha$, $\alpha \in |\Sigma|$. Set*

$$P^\sigma = \{g \in \mathrm{GL}(E) | gE^\alpha = E^\alpha, \forall \alpha \in |\sigma|\}, \qquad \sigma \in \Sigma.$$

*Then equivariant bundles on $X$ with fiber $E$ over the point $x_0$ which all have the same spectrum as $\mathscr{E}$ are parametrized by collections of elements $S_\sigma \in \mathrm{GL}(E)/P^\sigma$ such that $S_\sigma \equiv S_\tau \pmod{P^{\sigma\cap\tau}}$.*

Here we consider bundles with a fixed fiber $E = \mathscr{E}(x_0)$. If one is interested in bundles up to isomorphism, then one needs to factor out the action of the group $\mathrm{GL}(E): S_\sigma \mapsto gS_\sigma$.

PROOF. Let $\widetilde{E}^\alpha$, $\alpha \in |\Sigma|$, be a system of filtrations which determines the bundle $\widetilde{\mathscr{E}}$. Then the fact that the spectra of the bundles $\widetilde{\mathscr{E}}$ and $\mathscr{E}$ coincide means that for an arbitrary $\sigma \in \Sigma$ there exists an automorphism $S_\sigma \in \mathrm{GL}(E)$ such that $\widetilde{E}^\alpha = S_\sigma E^\alpha$ for all $\alpha \in |\sigma|$. The elements of $S_\sigma$ are determined uniquely modulo $P^\sigma$ and $S_\sigma E^\alpha = \widetilde{E}^\alpha = S_\tau E^\alpha$, $\forall \alpha \in |\sigma \cap \tau|$. Therefore $S_\sigma^{-1} S_\tau \in P^{\sigma \cap \tau}$.

Conversely, every such collection of elements $S_\sigma$ determines a compatible system of filtrations $\widetilde{E}^\alpha = S_\sigma E^\alpha$, and hence a bundle $\widetilde{\mathscr{E}}$ with the same spectrum as $\mathscr{E}$.

2.3. EXAMPLES. 1. *Line bundles.* In this case the filtrations $E^\alpha$ are determined by numbers $n_\alpha$ for which $E^\alpha(n_\alpha) = E$ and $E^\alpha(n_\alpha + 1) = 0$, and the entire bundle is determined by the function $f\colon |\Sigma| \to \mathbf{Z}$, $\alpha \to n_\alpha$, and denoted $\mathscr{O}(f)$. For a nonsingular variety the compatibility condition is automatically satisfied and the function $f$ can be arbitrary. In the general case one needs that $f$ extends over each cone $\sigma \in \Sigma$ to a linear function, integral-valued on $\sigma \cap \widehat{T}^0$ (cf. [1]–[3]).

2. *Bundles of rank two.* For simplicity we assume that the variety $X = X(\Sigma)$ is nonsingular. For a rank two toral bundle $\mathscr{E}$ we denote by $\Sigma(\mathscr{E}) \subset \Sigma$ the subcomplex consisting of the cones $\sigma \in \Sigma$ for which all the filtrations $E^\alpha(i)$, $\alpha \in |\sigma|$, contain a one-dimensional subspace. We identify this subspace with the point $p_\alpha \in \mathbf{P}(E) = \mathbf{P}^1$. The compatibility condition in Theorem 2.2.1 implies that the correspondence $f\colon \alpha \mapsto p_\alpha$ determines a simplicial map of the fan $\Sigma(\mathscr{E})$ to a one-dimensional complex in $\mathbf{P}^1$ (i.e. for all $\sigma \in \Sigma(\mathscr{E})$ the image $f(|\sigma|)$ consists of no more than two points). The bundle $\mathscr{E}$ splits if and only if the whole image $f(|\Sigma(\mathscr{E})|)$ contains no more than two points. In particular, for a nonsplittable rank two bundle over $X$ to exist it is necessary and sufficient that one can find three vertices $\alpha$, $\beta$, $\gamma \in |\Sigma|$ which do not belong to the same cone. An analogous statement holds for bundles of arbitrary rank (see 6.1.4).

3. *Bundles on the line* $\mathbf{P}^1$. In this case a bundle $\mathscr{E}$ is determined by a pair of filtrations $E^\alpha$ and $E^\beta$. It is well known that two filtrations are always generated by the same bigrading. Consequently, any toral bundle on $\mathbf{P}^1$ splits (toral version of Grothendieck's theorem).

4. *Smooth toral surfaces.* The distinguishing feature of this case consists of the fact that the compatibility condition is automatically satisfied (see the previous example). Therefore toral bundles can be described by an arbitrary collection of filtrations $E^\alpha$, $\alpha \in |\Sigma|$. One can view such filtrations as representations of quivers consisting of $N = \#|\Sigma|$ chains which meet at one point (in Figure 2 the quiver for the plane $\mathbf{P}^2$ is shown). The complete classification of the representations of such quivers is a very difficult problem. However, interesting information on their construction is contained in Kac's theorem [5], which we now recall.

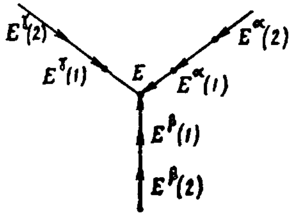


FIGURE 2

Let $\Gamma$ be a connected oriented graph with no cycles, with vertex set $\Pi$ which are called *simple roots*. We introduce on the free abelian group $\mathbf{Z}^{\Pi}$ the inner product

$$(\alpha, \beta) = \delta_{\alpha,\beta} - \tfrac{1}{2}(b_{\alpha\beta} + b_{\beta\alpha}), \qquad \alpha, \beta \in \Pi,$$

where $b_{\alpha\beta}$ is the number of arrows $\alpha \to \beta$. For a simple root $\alpha \in \Pi$ we define a fundamental reflection $r_\alpha: \mathbf{Z}^{\Pi} \to \mathbf{Z}^{\Pi}$, $r_\alpha(\lambda) = \lambda - 2(\lambda, \alpha)\alpha$, and call the group generated by them the Weyl group $W(\Gamma)$.

$W$-images of simple roots are called *real roots* $\Delta^{\mathrm{Re}}(\Gamma)$. Imaginary roots $\Delta^{\mathrm{Im}}(\Gamma)$ are defined as the $W$-images of elements of the set $M \cup -M$, where $M$ consists of vectors $\gamma \in \mathbf{Z}^{\Pi}$ with connected support for which $(\gamma, \alpha) \le 0$, $\forall \alpha \in \Pi$.

The root $\lambda \in \Delta(\Gamma) = \Delta^{\mathrm{Re}}(\Gamma) \cup \Delta^{\mathrm{Im}}(\Gamma)$ is called *positive* if all of its coordinates are nonnegative. The set of positive roots will be denoted by $\Delta_+(\Gamma) = \Delta_+^{\mathrm{Re}} \cup \Delta_+^{\mathrm{Im}}$.

A representation of the graph $\Gamma$ over a field $k$ is a collection of vector spaces $E^\alpha$, $\alpha \in \Pi$, over $k$ together with morphisms $\varphi_{\alpha\to\beta}: E^\alpha \to E^\beta$ for every arrow $\alpha \to \beta$. We call the vector $\lambda = \Sigma_{\alpha\in\Pi}(\dim E^\alpha)\alpha$ the *dimension of the representation.*

THEOREM (Kac [5]). *Assume that the ground field is algebraically closed. Then:*

i) *There exists an irreducible representation of a quiver $\Gamma$ of dimension $\lambda \in \mathbf{Z}^{\Pi}$ is unique if and only if $\lambda \in \Delta_+(\Gamma)$.*

ii) *An irreducible representation of dimension $\lambda$ is unique if and only if $\lambda \in \Delta_+^{\mathrm{Re}}(\Gamma)$.*

iii) *If $\lambda \in \Delta_+^{\mathrm{Im}}(\Gamma)$, then the maximal number of parameters on which an irreducible representation of dimension $\lambda$ can depend is equal to* $1 - (\lambda, \lambda) > 0$.

For (semi-) definite forms $(\alpha, \beta)$ this theorem is due to Gabriel [6] and Nazarova [7]. For example, it makes it possible to describe explicitly all irreducible bundles over $\mathbf{P}^2$ determined by three filtrations $E^\alpha$, $E^\beta$, and $E^\gamma$, the number of nonzero terms of which equals $a$, $b$, and $c$, and satisfies the inequality $a^{-1} + b^{-1} + c^{-1} \ge 1$. If $a^{-1} + b^{-1} + c^{-1} > 1$, then the dimensions of the terms of the filtrations $E^\alpha$, $E^\beta$, and $E^\gamma$ coincide with the coordinates of the positive roots of one of the systems $A$, $D$, or $E$. If $a^{-1} + b^{-1} + c^{-1} = 1$, then the dimensions are equal to the coordinates of the affine roots of the system $E$.

For the graph $\Gamma$ described above, consisting of $N$ chains which meet in a common vertex $\alpha$, not all irreducible representations of dimension $\lambda$ can be realized by filtrations. For this it is necessary and sufficient that the support of the root $\lambda$ contains the vertex $\alpha$.

5. As a last example we give for reference a *description of the tangent and cotangent bundles.* They are defined by filtrations of the spaces $\mathscr{T} = \widehat{T}^0 \otimes k$ and $\Omega = \widehat{T} \otimes k$ given by the formulas

$$\mathscr{T}^\alpha(i) = \begin{cases} \mathscr{T}, & i \le 0, \\ \langle\alpha\rangle, & i = 1, \\ 0, & i > 1. \end{cases} \qquad \Omega^\alpha(i) = \begin{cases} \Omega, & i < 0, \\ \ker\alpha = \{\omega | \langle\omega, \alpha\rangle = 0\}, & i = 0, \\ 0, & i > 0. \end{cases}$$

## §3. The canonical resolution and characteristic classes

3.1. Let $\mathscr{E}$ be an equivariant bundle on a complete nonsingular toral variety $X = X(\Sigma)$ of dimension $n$, and $E^\alpha$, $\alpha \in |\Sigma|$, the corresponding system of compatible filtrations of the space $E = \mathscr{E}(x_0)$. At this point we shall construct a canonical resolution of the bundle $\mathscr{E}$ which consists of splittable bundles

$$\mathscr{C}_f \colon 0 \to \mathscr{E} \to \mathscr{F}_0 \to \mathscr{F}_1 \to \cdots \to \mathscr{F}_n \to 0. \tag{3.1}$$

The resolution $\mathscr{C}_f$ depends on the function $f \colon |\Sigma| \to \mathbf{Z}$ which bounds the filtrations $E^\alpha$; $E^\alpha(i) = 0$ for $i > f(\alpha)$. Geometrically, this condition means that the bundle $\mathscr{E}^* \otimes \mathscr{O}(f)$ is generated at the generic point by $T$-invariant global sections.

Consider an exact sequence associated to the chain complex of the fan $\Sigma$ with coefficients in $E$:

$$C_f \colon 0 \to E \to \bigoplus_{\operatorname{codim}\sigma=0} \sigma \otimes E \xrightarrow{d} \bigoplus_{\operatorname{codim}\sigma=1} \sigma \otimes E \to \cdots \to \bigoplus_{\operatorname{codim}\sigma=n-1} \sigma \otimes E \xrightarrow{d} \varnothing \otimes E \to 0. \tag{3.2}$$

We assume that all the cones $\sigma \in \Sigma$ are oriented and that $d(\sigma)$ equals the sum of the faces of codimension 1 with the induced orientation. We shall define filtrations $(\ \ )^\alpha$, $\alpha \in |\Sigma|$, on the terms in the resolution (3.2) by putting

$$(\sigma \otimes E)^\alpha(i) = \begin{cases} \sigma \otimes E^\alpha(i), & \text{if } \alpha \in |\sigma|, \\ \sigma \otimes E, & \text{if } \alpha \notin |\sigma|,\ i \le f(\alpha), \\ 0, & \text{if } \alpha \notin |\sigma|,\ i > f(\alpha). \end{cases} \tag{3.3}$$

These filtrations satisfy the compatibility condition in Theorem 2.2.1 and are compatible with the differentials of the complex $C_f$. Consequently, by Theorem 2.2.1, $C_f$ determines a complex of equivariant bundles $\mathscr{C}_f$ with first term $\mathscr{E}$.

3.1.1. THEOREM. *$\mathscr{C}_f$ is a resolution of the bundle $\mathscr{E}$ consisting of splittable bundles.*

PROOF. Let $X = \bigcup_\delta U_\delta$ be an invariant affine covering and $U_\delta = \operatorname{Spec}[\hat{\delta}]$. We need to verify the exactness of the complexes of global sections $\Gamma(U_\delta, \mathscr{C}_f)$. Consider their isotypical components $C_f^\delta(\chi) = \Gamma(U_\delta, \mathscr{C}_f)_\chi$:

$$C_f^\delta(\chi) \colon 0 \to E^\delta(\chi) \to F_0^\delta(\chi) \to F_1^\delta(\chi) \to \cdots \to F_n^\delta(\chi) \to 0, \tag{3.4}$$

where $F_k$ is the $k$th term of the resolution (3.2) and $F_k^\delta(\chi) = \bigcap_{\alpha \in |\delta|} F_k^\alpha(\langle \chi, \alpha \rangle) = \Gamma(U_\delta, \mathscr{F}_k)_\chi$ (see §4). The complex $C_f^\delta(\chi)$ depends only on the filtrations with indices $\alpha \in |\delta|$. Since these filtrations are compatible on $E$, the filtered space $(E; E^\alpha, \alpha \in |\delta|)$ can be decomposed into a sum of one-dimensional filtered spaces. This allows one to assume that $\dim E = 1$ when checking the exactness of the complex (3.4). We put $\tau = \langle \alpha \in |\delta| \,|\, E^\alpha(\chi) = 0 \rangle$. Then

$$F_k^\delta(\chi) = \bigcap_{\alpha \in |\delta|} F_k^\alpha(\chi) = \bigoplus_{\sigma \cap \tau = 0;\, \operatorname{codim}\sigma = k} (\sigma \otimes E)$$

and the complex (3.4) can be interpreted as an augmented complex of homology groups of the simplicial scheme $\{\sigma \in \Sigma | \sigma \cap \tau = 0\}$. Consequently it is acyclic.

The splittability of the bundles $\mathscr{F}_k$ is equivalent to the splittability of the filtrations (3.3) and follows immediately from the compatibility condition of Theorem 2.2.1. We shall note for reference the explicit formula.

Let the filtrations $E^\alpha$, $\alpha \in |\delta|$, be generated by the grading $E = \bigoplus_{\chi\in\widehat{T}_\delta} E^{[\delta]}(\chi)$ in Theorem 2.2.1. We define functions $f_\chi^\delta : |\Sigma| \to \mathbf{Z}$ by the formulas

$$f_\chi^\delta(\alpha) = \begin{cases} f(\alpha), & \alpha \notin |\delta|, \\ \langle \chi, \alpha\rangle, & \alpha \in |\delta|, \end{cases} \tag{3.5}$$

and denote the one-dimensional filtered space associated to the line bundle $\mathscr{O}(f_\chi^\delta)$ by $O(f_\chi^\delta)$ (see Example 2.3.1). Then there is an isomorphism of filtered spaces

$$\delta \otimes E \simeq \bigoplus_\chi E^{[\delta]}(\chi) \otimes O(f_\chi^\delta).$$

In other words, the terms of the canonical resolution (3.1) take the form

$$\mathscr{F}_k \simeq \bigoplus_{\chi\equiv\widehat{T}_\delta\,;\,\operatorname{codim}\delta=k} E^{[\delta]}(\chi) \otimes \mathscr{O}(f_\chi^\delta). \tag{3.6}$$

**3.2.** We apply the canonical resolution (3.1) to calculate the characteristic classes of toral bundles.

Denote the closure of the orbit $O_\delta$ in $X$ by $X_\delta = \overline{O}_\delta$. The variety $X_\delta$ is nonsingular and can be represented in the form of a complete intersection of divisors $X_\delta = X_{\alpha_1} \cdots X_{\alpha_k}$, $\delta = \langle \alpha_1, \dots, \alpha_k\rangle$. We shall reserve the notation $X_\delta$ for the class of the variety $X_\delta$ in the Chow ring $\mathrm{CH}(X)$ or in the cohomology ring $H^*(X, \mathbf{Z})$.

Let $\mathscr{E}$ be a toral bundle over $X$, $T_\delta \subset T$ the stabilizerr of an arbitrary point $x_\delta \in O_\delta$, and $m(\chi, \mathscr{E}(x_\delta))$ the multiplicity of the character $\chi \in \widehat{T}_\delta$ in the fiber $\mathscr{E}(x_\delta)$. This multiplicity can be calculated by considering the multigradings $E^{[\delta]}(\chi)$ from the compatibility condition in Theorem 2.2.1:

$$m(\chi, \mathscr{E}(x_\delta)) = \dim E^{[\delta]}(\chi). \tag{3.7}$$

Recall that if $\tau \subset \delta$ then $\mathscr{E}(x_\tau) \simeq \mathscr{E}(x_\delta)|_{T_\tau}$ (see §2.2.6). Therefore, in the case of a complete variety the entire information about the representations $\mathscr{E}(x_\delta)$ is contained in the fixed fibers of $\mathscr{E}(x_\Delta)$.

3.2.1. THEOREM. *The total Chern class of a toral bundle $\mathscr{E}$ over a complete nonsingular variety $X = X(\Sigma)$ equals*

$$c(\mathscr{E}) = \prod_{\delta\in\Sigma} \prod_{\chi\in\widehat{T}_\delta} \left(1 + \sum_{\alpha\in|\delta|} \langle\chi, \alpha\rangle X_\alpha\right)^{(-1)^{\operatorname{codim}\delta} m(\chi, \mathscr{E}(x_\delta))}$$
$$= \prod_{\delta\in\Sigma} \det\left(1 + \sum_{\alpha\in|\delta|} \alpha X_\alpha | \mathscr{E}(x_\delta)\right)^{(-1)^{\operatorname{codim}\delta}},$$

*where in the last formula, true over the complex number field, we consider the vector $\alpha \in |\delta| \subset \widehat{T}_\delta^0 \otimes \mathbf{R}$ as an element of the Lie algebra $\mathscr{L}\!ie\, T_\delta = \widehat{T}_\delta^0 \otimes \mathbf{C}$ acting on the fiber $\mathscr{E}(x_\delta)$.*

PROOF. From formula (3.6) for terms of the canonical resolution (3.1) and from the multiplicativity property of characteristic classes we get

$$c(\mathscr{E}) = \prod_{\delta\in\Sigma}\prod_{\chi\in\widehat{T}_\delta}\left(1+\sum_{\alpha\in|\Sigma|} f_\chi^\delta(\alpha)X_\alpha\right)^{(-1)^{\operatorname{codim}\delta}\dim E^{[\delta]}(\chi)}, \tag{3.8}$$

where $f\colon |\Sigma| \to \mathbf{Z}$ is an arbitrary function which bounds the filtrations $E^\alpha$ determining the bundle $\mathscr{E}$, $E^\alpha(i) = 0$ for $i > f(\alpha)$, and the functions $f_\chi^\delta\colon |\Sigma| \to \mathbf{Z}$ are given by (3.5). Formally, in (3.8), the total Chern class is written as a polynomial in $f(\alpha)$, $\alpha \in |\Sigma|$, and for sufficiently large $f(\alpha)$ (such that $E^\alpha(i) = 0$ for $i > f(\alpha)$) it should be the constant function. This implies that the right-hand side of (3.8) does not depend at all on the choice of $f$. By putting $f \equiv 0$ and using (3.7) we get the theorem.

3.2.2. Corollary. *Characteristic classes of a toral variety $\mathscr{E}$ depend only on the spectrum of the representation of the torus at the fixed fibers $\mathscr{E}(x)$, $x \in X^T$.*

3.2.3. Example. In the case of the tangent bundle $\mathscr{T}$, the spectrum of a representation at a fixed point $\mathscr{T}(x_\Delta)$ consists of characters which belong to the basis $\Delta^*$ of the lattice $\widehat{T}$ dual to $|\Delta|$. In this case Theorem 2.2.1 gives

$$c(X) = \prod_{\delta\in\Sigma}\prod_{\alpha\in|\delta|}(1+X_\alpha)^{(-1)^{\operatorname{codim}\delta}} = \prod_{\alpha\in|\Sigma|}(1+X_\alpha)^{\sum_{\delta\ni\alpha}(-1)^{\operatorname{codim}\delta}} = \prod_{\alpha\in|\Sigma|}(1+X_\alpha)$$

($\sum_{\delta\ni\alpha}(-1)^{\operatorname{codim}\delta} = 1$ is the Euler characteristic).

Since the divisors $X_\alpha$ and $X_\beta$ do not intersect if $\langle\alpha, \beta\rangle \notin \Sigma$ and $X_\delta = \prod_{\alpha\in|\delta|} X_\alpha$, from the previous formula we get

$$c_k(X) = \sum_{\dim\delta=k} X_\delta.$$

3.2.4. Remark. For practical purposes it is convenient to write the formula for the Chern class in the form

$$c(\mathscr{E}) = \prod_{\sigma\in\Sigma} P_\sigma(\mathscr{E}),$$

where

$$P_\sigma(\mathscr{E}) = \prod_{\tau\subset\sigma}\det\left(1+\sum_{\alpha\in|\tau|}\alpha X_\alpha|\mathscr{E}(x_\sigma)\right)^{(-1)^{\dim\sigma-\dim\tau}}.$$

It follows from the definition that $P_\sigma \equiv 1 \pmod{\deg\dim\sigma}$. This allows one to shorten calculations when computing the first Chern class. For example

$$\begin{aligned} c_1(\mathscr{E}) &= \kappa_1\prod_{\alpha\in|\Sigma|}P_\alpha(\mathscr{E}) = \kappa_1\prod_{\alpha\in|\Sigma|}\det(1+\alpha X_\alpha|\mathscr{E}(x_\alpha)) \\ &= \sum_{\alpha\in|\Sigma|;\,\chi\in\widehat{T}_\alpha}\langle\chi, \alpha\rangle m(\chi, \mathscr{E}(x_\alpha))X_\alpha = \sum_{i,\alpha} i\dim E^{[\alpha]}(i)\cdot X_\alpha, \end{aligned}$$

where $E^{[\alpha]}(i) = E^\alpha(i)/E^\alpha(i+1)$.

## §4. Cohomology groups and the trace formula

**4.1.** In this section we shall calculate the cohomology groups of equivariant bundles on a nonsingular toral variety $X = X(\Sigma)$. Let the bundle $\mathscr{E}$ be determined by the filtrations $E^\alpha$, $\alpha \in |\Sigma|$, of the space $E$. For each cone $\sigma \in \Sigma$ and character $\chi \in \widehat{T}$ we put

$$E^\sigma(\chi) = \bigcap_{\alpha \in |\sigma|} E^\alpha(\langle \chi, \alpha \rangle), \qquad E_\sigma(\chi) = E / \sum_{\alpha \in |\sigma|} E^\alpha(\langle \chi, \alpha \rangle) \tag{4.1}$$

and consider the complex

$$C^*(\mathscr{E}, \chi) \colon 0 \to E \xrightarrow{d} \bigoplus_{\dim \sigma = 1} E_\sigma(\chi) \xrightarrow{d} \bigoplus_{\dim \sigma = 2} E_\sigma(\chi) \to \cdots \to \bigoplus_{\dim \sigma = n} E_\sigma(\chi) \to 0 \tag{4.2}$$

with the differential $d(a_\sigma) = \sum_{\tau \supset \sigma;\, \dim \tau = \dim \sigma + 1} (a_\sigma)_\tau$, $a_\sigma \in E_\sigma(\chi)$, where $(\ )_\tau \colon E_\sigma(\chi) \to E_\tau(\chi)$ is the canonical projection. We assume all the cones from the fan $\Sigma$ are oriented and in the formula for the differential the orientations of $\sigma$ and $\tau$ are compatible.

The torus $T$ acts canonically on the cohomology groups $H^*(X, \mathscr{E})$ of equivariant bundles. We denote the isotypical component corresponding to the character $\chi$ by $H^*(X, \mathscr{E})_\chi$.

4.1.1. THEOREM. *Cohomology groups of the complex $C^*(\mathscr{E}, \chi)$ are canonically isomorphic to the $\chi$-component of cohomology groups of the bundle $\mathscr{E}$:*

$$H(C^*(\mathscr{E}, \chi)) \simeq H(X, \mathscr{E})_\chi.$$

PROOF. We shall calculate the cohomology groups of the bundle $\mathscr{E}$ using the open cover

$$\mathscr{U} \colon X = \bigcup U_\delta; \qquad U_\delta = \operatorname{Spec} k[\delta], \ \delta \in \Sigma.$$

Since all the intersections $U_{\delta_1} \cap \cdots \cap U_{\delta_k} = U_{\delta_1 \cap \cdots \cap \delta_k}$ are acyclic, the cohomology groups $H(X, \mathscr{E})_\chi$ coincide with the cohomology groups of the complex

$$0 \to \prod_\delta \Gamma(U_\delta, \mathscr{E})_\chi \to \prod_{\delta, \sigma} \Gamma(U_\delta \cap U_\sigma, \mathscr{E})_\chi \to \prod_{\delta, \sigma, \tau} \Gamma(U_\delta \cap U_\sigma \cap U_\tau, \mathscr{E})_\chi \to \cdots \tag{4.3}$$

with standard differentials. One can view (4.3) as a cohomology complex for the nerve $\Sigma^* = \{(\delta_1, \dots, \delta_k) | \delta_i \in \Sigma\}$ of the cover $\mathscr{U}$ with a system of coefficients

$$\mathscr{E}^*(\chi) \colon (\delta_1, \dots, \delta_k) \to \Gamma(U_{\delta_1} \cap \cdots \cap U_{\delta_k}, \mathscr{E})_\chi = E^{\delta_1 \cap \cdots \cap \delta_k}(\chi) \tag{4.4}$$

(see [8], 1.3.3). Thus $H(X, \mathscr{E})_\chi = H(\Sigma^*, \mathscr{E}^*(\chi))$.

On the other hand the complex $C^*(\mathscr{E}, \chi)$ can be considered as a complex of alternating cochains of the simplicial scheme $\Sigma$ with a system of coefficients $\mathscr{E}(\chi) \colon \delta \mapsto E_\delta(\chi)$. Therefore the theorem reduces to proving the equality

$$H(\Sigma^*, \mathscr{E}^*(\chi)) = H(\Sigma, \mathscr{E}(\chi)),$$

which follows from the following result of Leray ([8], Chapter II, 5.2.4).

Let $\mathfrak{M} = \{M_i\}$ be a family of subcomplexes of the simplicial complex $K = \bigcup_i M_i$, and $\mathscr{E}$ be the coefficient system on $K$. Consider the coefficient system $\mathscr{H}^q(\mathscr{E})$ on the nerve of the covering $\mathfrak{M}$ defined by $(M_1, \dots, M_k) \mapsto H^q(M_1 \cap \cdots \cap M_k, \mathscr{E})$, and let $H^p(\mathfrak{M}, \mathscr{H}^q(\mathscr{E}))$ be the cohomology groups of this system. Then there is a spectral sequence with second term $H^p(\mathfrak{M}, \mathscr{H}^q(\mathscr{E}))$ which converges to $H^{p+q}(K, \mathscr{E})$.

In our situation the complex $\Sigma^*$ can be considered as the nerve of the covering of the fan $\Sigma$ by cones $\delta \in \Sigma$ on which the coefficient system $\mathscr{E}(\chi)$ is acyclic in all dimensions save zero, and $\mathscr{H}^0(\mathscr{E}(\chi)) = \mathscr{E}^*(\chi)$. This follows from the compatibility condition of the filtrations $E^\alpha$, $\alpha \in |\Sigma|$, which allows one to reduce to the case $\dim E = 1$. Here we consider $\Sigma$ as a simplicial scheme with the set of vertices $|\Sigma|$ whose simplices form subsets $|\sigma|$, $\sigma \in \Sigma$.

The Leray spectral sequence degenerates and gives the necessary isomorphism $H(\Sigma^*, \mathscr{E}^*(\chi)) \simeq H(\Sigma, \mathscr{E}(\chi))$.

4.1.2. REMARK. In some cases the dual complex for the dual Serre bundle $\mathscr{E}^* \otimes \Omega^n$ is more convenient. The filtration on the space $E^*$ for the dual bundle $\mathscr{E}^*$ is given by the formula $E^{*\alpha}(i) = (E/E^\alpha(1-i))^*$. For the Serre dual bundle $\mathscr{E}^* \otimes \Omega^n$ we get $(E^* \otimes \Omega^n)^\alpha(i) = (E/E^\alpha(-i))^*$. Finally, the "homology" complex of the bundle $\mathscr{E}$, dual to $C^*(\mathscr{E}^* \otimes \Omega^n, \chi^{-1})$, takes the form

$$C_*(\mathscr{E}, \chi): 0 \leftarrow E \leftarrow \bigoplus_{\dim\delta=1} E^\delta(\chi) \leftarrow \bigoplus_{\dim\delta=2} E^\delta(\chi) \leftarrow \cdots \leftarrow \bigoplus_{\dim\delta=n} E^\delta(\chi) \leftarrow 0. \quad (4.5)$$

The differential is given by the usual formula

$$d(a^\delta) = \sum_i (-1)^i (a^\delta)_i,$$

$$\delta = \langle \alpha_1, \dots, \alpha_k \rangle, \qquad \delta_i = \langle \alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_k \rangle,$$

where $(a^\delta)_i$ is an image of the element $a^\delta \in E^\delta(\chi)$ in the space $E^{\delta_i}(\chi)$.

From Serre duality the cohomology groups of the bundle $\mathscr{E}$ can be expressed in terms of the cohomology groups of the complex $C_*(\mathscr{E}, \chi)$:

$$H^p(X, \mathscr{E})_\chi = H_{n-p}(C_*(\chi, \mathscr{E})). \quad (4.6)$$

This equality is satisfied for complete varieties.

4.1.3. COROLLARY. *The following equalities hold*:

i) $H^0(X, \mathscr{E})_\chi = \bigcap_{\alpha\in|\Sigma|} E^\alpha(\chi)$.

ii) $H^n(X, \mathscr{E})_\chi = E/\sum_{\alpha\in|\Sigma|} E^\alpha(\chi)$; $n = \dim X$, $X$ *complete.*

iii)
$$\sum_i (-1)^i \dim H^i(X, \mathscr{E})_\chi = \sum_{\delta\in\Sigma} (-1)^{\dim\delta} \dim E_\delta(\chi) = \sum_{\delta\in\Sigma} (-1)^{\operatorname{codim}\delta} \dim E^\delta(\chi).$$

In order to prove i) and the first equality in iii) it suffices to use the complex $C^*(\mathscr{E}, \chi)$, and for ii) and the second equality in iii) the complex $C_*(\mathscr{E}, \chi)$.

**4.2.** Formula iii) for the Euler characteristic has another useful interpretation.

4.2.1. THEOREM (the trace formula). *Let $\mathscr{E}$ be an equivariant bundle on a complete nonsingular toral variety $X = X(\Sigma)$. Then, for all $t \in T$,*

$$\sum_i (-1)^i \operatorname{Tr}(t|H^i(X, \mathscr{E})) = \sum_\Delta \operatorname{Tr}(t|\mathscr{E}(x_\Delta)) / \prod_{\chi\in\Delta^*} (1 - \chi^{-1}(t)), \quad (4.7)$$

*where $\Delta$ is taken over cones of maximal dimension $n = \dim X$ in the fan $\Sigma$, $x_\Delta \in X^T$ is the fixed point corresponding to $\Delta$, and $\Delta^*$ is the basis of the character group $\widehat{T}$ dual to $|\Delta|$.*

PROOF. Let $\sum_\chi m_\Delta(\chi) \cdot \chi$ be the formal character of a representation of the torus $T$ in the fixed fiber $\mathscr{E}(x_\Delta)$. The multiplicities $m_\Delta(\chi)$ are related to the filtrations

$E^\alpha$ by the following formula (see §2.2.6):

$$\dim E_\delta(\chi) = \sum_{\langle\psi,\delta\rangle<\langle\chi,\delta\rangle} m_\Delta(\psi),$$

where $\Delta$ is an arbitrary cone of maximal dimension containing $\delta$ and $\langle\psi,\delta\rangle < \langle\chi,\delta\rangle$ is an abbreviation for $\langle\psi,\alpha\rangle < \langle\chi,\alpha\rangle$, $\forall\alpha\in|\delta|$.

Substituting this value into the formula for the Euler characteristic 4.1.3.iii), we get

$$\begin{aligned}\operatorname{Tr}(t|H^*(X,\mathscr{E})) &= \sum_{\chi,\delta}(-1)^{\dim\delta}\dim E_\delta(\chi)\cdot\chi \\ &= \sum_{\langle\psi,\delta\rangle<\langle\chi,\delta\rangle}(-1)^{\dim\delta}m_\Delta(\psi)\chi = \sum_{\langle\theta,\delta\rangle>0}(-1)^{\dim\delta}m_\Delta(\psi)\psi\theta \quad (4.8)\end{aligned}$$

(in the last equality we hae substituted $\theta = \chi\psi^{-1}$).

If $\delta$ is a proper face of the cone $\Delta$, then the polynomial $\prod_{\chi\in\Delta^*}(1-\chi^{-1})$ annihilates the Laurent series $\sum_{\langle\theta,\delta\rangle>0}\theta$. Therefore, multiplying both sides of (4.8) by $P=\prod_\Delta\prod_{\chi\in\Delta^*}(1-\chi^{-1})$, we get

$$\begin{aligned}P\cdot\operatorname{Tr}(t|H^*(X,\mathscr{E})) &= P\cdot\sum_{\Delta,\psi}(-1)^{\dim X}m_\Delta(\psi)\psi\sum_{\langle\theta,\Delta\rangle>0}\theta \\ &= P\sum_{\Delta,\psi}\frac{m_\Delta(\psi)\psi}{\prod_{\chi\in\Delta^*}(1-\chi^{-1})},\end{aligned}$$

which is equivalent to the statement of the theorem.

4.2.2. Corollary. *Under the assumptions of the theorem the Poincaré polynomial of the variety $X$ is given by*

$$P_X(s) = \sum_q s^q \dim H^{2q}(X,\mathbf{C}) = \sum_\Delta\prod_{\chi\in\Delta^*}\left(\frac{1-s\chi}{1-\chi}\right). \quad (4.9)$$

Formally the right-hand side of (4.9) is a rational function on the torus, which is actually constant! It is interesting to compare (4.9) to the more customary

$$P_X(s) = \sum_{k=0}^n |\Sigma^{(k)}|(s-1)^{n-k}.$$

Proof. It is known that the cohomology ring $H^*(X,\mathbf{C})$ is generated by the algebraic cycles $X_\delta$. Therefore, the Hodge structure in the cohomology groups is diagonal:

$$H^p(X,\Omega^q) = \begin{cases}0, & p\neq q,\\ H^{2q}(X,\mathbf{C}), & p=q.\end{cases}$$

We apply Theorem 4.2.1 to the bundle $\Omega^q$ (see Example 2.3.5):

$$\operatorname{Tr}(t|H^*(X,\Omega^q)) = (-1)^q\dim H^{2q}(X,\mathbf{C}) = \sum_\Delta\frac{\sigma_q(\chi^{-1}|\chi\in\Delta^*)}{\prod_{\chi\in\Delta^*}(1-\chi^{-1})},$$

where $\sigma_q$ is the elementary symmetric function. We get

$$P_X(s) = \sum_\Delta\prod_{\chi\in\Delta^*}\left(\frac{(1-s\chi^{-1}}{1-\chi^{-1}}\right) = \sum_\Delta\prod_{\chi\in\Delta^*}\left(\frac{1-s\chi}{1-\chi}\right).$$

In the last equation we have used Poincaré duality $P(s) = s^n P(s^{-1})$.

4.2.3. EXAMPLE. Formula (4.9) leads to a sequence of exotic identities related to toral varieties. Consider the simplest case of projective space $\mathbf{P}^n$ (Example 1.1.1). For $\mathbf{P}^n$ the Poincaré polynomial is well known:

$$P(s) = 1 + s + \cdots + s^n \underset{(4.9)}{=} \sum_{i=0}^{n} \prod_{j,\, j\neq i} \left(\frac{x_j - sx_i}{x_j - x_i}\right).$$

In the case when $x_i = q^{-i}$ for $n \to \infty$ leads to Cauchy's identity

$$\frac{1}{1-s} \prod_{i=1}^{\infty} \left(\frac{1-q^i}{1-sq^i}\right) = \sum_{i=0}^{\infty} \frac{(s-q)(s-q^2)\cdots(s-q^i)}{(1-q)(1-q^2)\cdots(1-q^i)},$$

known for $s = 0$ as Euler's identity.

4.2.4. REMARK. Formula (4.7) in Theorem 4.2.1 can also be written in the form

$$\operatorname{Tr}(t|H^*(X, \mathscr{E})) = \sum_{x\in X^T} \frac{\operatorname{Tr}(t|\mathscr{E}(x))}{\det(1 - t^{-1}|\mathscr{T}(x))},$$

where $X^T$ is the set of fixed points of the torus $T$ and $\mathscr{T}(x)$ is the tangent space at the point $x$. In this form the theorem still makes sense, and remains valid for a wide class of varieties on which a torus acts with isolated fixed points [9].

The trace formula (4.7) allows us to express a virtual representation of the torus in the cohomology groups $H^*(X, \mathscr{E})$ by its representation in fixed fibers $\mathscr{E}(x_\delta)$. Conversely, it is not difficult to express the representations $\mathscr{E}(x_\Delta)$ by cohomology groups.

4.2.5. PROPOSITION. *Let $\mathscr{E}$ be an equivariant bundle on a complete nonsingular toral variety $X$. Then, in the representation ring of the torus,*

$$\mathscr{E}(x_\Delta) = \sum_{\delta\subset\Delta} (-1)^{\dim\delta} H^*(X, \mathscr{E}(-\delta)), \tag{4.10}$$

*where $\mathscr{E}(-\delta) = \mathscr{E} \otimes \mathscr{O}(-\sum_{\alpha\in|\delta|} X_\alpha)$.*

PROOF. Denote the class of the point $x_\Delta$ in $K_T(X)$ by $[x_\Delta]$. Then $\mathscr{E}(x_\Delta) = H^*(X, \mathscr{E} \otimes [x_\Delta])$. On the other hand, $[X_\alpha] = 1 - \mathscr{O}(-\alpha)$, and so

$$[x_\Delta] = \prod_{\alpha\in|\Delta|} [X_\alpha] = \prod_{\alpha\in|\Delta|} (1 - O(-\alpha)) = \sum_{\delta\subset\Delta} (-1)^{\dim\delta} \mathscr{O}(-\delta).$$

The trace formula can be understood as a version of the simpler formula (4.10).

**4.3.** One can obtain explicit formulas for the cohomology groups of equivarient bundles over projective spaces.

From the description of the fan $\Sigma(\mathbf{P}^n)$ (Example 1.1.1) and Theorem 2.2.1 it follows that toral bundles over $\mathbf{P}^n$ are parametrized by collections of filtrations $E^\alpha$, $\alpha = 0, \ldots, n$, any $n$ of which are splittable (i.e. satisfy conditions i)–iii) of 2.2.2).

4.3.1 THEOREM. *Let $\mathscr{E}$ be a toral bundle over $\mathbf{P}^n$ determined by a family of filtrations $E^\alpha$, $\alpha = 0, \dots, n$. Then*

$$H^0(\mathbf{P}^n, \mathscr{E})_\chi = \bigcap_\alpha E^\alpha(\chi); \qquad H^n(\mathbf{P}^n, \mathscr{E})_\chi = E/\sum_\alpha E^\alpha(\chi);$$

$$H^p(\mathbf{P}^n, \mathscr{E})_\chi = \bigcap_{k\geq p}(E^0(\chi)+\cdots+E^{p-1}(\chi) + E^k(\chi))/(E^0(\chi)+\cdots+E^{p-1}(\chi)+\bigcap_{k\geq p}E^k(\chi)), \tag{4.11}$$

*where $E^\alpha(\chi)$ is determined by* (4.1).

PROOF. Consider the exact sequence of complexes in which $C(\chi) = C^*(\mathscr{E}, \chi)$:

$$\begin{array}{ccccccccc}
 & 0 & & & & & & & \\
 & \downarrow & & & & & & & \\
C'(\chi): & 0\to 0 & \to & E/E^0(\chi) & \to & \bigoplus_{\beta\neq 0} E/E^0(\chi)+E^\beta(\chi) & \to & \bigoplus_{\beta,\gamma\neq 0} E/E^0(\chi)+E^\beta(\chi)+E^\gamma(\chi) & \to\cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
C(\chi): & 0\to E & \to & \bigoplus_\alpha E/E^\alpha(\chi) & \to & \bigoplus_{\alpha,\beta} E/E^\alpha(\chi)+E^\beta(\chi) & \to & \bigoplus_{\alpha,\beta,\gamma} E/E^\alpha(\chi)+E^\beta(\chi)+E^\gamma(\chi) & \to\cdots \\
 & \downarrow & \downarrow & \downarrow & & \downarrow & & \downarrow & \\
C''(\chi): & 0\to E & \to & \bigoplus_{\alpha\neq 0} E/E^\alpha(\chi) & \to & \bigoplus_{\alpha,\beta\neq 0} E/E^\alpha(\chi)+E^\beta(\chi) & \to & \bigoplus_{\alpha,\beta,\gamma\neq 0} E/E^\alpha(\chi)+E^\beta(\chi)+E^\gamma(\chi) & \to\cdots \\
 & \downarrow & & & & & & & \\
 & 0 & & & & & & &
\end{array}$$

According to Theorem 4.1.1, $H^*(C(\chi)) = H^*(\mathbf{P}^n, \mathscr{E})_\chi$.

The complex $C''(\chi)$ is acyclic in all dimensions save zero and $n-1$, since the filtrations $E^\alpha$, $\alpha \neq 0$, split due to the compatibility condition of Theorem 2.2.1.

Finally, the complex $C'(\chi)$ can be considered as the standard complex of Theorem 4.1.1 associated to a bundle with fiber $E/E^0(\chi)$ over $\mathbf{P}^{n-1}$ and grading increased by one. By induction one can assume that the cohomology groups of the complex $C'(\chi)$ are given by the formulas of our theorem. Then from the exact cohomology sequence

$$\begin{array}{ccccccc}
\cdots\to & H^{k-1}(C''(\chi)) & \to H^k(C'(\chi)) \to H^k(C(\chi)) \to & H^k(C''(\chi)) & \to\cdots \\
 & \| & & \| & \\
 & 0 & & 0 &
\end{array}$$

we obtain the necessary formulas for $H^k(C(\chi)) = H^k(\mathbf{P}^n, \mathscr{E})_\chi$.

4.3.2 REMARK. If we use the homology complex $C_*(\mathscr{E}, \chi)$ from §4.1.2 instead of the cohomology complex $C^*(\mathscr{E}, \chi)$, then by a similar computation we can prove that

$$H^{n-k}(\mathbf{P}^n, \mathscr{E})_\chi = E^0(\chi)\cap\cdots\cap E^{k-1}(\chi)\cap\sum_{i\geq k}E^i(\chi) \Big/ \sum_{i\geq k} E^0(\chi)\cap\cdots\cap E^{k-1}(\chi)\cap E^i(\chi).$$

Note also that the dependence of these formulas on the ordering of the filtrations is only illusory. For example, it is well known that the ratios

$$\frac{E^\alpha\cap(E^\beta+E^\gamma)}{E^\alpha\cap E^\beta + E^\alpha\cap E^\gamma} = H^1(\mathbf{P}^2, \mathscr{E})$$

are canonically isomorphic for an arbitrary permutation $\alpha, \beta, \gamma$ of the indices $1, 2, 3$. Nevertheless, one can write down a symmetric formula for the double cohomology groups:

$$H^1(\mathbf{P}^2, \mathscr{E}) \oplus H^1(\mathbf{P}^2, \mathscr{E}) \simeq \frac{(E^\alpha + E^\beta) \cap (E^\beta + E^\gamma) \cap (E^\gamma + E^\alpha)}{E^\alpha \cap E^\beta + E^\beta \cap E^\gamma + E^\gamma \cap E^\alpha}.$$

For similar relations for $\binom{n}{p} H^p(\mathbf{P}^n, \mathscr{E})$, see §6.4.3.

4.3.3. Corollary (Horrock's criterion). *A toral bundle $\mathscr{E}$ over $\mathbf{P}^n$ splits if and only if $H^p(\mathbf{P}^n, \mathscr{E}(k)) = 0$, $k \in \mathbf{Z}$, $0 < p < n$.*

Indeed, according to condition iii) of §2.2.2 all the cohomology groups $H^p(\mathbf{P}^n, \mathscr{E}(k))$ are equal to zero if and only if the filtrations $E^\alpha$, $\alpha = 0, \dots, n$, generate a distributive lattice. From this it follows that the bundle $\mathscr{E}$ splits (Corollary 2.2.3).

4.3.4. Corollary. *For an arbitrary character $\chi$,*

$$\sum_i \dim H^i(\mathbf{P}^n, \mathscr{E})_\chi \le \operatorname{rk} \mathscr{E},$$

*with equality only in the case $E^\alpha(\chi) = E^\beta(\chi)$, $\forall \alpha, \beta$. Moreover, $H^i(\mathbf{P}^n, \mathscr{E})_\chi = 0$, $0 < i < n$, and $\mathscr{E}$ has a subbundle of rank $r = \dim H^0(\mathbf{P}^n, \mathscr{E})_\chi$.*

Proof. The inequality follows from the fact that every subsequent term in (4.11) lies higher than the previous one:

$$E^0(\chi) + \cdots + E^p(\chi) + \bigcap_{i>p} E^i(\chi) \supset \bigcap_{i \ge p} (E^0(\chi) + \cdots + E^{p-1}(\chi) + E^i(\chi)). \qquad (4.12)$$

Equality in (4.12) can happen for all $p$ only if all the spaces $E^i(\chi)$, $i = 0, \dots, n$, coincide. Then the subspace $\widetilde{E} = E^i(\chi) \subset E$ with the induced filtrations defines a subbundle of rank $r = \dim \widetilde{E} = \dim \bigcap_i E^i(\chi) = \dim H^0(\mathbf{P}^n, \mathscr{E})_\chi$.

The inequality in Corollary 4.3.4 can be significantly strengthened (see Corollary 6.4.3).

4.3.5. Example. *The cohomology groups of general bundles over $\mathbf{P}^2$.*

Consider a bundle $\mathscr{E}$ of rank over $\mathbf{P}^2$ defined by three filtrations $E^\alpha$, $E^\beta$, and $E^\gamma$ which are in general position. We shall denote the sum of dimensions of the spaces $E^\alpha(\chi)$, $E^\beta(\chi)$, and $E^\gamma(\chi)$ by $d(\mathscr{E}, \chi)$. Then by Theorem 4.3.1 we have

$$\dim H^0(\mathbf{P}^2, \mathscr{E})_\chi = \dim E^\alpha(\chi) \cap E^\beta(\chi) \cap E^\gamma(\chi) = d(\mathscr{E}, \chi) - 2r, \qquad d(\mathscr{E}, \chi) \ge 2r.$$

$$\dim H^1(\mathbf{P}^2, \mathscr{E})_\chi = \dim \frac{E^\alpha(\chi) \cap (E^\beta(\chi) + E^\gamma(\chi))}{E^\alpha(\chi) \cap E^\beta(\chi) + E^\alpha(\chi) \cap E^\gamma(\chi)} = d(\mathscr{E}, x) - r,$$

$$2r > d(\mathscr{E}, \chi) \ge r.$$

$$\dim H^2(\mathbf{P}^2, \mathscr{E})_\chi = \dim E/E^\alpha(\chi) + E^\beta(\chi) + E^\gamma(\chi) = r - d(\mathscr{E}, \chi), \qquad r \ge d(\varepsilon, \chi).$$

The remaining cohomology groups vanish.

In particular, the representations of a torus on the cohomology groups $H^i(\mathbf{P}^2, \mathscr{E})$, $i = 0, 1, 2$, are pairwise disjoint, and each character with $d(\mathscr{E}, \chi) \ne r, 2r$ appears in some space $H^i(\mathbf{P}^2, \mathscr{E})$.

## §5. Intersection indices and the Chow ring

**5.1.** An as application of the technique developed above we shall obtain explicit formulas for the intersection index of cycles and for the Chern numbers of toral bundles.

Let $X = X(\Sigma)$ be a complete nonsingular toral variety and $Z = \sum_{\dim\delta=d} m_\delta X_\delta$ an invariant cycle on $X$ of codimension $d$. We shall associate to $Z$ a collection of polynomial functions $Z_\Delta(h)$, $\Delta \in \Sigma^{(n)}$, of degree $d$ on the space $\widehat{T}^0_{\mathbf{R}}$:

$$Z_\Delta(h) = \sum_{\delta\subset\Delta} m_\delta \prod_{\alpha\in|\delta|} \langle\alpha^*, h\rangle, \qquad h \in \widehat{T}^0_{\mathbf{R}}, \tag{5.1}$$

where $\alpha^*$, $\alpha \in |\Delta|$, are elements of the basis $\Delta^*$ of the character lattice $\widehat{T}$ dual to the basis $|\Delta|$ of the lattice $\widehat{T}^0$.

Note that the polynomials $Z_\Delta(h)$, considered as functions on the cones $\Delta$, are compatible on common faces and patch together to give a single continuous function $Z(h)$ on $\widehat{T}^0_{\mathbf{R}}$.

5.1.1. THEOREM. *Let $Z^i$, $i = 1, \dots, m$, be invariant cycles on $X$ and $\sum_i \operatorname{codim} Z^i = \dim X$. Then their intersection index equals*

$$(Z^1 \cdots Z^m) = \sum_\Delta \prod_i Z^i_\Delta(h) \Big/ \prod_{\chi\in\Delta^*} \langle\chi, h\rangle, \tag{5.2}$$

*where the summation is taken over simplices $\Delta$ of maximal dimension $n = \dim X$, $\Delta^*$ is a basis of the character lattice $\widehat{T}$ dual to $|\Delta|$, and the polynomials $Z^i_\Delta(h)$ are defined by* (5.1).

In the theorem the intersection index is written in the form of a rational function on $\widehat{T}^0_{\mathbf{R}}$ which is actually constant (see 4.2.2).

PROOF. Since both sides of (5.2) are multilinear, it suffices to prove it for the closures of orbits $X_\delta = \overline{O}_\delta$. Now $X_\delta$ is a normal intersection of divisors $X_\delta = \prod_{\alpha\in|\delta|} X_\alpha$. Therefore, it suffices to consider the case of codimension one orbits. Let $J_\alpha$, $\alpha \in |\Sigma|$, denote the ideal of functions which vanish on $X_\alpha$. It is isomorphic to the invertible sheaf $\mathcal{O}(-X_\alpha)$. Consequently, we have the resolution

$$0 \leftarrow \mathcal{O}_X/J_\alpha \leftarrow \mathcal{O}_X \leftarrow \mathcal{O}(-X_\alpha) \leftarrow 0.$$

In order to find the intersection index we should, by Serre [10], tensor the resolutions $\mathcal{O}_X \leftarrow \mathcal{O}(-X_\alpha) \leftarrow 0$ together and take the Euler characteristic:

$$(X_{\alpha_1} \cdots X_{\alpha_n}) = \sum_{I\subset\{1,\dots,n\}} (-1)^{|I|} \chi\left(\mathcal{O}\left(-\sum_{i\in I} X_{\alpha_i}\right)\right).$$

In order to calculate the Euler characteristic we use the trace formula (§4.2.1):

$$\begin{aligned}(X_{\alpha_1} \cdots X_{\alpha_n}) &= \lim_{t\to 1} \sum_{I\subset\{1,\dots,n\}} (-1)^{|I|} \sum_\Delta \prod_{i\in I;\,\alpha_i\in|\Delta|} \alpha^{*-1}_{i\Delta} \Big/ \prod_{\chi\in\Delta^*} (1-\chi^{-1}) \\ &= \lim_{t\to 1} \sum_{\Delta\supset\{\alpha_1,\dots,\alpha_n\}} \prod_{i=1}^n (1-\alpha^{*-1}_{i\Delta}) \Big/ \prod_{\chi\in\Delta^*} (1-\chi^{-1}),\end{aligned}$$

where, for an element $\alpha \in |\Delta|$ we let $\alpha^*_\Delta = \alpha^*_\Delta(t)$, $t \in T$, denote the corresponding element of the dual basis $\Delta^*$ of the character lattice $\widehat{T}$. If in the last formula we pass to the limit along $h \in \widehat{T}^0_{\mathbf{R}}$, then we get

$$(X_{\alpha_1} \cdots X_{\alpha_n}) = \sum_{\Delta \supset \{\alpha_1, \dots, \alpha_n\}} \prod_i \langle \alpha^*_{i\Delta}, h\rangle \Big/ \prod_{\chi \in \Delta^*} \langle \chi, h\rangle.$$

This equality coincides, in the case we are considering, with the conclusion of the theorem.

**5.2.** We shall apply the formula for the intersection index to calculate the Chern numbers of toral bundles.

We consider the graded equivariant bundle $\mathscr{E} = \mathscr{E}_1 \oplus \cdots \oplus \mathscr{E}_m$. We shall call the following polynomials its $k$th *Chern class*

$$c_k(\mathscr{E}, z) = \sum_{k_1 + \cdots + k_m = k} c_{k_1}(\mathscr{E}_1) \cdots c_{k_m}(\mathscr{E}_m) z_1^{k_1} \cdots z_m^{k_m}.$$

For conciseness, we denote the product $\prod_i \det(1 + hz_i|\mathscr{E}_i(x))$, $x \in X^T$, $h \in \widehat{T}^0_{\mathbf{R}} \subset \mathscr{L}ie\, T$, by $\det(1 + hz|\mathscr{E}(x))$. We shall prove the following formula for the top Chern class.

5.2.1. THEOREM. *Let $\mathscr{E}$ be a graded equivariant bundle on a complete nonsingular toral variety $X = X(\Sigma)$ of dimension $n$. Then*

$$c_n(\mathscr{E}, z) = \kappa_n \sum_\Delta \frac{\det(1 + hz|\mathscr{E}(x_\Delta))}{\prod_{\chi \in \Delta^*}} \langle \chi, h\rangle, \tag{5.3}$$

*where $\kappa_n(\ )$ is the component of total degree $n$ with respect to $z$.*

For the proof we use the graded version of the formula for the total Chern class (§3.2.1):

$$c(\mathscr{E}, z) = \prod_i c(\mathscr{E}_i, z_i) = \prod_i \sum_{\delta \in \Sigma} \det\left(1 + z_i \sum_{\alpha \in |\delta|} \alpha X_\alpha |\mathscr{E}_i(x_\delta)\right)^{(-1)^{\operatorname{codim} \delta}}$$

$$= \prod_{\delta \in \Sigma} \det\left(1 + z \sum_{\alpha \in |\delta|} \alpha X_\alpha |\mathscr{E}(x_\delta)\right)^{(-1)^{\operatorname{codim} \delta}} \tag{5.4}$$

(the last term is just an abbreviation for the preceeding one).

The top Chern class can be derived from (5.4) using the formula for multiplying cycles (5.2). In this formula we need to replace every $X_\alpha$ by $\langle \alpha^*_\Delta, h\rangle$ if $\alpha \in |\Delta|$ and by zero if $\alpha \notin |\Delta|$, divide by $\prod_{\chi \in \Delta^*} \langle \chi, h\rangle$, and sum over all cones in $\Delta$ of maximal dimension:

$$c_n(\mathscr{E}, z) = \kappa_n \sum_\Delta \prod_{\delta \in \Sigma} \det\left(1 + z \sum_{\alpha \in |\delta \cap \Delta|} \alpha \langle \alpha^*_\Delta, h\rangle |\mathscr{E}(x_\delta)\right)^{(-1)^{\operatorname{codim} \delta}} \Big/ \prod_{\chi \in \Delta^*} \langle \chi, h\rangle. \tag{5.5}$$

If one fixes $\tau = \delta \cap \Delta$ in the product, then the power of the term $\det(1 + z \sum_{\alpha \in |\tau|} \alpha \langle \alpha^*_\Delta, h\rangle |\mathscr{E}(x_\tau))$ will be equal to

$$\sum_{\delta \cap \Delta = \tau} (-1)^{\operatorname{codim} \delta} = \begin{cases} 0, & \text{if } \tau \neq \Delta \\ 1, & \text{if } \tau = \Delta. \end{cases} \tag{5.6}$$

This and the equality $h = \sum_{\alpha\in|\Delta|}\alpha\langle\alpha_\Delta^*, h\rangle$ show that the numerator in (5.5) is $\det(1+zh|\mathscr{E}(x_\Delta))$. This proves the theorem.

5.2.2. COROLLARY. *Let $P(c_1, \dots, c_n)$ be a weighted homogeneous polynomial of degree $n = \dim X$ of the characteristic classes $c_i = c_i(\mathscr{E})$, $\deg c_i = i$, of an equivariant bundle $\mathscr{E}$. Then*

$$P(c_1, \dots, c_n) = \sum_\Delta P(c_{1\Delta}, \dots, c_{n\Delta}) \Big/ \prod_{\chi\in\Delta^*}\langle\chi, h\rangle, \tag{5.7}$$

*where the $c_{i\Delta} = c_{i\Delta}(t)$ are the coefficients of the characteristic polynomial*

$$\det(1+zh|\mathscr{E}(x_\Delta)) = \sum_i c_{i\Delta}(h)z^i, \qquad h\in \widehat{T}^0_{\mathbf{R}} \subset \mathscr{L}ie\, T.$$

The proof reduces to applying the theorem to a graded bundle all components of which are isomorphic to $\mathscr{E}$.

5.2.3. REMARK. Formulas similar to (5.7) can also be obtained from the Bott residue formulas ([11], Chapter 3, §4).

5.2.4. EXAMPLE. Since toral varieties are rational, their Todd genus equals one: $\mathrm{Td}(c_1, \dots, c_n) = \chi(\mathscr{O}_X) = 1$. For a toral surface $X$ this gives $c_1^2 + c_2 = 12$, from which, applying (5.7), we obtain

$$\sum_{i=1}^N \frac{(x_i+y_i)^2 + x_iy_i}{x_iy_i} = 12 \Leftrightarrow \sum_i\left(\frac{x_i}{y_i} + \frac{y_i}{x_i}\right) = 12 - 3N,$$

where $(x_i, y_i)$ are the coordinates of an arbitrary vector $h\in\widehat{T}^0_{\mathbf{R}}$ in the $i$th basis of the fan $\Sigma(X)$ and $N$ is the number of bases. For three-dimensional varieties it follows from the relation $c_1c_2 = 24$ that

$$\sum_i\left(\frac{x_i}{y_i} + \frac{y_i}{x_i} + \frac{y_i}{z_i} + \frac{z_i}{y_i} + \frac{z_i}{x_i} + \frac{x_i}{z_i}\right) = 24 - 3N.$$

**5.3.** To conclude this section we consider the construction of the Chow ring $\mathrm{CH}(X)$ and the Grothendieck ring $K(X)$ of a toral variety $X$. We would like to construct a "canonical" basis of the additive groups of these rings. Along with all formulas of this subsection it will depend on a choice of an element $h\in\widehat{T}^0_{\mathbf{R}}$ which is in general position with respect to the fan $\Sigma = \Sigma(X)$; this means that all the coordinates of the vector $h$ in an arbitrary basis $|\Delta|$, $\Delta\in\Sigma$, are nonzero. We shall call such an element $h$ a *polarization vector.* We shall let $\Delta(h)\subset\Delta$ denote the face of the cone $\Delta$ spanned by vectors of the basis $|\Delta|$ for which the corresponding coordinates of the vector $h$ are negative. We shall call cones of the form $\Delta(h)$ together with cycles $X_{\Delta(h)}$ and divisors $D_{\Delta(h)} = \sum_{\alpha\in|\Delta(h)|}X_\alpha$ corresponding to them *distinguished* (with respect to the polarization $h$).

5.3.1. THEOREM. *Let $X$ be a smooth projective toral variety. Then for an arbitrary polarization $h\in\widehat{T}^0_{\mathbf{R}}$ the following conditions are satisfied.*

i) *The distinguished cycles form a basis of the additive group of the Chow ring* $\mathrm{CH}(X)$.

ii) *The invertible sheaves $\mathscr{O}(-\delta) = \mathscr{O}(-D_\delta)$ corresponding to distinguished divisors $D_\delta$, $\delta\in\Sigma$, form a basis of the group $K(X)$.*

5.3.2. COROLLARY. *The rank of the group* $\mathrm{CH}^d(X)''$ *equals the number of bases* $|\Delta|$, $\Delta \in \Sigma$, *in which the polarization vector has exactly* $d$ *negative coordinates.*

It is interesting to compare this corollary with the formula for the Poincaré polynomial (4.2.2). The theorem is consistent with the general philosophy of this section: in order to obtain explicit formulas for the intersection index, Chern numbers, etc., it is necessary to fix a polarization or, equivalently, an invariant vector field on $X$.

PROOF OF THE THEOREM. i) We first introduce some useful definitions and notation. Let $X_f = \sum_\alpha f(\alpha)X_\alpha$ be an ample divisor on $X$. By the Demazure criterion [1], [3], this is equivalent to saying that the convex polyhedron $\Gamma = \{x \in \widehat{T}_{\mathbf{R}} | \langle x, \alpha\rangle \leq f(\alpha)\}$ generates a fan $\Sigma$, i.e. the map $\delta \mapsto \Gamma_\delta = \{x \in \Gamma | \langle x, \alpha\rangle = f(\alpha), \forall\alpha \in |\delta|\}$ defines a one-to-one correspondence between the cones $\delta \in \Sigma$ and the faces $\Gamma_\delta \subset \Gamma$; $\dim \delta = \operatorname{codim} \Gamma_\delta$. Corresponding to the polarization vector $h$ there is a linear form $H$ on the space $\widehat{T}_{\mathbf{R}}$ which takes different values on an arbitrary pair of adjacent vertices of $\Gamma$. We call $H$ the *height function*. Each face $\Gamma_\delta$ has a unique vertex $v_\delta \in \Gamma_\delta$ of maximal height. Define the height of a face $H(\Gamma_\delta)$ to be $H(v_\delta)$. Amongst the faces with a given maximal vertex $v$ there is a largest one $\Gamma_\delta$. It corresponds to the distinguished simplex $\delta$ and will also be called the distinguished vertex.

STEP 1. *It suffices to prove that distinguished cycles generate the group* $\mathrm{CH}(X)$.

Indeed, it is well known that $\operatorname{rk}\mathrm{CH}(X) = |\sum^{(n)}|$ (cf. §4.2.2). This coincides with the number of distinguished cycles.

STEP 2. *The faces* $\Gamma_\delta \subset \Gamma$ *of a fixed dimension* $d$ *which have minimal height are distinguished.*

Indeed, let $\Gamma_\tau \supset \Gamma_\delta$ be a distinguished face with the maximal vertex $v_\delta$. If $\Gamma_\tau \neq \Gamma_\delta$ then $\Gamma_\tau$ has a face of dimension $d$ which does not contain the maximal vertex $v_\delta$. It has a smaller height than $\Gamma_\delta$, which is impossible by assumption. Consequently, the face $\Gamma_d = \Gamma_\tau$ is distinguished.

STEP 3. *For any face* $\Gamma_\tau \subset \Gamma$ *and vertex* $v \in \Gamma_\tau$ *the cycles* $X_\delta$ *which correspond to the faces* $\Gamma_\delta \subset \Gamma_\tau$ *containing* $v$ *can be expressed linearly, in the Chow ring, by the cycles* $X_\rho$ *corresponding to the faces* $\Gamma_\rho \subset \Gamma_\tau$ *not containing* $v$.

Dually, the basis cone $\Delta \supset \tau$ corresponds to the vertex $v \in \Gamma_\tau$. We need to prove that the cycles $X_\delta$, $\tau \subset \delta \subset \Delta$, are linear combinations of the cycles $X_\rho$, $\tau \subset \rho \not\subset \Delta$.

Consider the case $\delta = \langle \tau, \alpha\rangle$, $\alpha \in |\Sigma|$. Choose a character $\chi \in \widehat{T}$, for which $\langle \chi, \alpha\rangle = 1$ and $\langle \chi, \beta\rangle = 0$ for $\alpha \neq \beta \in |\Delta|$. In the Chow ring we have the relation $\sum_{\gamma\in|\Sigma|}\langle \chi, \gamma\rangle X_\gamma = 0$ (see [1]). By multiplying this by $X_\tau$ and using the relation $X_\alpha X_\tau = X_{\langle\alpha, \tau\rangle} = X_\delta$ we obtain the required representation of the cycle $X_\delta$. The general case reduces to this if in the argument one replaces the element $\tau$ by an arbitrary face $\tau' \supset \tau$ of codimension 1 in $\delta$.

STEP 4. *Any cycle is a linear combination of distinguished cycles.*

We fix the dimension of the cycles $X_\delta$ and argue by induction on the height $H(\Gamma_\delta)$. Step 2 provides the initial step of the induction.

If the face $\Gamma_\delta$ is distinguished, then there is nothing to prove. For the opposite case let $\Gamma_\tau \supset \Gamma_\delta$ be a distinguished face and $v_\delta$ a general maximal vertex of the faces $\Gamma_\delta$ and $\Gamma_\tau$. By Step 3, $X_\delta$ can be expressed in the Chow ring by cycles $X_\rho$ corresponding to faces $\Gamma_\rho \subset \Gamma_\tau$ which do not contain $v_\delta$. These faces have a smaller height. Therefore, by the induction hypothesis, the cycles $X_\rho$ are linear combinations of distinguished cycles.

Assertion i) of the theorem is now proved. In order to prove the second assertion it is sufficient to express the classes of the distinguished cycles $X_\delta$ in $K(X)$ by the distinguished invertible sheaves $\mathscr{O}(-\delta)$ [18]. It is easy to do this by induction on $\dim\delta$ using the formulas

$$X_\delta = \prod_{\alpha\in|\delta|} X_\alpha = \prod_{\alpha\in|\delta|}(1-\mathscr{O}(-\alpha)) = \sum_{\sigma\subset\delta}(-1)^{\dim\sigma}\mathscr{O}(-\sigma),$$
$$\mathscr{O}(-\sigma) = \prod_{\alpha\in|\sigma|}(1-X_\alpha) = \sum_{\tau\subset\sigma}(-1)^{\dim\tau}X_\tau.$$

5.3.3. Remark. Another proof of this theorem with the Chow ring replaced by homology groups can be obtained by using Atiyah's arguments in [19] (cf. [15] and [16]). To a polarization vector $h$ one associates a Hamiltonian vector field on $X$ whose zeros coincide with the fixed points $x_\Delta \in X$ of the torus. The Hamitonian function of this field is an exact Morse function, and the index of the critical point $x_\Delta$ equals twice the number of negative coordinates of the vector $h$ in the basis $|\Delta|$. Morse theory, in this situation, allows one to find the Betti numbers and to construct a basis of the homology groups. The proof presented in the main text makes use of a combinatorial version of these ideas.

## §6. Structure results

**6.1.** In this section we shall prove several general results on construction of toral bundles based on their description in terms of filtrations given in §2. The central role in this description is played by the compatibility condition of Theorem 2.2.1. We shall deal with nonsingular varieties and therefore shall adopt the following definition (see §2.2.2, i)–iii)).

6.1.1. Definition. We shall call the family of subspaces $E^\alpha \subset E$, $\alpha \in A$, *splittable* if it generates a distributive lattice. A family of filtrations $E^\alpha(i)$, $\alpha\in A$, is splittable if the family of subspaces $E^\alpha(i)$, $\alpha\in A$, $i\in\mathbf{Z}$, is splittable.

Splitable systems can be represented in the form of a direct sum of systems of rank one $(=\dim E)$.

It is convenient to write the splittability condition of the filtrations $E^\alpha(i)$ in terms of the parabolic subgroups $P^\alpha = \{g\in \mathrm{GL}(E) | gE^\alpha(i) = E^\alpha(i)\}$:

$$\{E^\alpha | \alpha\in A\} \text{ is splittable } \Leftrightarrow \bigcap_{\alpha\in A} P^\alpha \text{ contains a maximal torus.} \qquad (6.1)$$

Relation (6.1) together with Theorem 2.2.1 show that the study of toral bundles with fiber $E$ on a nonsingular variety $X = X(\Sigma)$ is essentially equivalent to describing simplicial maps from the fan $\Sigma$ to the complex $\mathscr{P}(E)$ whose vertices are the parabolic subgroups $P\subset \mathrm{GL}(E)$ and whose simplices form families $P^\alpha\subset\mathrm{GL}(E)$, $\alpha\in A$, containing a common maximal torus. In this way the complex $\mathscr{P}(E)$ plays the role of a classifying space for toral bundles. Its study is the principal aim of this section.

We note the formal analogy between $\mathscr{P}(E)$ and the Brauer-Tits complex in which the incidence relation between parabolic subgroups is defined by the presence of a general Borel subgroup.

We start from the following analog for $\mathscr{P}(E)$ of Helly's theorem about convex sets.

6.1.2. Theorem. *A family of subspaces $E^\alpha$, $\alpha\in A$, of anm m-dimensional space $E$ is splittable if and only if each $(m+1)$-element subfamily is splittable.*

AN EQUIVALENT FORMULATION. *For a family of parabolic subgroups $P^\alpha \subset \mathrm{GL}(m)$ to have a common maximal torus it is necessary and sufficient that any $m+1$ subgroups of this family contain a maximal torus.*

6.1.3. COROLLARY. *Let $\mathscr{E}$ be an equivariant bundle on an open $T$-invariant subset $Y$ of a smooth toral variety $X$, and let $\operatorname{rk}\mathscr{E} < \operatorname{codim}(X\backslash Y)$. Then $\mathscr{E}$ extends to a toral bundle on $X$.*

The corollary follows from the fact that under our assumptions the fan $\Sigma(Y)$ contains an $(m+1)$-dimensional skeleton of the fan $\Sigma(X)$, $m = \operatorname{rk}\mathscr{E}$. Therefore the filtrations $E^\alpha$, $\alpha \in |\Sigma(Y)| = |\Sigma(X)|$, which define the bundle $\mathscr{E}$ will be compatible on $\Sigma(X)$ and so determine an extension of $\mathscr{E}$ over $X$.

The proof of the theorem is based on the following criterion of Johnson distributivity (cf. §2.2.2, i)–iii)):

$$(E^\alpha | \alpha \in A) \text{ is splittable } \Leftrightarrow \forall B, C \subset A \left(\sum_{\beta\in B} E^\beta\right) \bigcap_{\gamma\in C} E^\gamma = \sum_{\beta\in B}\left(E^\beta \bigcap_{\gamma\in C} E^\gamma\right). \tag{6.2}$$

We need to check that if the equality on the right-hand side of (6.2) is satisfied for $|B| + |C| \le m+1$, then it is satisfied for all $B, C \subset A$.

We argue by induction on $m = \dim E$.

STEP 1. For all $B \subset A$ and $\alpha \in A$

$$\left(\sum_{\beta\in B} E^\beta\right) \cap E^\alpha = \sum_{\beta\in B} E^\beta \cap E^\alpha. \tag{6.3}$$

Indeed, the inclusion $\supset$ is obvious. Conversely, since $\dim E = m$, there exists a subset $B' \subset B$, $|B'| \le m$, for which $\sum_{\beta\in B'} E^\beta = \sum_{\beta\in B} E^\beta$. Then

$$\left(\sum_{\beta\in B} E^\beta\right) \cap E^\alpha = \left(\sum_{\beta\in B'} E^\beta\right) \cap E^\alpha \underset{(6.2)}{=} \sum_{\beta\in B'} E^\beta \cap E^\alpha \subset \sum_{\beta\in B} E^\beta \cap E^\alpha.$$

STEP 2. *For any eigensubspace $E^\alpha \subset E$ the family $E^\beta \cap E^\alpha$, $\beta \in A$, is splittable in $E^\alpha$.*

Indeed, for $|B| + |C| \le m$ by the assumption of the theorem we have

$$\left(\sum_{\beta\in B} E^\beta \cap E^\alpha\right) \bigcap_{\gamma\in C} E^\gamma \cap E^\alpha \underset{(6.3)}{=} \left(\sum_{\beta\in B} E^\beta\right) \bigcap_{\gamma\in C} E^\gamma \cap E^\alpha \underset{(6.2)}{=} \sum_{\beta\in B} E^\beta \bigcap_{\gamma\in C} E^\gamma \cap E^\alpha. \tag{6.4}$$

Since $\dim E^\alpha < \dim E = m$, the inductive hypothesis implies that the splittability of the family $E^\beta \cap E^\alpha$, $\beta \in A$, follows from (6.4).

STEP 3. *The equality on the right-hand side of the Johnson criterion* (6.2) *is satisfied for any $B, C \subset A$.*

Indeed, let $E^\alpha \subset E$, $\alpha \in C$, be an eigenspace in $E$ (if there is no such $\alpha$, (6.2) is obviously satisfied). Then

$$\begin{aligned}\left(\sum_{\beta\in B} E^\beta\right) \bigcap_{\gamma\in C} E^\gamma &= \left(\sum_{\beta\in B} E^\beta\right) \cap E^\alpha \bigcap_{\gamma\in C} E^\gamma \\ &\underset{(6.3)}{=} \left(\sum_{\beta\in B} E^\beta \cap E^\alpha\right) \bigcap_{\gamma\in C} E^\gamma \cap E^\alpha = \sum_{\beta\in B} E^\beta \bigcap_{\gamma\in C} E^\gamma.\end{aligned}$$

For the last equality we used the splittability of the family $E^\beta \cap E^\alpha$, $\beta \in A$ (Step 2). The theorem is proved.

6.1.4. COROLLARY. *The following conditions on a nonsingular toral variety* $X = X(\Sigma)$ *are equivalent:*
i) *All toral bundles of rank* $m$ *over* $X$ *split.*
ii) *Any* $m+1$ *vectors in* $|\Sigma|$ *generate a cone in* $\Sigma$.

PROOF. ii) $\Rightarrow$ i). Let $E^\alpha$, $\alpha \in |\Sigma|$, be a family of filtrations of the space $E$ defining a bundle $\mathscr{E}$ of rank $m$. Then, by the compatibility condition of Theorem 2.2.1, any $m+1$ filtrations of this family are splittable. Consequently, by Theorem 6.1.2, the whole family $E^\alpha$, $\alpha \in |\Sigma|$, is splittable. This is equivalent to the splittability of the bundle $\mathscr{E}$ (Corollary 2.2.3).

i) $\Rightarrow$ ii). Assume that the vectors $\alpha \in A \subset |\Sigma|$, $|A| = m+1$, do not generate a cone in $\Sigma$. Consider a family of filtrations $E^\alpha$, $\alpha \in |\Sigma|$, such that
1. $E^\alpha(i) = 0$, or $E$, if $\alpha \notin A$, and
2. $E^\alpha(i) = 0$, $L_\alpha$, or $E$, if $\alpha \in A$, where the $L_\alpha \subset E$, $\alpha \in A$, are one-dimensional subspaces in general position in the space $E$.

According to Theorem 2.2.1 the filtrations $E^\alpha$, $\alpha \in |\Sigma|$, determine a nonsplittable bundle of rank $m$.

6.1.5. COROLLARY. *Let* $X$ *be a nonsingular toral variety of dimension* $n$. *Then:*
i) *All equivariant bundles of rank* $n$ *on* $X$ *are splittable if and only if* $X$ *is affine.*
ii) *All equivariant bundles of rank* $n-1$ *split if and only if* $X = \mathbf{P}^n$ *or* $X$ *is affine.*

The proof follows immediately from 6.1.4.

**6.2.** In the case where for each $\alpha \in A$ the subspaces $E^\alpha(i)$, $i \in \mathbf{Z}$, form a full flag, Theorem 6.1.2 can be considerably strengthened. Since the order is of no importance in this context, we shall assume that $\operatorname{codim} E^\alpha(i) = i$, $i = 0, \dots, m$. Recall that the configuration of a pair of full flags $E^\alpha$ and $E^\beta$ can be characterized by the permutation $\pi_{\alpha|\beta} \in S_m$, where the index $j = \pi_{\alpha|\beta}(i)$ is determined from the condition that $E^\alpha(i-1) \subset E^\beta(j-1) + E^\alpha(i)$ and $E^\alpha(i-1) \not\subset E^\beta(j) + E^\alpha(i)$.

6.2.1. THEOREM. *Let* $E^\alpha$, $\alpha \in A$, *be a family of full flags on an* $m$*-dimensional space* $E$. *Then the following conditions are equivalent:*
i) *The family* $E^\alpha$, $\alpha \in A$, *is splittable.*
ii) *The permutations* $\pi_{\alpha|\beta}$ *of flags* $E^\alpha$ *and* $E^\beta$ *form a cocycle:* $\pi_{\alpha|\beta}\pi_{\beta|\gamma}\pi_{\gamma|\alpha} = 1$, $\forall \alpha, \beta, \gamma \in A$.
iii) *This cocycle is principal:* $\pi_{\alpha|\beta} = s_\alpha^{-1} s_\beta$, $s_\alpha \in S_m$, $\alpha \in A$.

6.2.2. COROLLARY. *A necessary and sufficient condition for a family of full flags* $E^\alpha$, $\alpha \in A$, *to be splittable is that any triple of flags in this family is splittable.*

AN EQUIVALENT FORMULATION. *A family of Borel subgroups* $B^\alpha \subset \mathrm{GL}(E)$, $\alpha \in A$, *contains a common maximal torus if and only if an intersection of any three subgroups of this family contains a maximal torus.*

PROOF. i) $\Rightarrow$ iii). The splittability of the family of full flags $E^\alpha$, $\alpha \in A$, means that they are generaed by different orderings of the same basis $(e_1, \dots, e_m)$, i.e. $E^\alpha(i) = \langle e_j | s_\alpha(j) > i \rangle$, $s_\alpha \in S_m$. In this case the permutation of the flags $E^\alpha$ and $E^\beta$ is $\pi_{\alpha|\beta} = s_\alpha^{-1} s_\beta$.

iii) $\Leftrightarrow$ ii). Obviously, iii) $\Rightarrow$ ii). For the proof of the reverse implication one just has to fix $\gamma$ and put $s_\alpha = \pi_{\gamma|\alpha}$. Then $\pi_{\alpha|\beta} = \pi_{\alpha|\gamma} \cdot \pi_{\gamma|\beta} = s_\alpha^{-1} s_\beta$.

iii) $\Rightarrow$ i). Unfortunately no simple proof of this implication is known to the author. First we shall check it for a family of three flags, and then we shall reduce everything to this case, independently proving Corollary 6.2.2.

STEP 1. Let $B^\alpha$, $B^\beta$, $B^\gamma \subset \mathrm{GL}(E)$ be three Borel subgroups containing a common maximal torus. Then

$$B^\alpha \cap B^\beta B^\gamma = (B^\alpha \cap B^\beta)(B^\alpha \cap B^\gamma). \tag{6.5}$$

It suffices to prove that $B^\alpha \cap B^\beta B^\gamma \subset (B^\alpha \cap B^\beta)(B^\alpha \cap B^\gamma)$, since the reverse inclusion is obvious.

Let $B^\alpha$ be the group of lower triangular matrices $X = (x_{pq})$ with $x_{pq} = 0$ for $p < q$.

Let $B^\beta$ be the group matrices $Y = (y_{pq})$ with $y_{pq} = 0$ for $i_p < i_q$.

Let $B^\gamma$ be the group of matrices $Z = (z_{pq})$ with $z_{pq} = 0$ for $j_p < j_q$.

Here $(i_1, \dots, i_m)$ and $(j_1, \dots, j_m)$ are premutations of the indices $1, \dots, m$.

Consider a matrix $X \in B^\alpha \cap B^\beta B^\gamma$. Then $X \in B^\alpha$ and $YX \in B^\gamma$ for some $Y \in B^\beta$, or, on the level of matrix elements,

$$\sum_q y_{pq} x_{qr} = 0 \quad \text{for } j_p < j_r. \tag{6.6}$$

We shall assume in (6.6) that $i_p \geq i_q$, since $y_{pq} = 0$ in the other cases. Then (6.6) is equivalent to the fact that in the matrix

$$X_p = (x_{qr})_{i_q \leq i_p;\, j_r > j_p}$$

the $p$th row is a combination of the remaining rows.

We claim that the $p$th row of this matrix is, in fact, a combination of rows with indices $k < p$. This implies that $\widetilde{Y} X = \widetilde{Z} \in B^\gamma$ for some $\widetilde{Y} \in B^\alpha \cap B^\beta$. This leads us to the desired result: $X = \widetilde{Y}^{-1} \widetilde{Z} \in (B^\alpha \cap B^\beta)(B^\alpha \cap B^\gamma)$.

We prove the claim by induction on the number of rows of the matrix $X_p$ $(= i_p)$. Consider the $k$th row for $k > p$. Then $i_k < i_p$, and so the matrix $X_k$ has fewer rows than the matrix $X_p$ and does not contain its $p$th row.

We consider two cases.

1) $j_k < j_p$. Then $X_k$ has more columns than $X_p$. By the induction hypothesis the $k$th row of $X_k$ is a combination of the preceding rows. Therefore, this is true for the $k$th row of $X_p$. In this way we can exclude from the decomposition of the $p$th row all the rows whose indices satisfy 1) and write it down in the form of a linear combination of rows with index less than $p$ and rows whose indices satisfy the inequality

2) $j_k > j_p$; $k > p$. This inequality means that the diagonal element $x_{kk} \neq 0$ of the $k$th row belongs to the matrix $X_p$. An arbitrary nontrivial combination of such rows contains a nonzero element in a column of index $k > p$. Therefore, they cannot be included in our decomposition of the $p$th row of $X_p$.

STEP 2. If the triple of maximal flags $E^\alpha$, $\alpha = 1, 2, 3$, satisfies condition iii), then it is splittable.

Indeed, it follows from iii) that there exists a triple of splittable flags $F^\alpha$, $\alpha = 1, 2, 3$, which are in the same configuration as the corresponding $E^\alpha$. Let $B^\alpha$ be the stabilizer of $F^\alpha$. Then for any permutation $\alpha, \beta, \gamma$ of the indices $1, 2, 3$ we have

$$E^\beta = S_\alpha F^\beta, \qquad E^\gamma = S_\alpha F^\gamma, \qquad S_\alpha \in \mathrm{GL}(E)/B^\beta \cap B^\gamma, \qquad S_\alpha \equiv S_\beta \pmod{B^\gamma}$$

(see Proposition 2.2.5). From this we get

$$S_\gamma^{-1} S_\beta = (S_\gamma^{-1} S_\alpha)(S_\alpha^{-1} S_\beta) \in B^\alpha \cap B^\beta B^\gamma \underset{\text{Step 1}}{=} (B^\alpha \cap B^\beta)(B^\alpha \cap B^\gamma).$$

Since the elements $S_\beta$ and $S_\gamma$ are determined modulo $B^\alpha \cap B^\gamma$ and $B^\alpha \cap B^\beta$, we can assume that $S_\gamma^{-1} S_\beta = 1$, i.e. $S_\beta = S_\gamma$. Then $S_\alpha^{-1} S_\gamma = S_\alpha^{-1} S_\beta \in B^\beta \cap B^\gamma$ and again, without loss of generality, one can put $S_\alpha = S_\beta = S_\gamma = S$. Hence the triple $E^\alpha = SF^\alpha$, $\alpha = 1, 2, 3$, is adjoint to the splittable triple $F^\alpha$, and, consequently, is itself splittable.

STEP 3. The family of flags $E^\alpha$, $\alpha \in A$, of the space $E$ is splittable if and only if any triple of flags $E^\alpha$, $E^\beta$, $E^\gamma$, $\alpha, \beta, \gamma \in A$, is splittable.

We argue by induction on $|A|$. Let $|A| = n + 1$ and suppose any subfamily of $n$ flags is splittable. Then, by Theorem 2.2.1, the family of filtrations $E^\alpha$, $\alpha \in A$, determines an equivariant bundle on $\mathbf{P}^n$. We need to prove that this bundle splits (see Corollary 2.2.3). We use Horrocks' theorem ([12], Chapter 1, 2.3.2) on the splittability of the bundle $\mathscr{E}$ on $\mathbf{P}^n$ when its restriction to some subspace $\mathbf{P}^k \subset \mathbf{P}^n$, $k \geq 2$, is splittable. Consider a coordinate hyperplane $\mathbf{P}^{n-1}_\alpha \subset \mathbf{P}^n$. The stabilizer of a general point $x_\alpha \in \mathbf{P}^{n-1}_\alpha$ is a one-dimensional torus $T_\alpha \subset T$. The fullness of the flag $E^\alpha$ is equivalent to the simplicity of the spectrum of the representation of the torus $T_\alpha$ in the fiber $\mathscr{E}(x)$, $x \in \mathbf{P}^{n-1}_\alpha$. Therefore the restriction $\mathscr{E}|\mathbf{P}^{n-1}_\alpha$ decomposes into one-dimensional isotypical components. From Horrocks' theorem it follows that the bundle $\mathscr{E}$ also splits, which is equivalent to the splittability of the family of filtrations $E^\alpha$, $\alpha \in A$.

6.2.3. COROLLARY. *Let $Y \subset X$ be an open $T$-invariant subset of a smooth toral variety $X$, $\operatorname{codim}(X \backslash Y) \geq 3$, and $\mathscr{E}$ an equivariant bundle with simple spectrum on $Y$ (this means that the representation of the stabilizer of the point $T_y$ in the fiber $\mathscr{E}(y)$ has a simple spectrum if $\dim T_y > 0$). Then $\mathscr{E}$ extends to an equivariant bundle on $X$.*

The proof is word for word the same as the arguments in §6.1.3.

It follows from Theorem 6.2.1 that the splittability of the family of full flags $E^\alpha$, $\alpha \in A$, is determined by their configuration. However, in the general case this does not happen. For example, three subspaces of codimension 1 which are in general position in a three-dimensional space $E$ form a splittable system, and three different subspaces containing a common straight line are not splittable although they have the same configuration of pairs. Nevertheless, the following local version of Theorem 6.2.1 is true.

6.2.4. THEOREM. *Let $\mathscr{E} = (E^\alpha, \alpha \in A)$ be a splittable system of filtrations of the space $E$ and $\mathfrak{E} = \mathfrak{E}(\mathscr{E})$ the set of systems $\mathscr{F} = (F^\alpha, \alpha \in A)$ with the same configurations as $\mathscr{E}$. Then the systems which are conjugate to $\mathscr{E}$ form an open subset in $\mathfrak{E}$.*

This theorem means that the simplices of the complex $\mathscr{P}(E)$ are rigid: a small deformation of the simplex $(P^\alpha, \alpha \in A)$ which maps every face $(P^\alpha, P^\beta)$ to a congruent one $(\widetilde{P}^\alpha, \widetilde{P}^\beta)$ is a translation by some element $g \in \mathrm{GL}(E)$.

PROOF. We argue by induction on the number of elements of $A$. Let $|A| = n+1$, and assume that for all proper subfamilies of the family $\mathscr{E} = (E^\alpha, \alpha \in A)$ the theorem holds. Then there exists a neighborhood $\mathfrak{U} \ni \mathscr{E}$ consisting of systems $\mathscr{F} = (F^\alpha, \alpha \in A)$ any proper subsystem $(F^\alpha, \alpha \in A\backslash\beta)$ of which is conjugate to the family $(E^\alpha, \alpha \in A\backslash\beta)$.

We shall interpret families of filtrations $\mathscr{F} \in \mathfrak{U}$ as bundles on $\mathbf{P}^n$. Since the bundle $\mathscr{E}$ is splittable, we have $H^1(\mathbf{P}^n, \mathscr{E}nd\,\mathscr{E}) = 0$, i.e. $\mathscr{E}$ is rigid. Consequently, it cannot arise in a nonconstant family. Therefore, all bundles $\mathscr{F} \in \mathfrak{U}_0 \subset \mathfrak{U}$ in some neighborhood $\mathfrak{U}_0 \ni \mathscr{E}$ are isomorphic to $\mathscr{E}$. This means that the corresponding filtrations $\mathscr{F} = (F^\alpha, \alpha \in A)$ are conjugate to $\mathscr{E}(E^\alpha, \alpha \in A)$. This implies that the orbit $\mathrm{GL}(E)\mathscr{E} = \mathrm{GL}(E)\mathfrak{U}_0$ is open in $\mathfrak{E}$.

6.2.5. REMARK. For any group $G$, let $\mathscr{P}(G)$ denote the simplicial complex whose vertices are the parabolic subgroups $P \subset G$ and whose simplices form families of subgroups containing a common maximal torus. Then simplicial maps of a nonsingular fan $\Sigma$ in $\mathscr{P}(G)$ can be interpreted in the language of toral bundles on $X(\Sigma)$ with structure group $G$. From this point of view it is interesting to generalize statements of type 6.1.3 or 6.2.1 to arbitrary semisimple groups.

**6.3.** In this subsection we prove a general fact about constructing restrictions of toral bundles to closures of orbits.

Let $X = X(\Sigma)$ be a nonsingular toral variety. Then the closure of an orbit $X_\sigma = \overline{O}_\sigma$ is also a smooth toral variety with respect to the action of the torus $T^\sigma = T/T_\sigma$, where $T_\sigma$ is the stabilizer of the point $X_\sigma \in O_\sigma$. Its fan $\Sigma_\sigma = \Sigma(X_\sigma)$ consists of images of cones $\tau \supset \sigma$ in $\widehat{T}^0_{\mathbf{R}}/\mathbf{R}\sigma$, where $\mathbf{R}_\sigma$ is the subspace spanned by $\sigma$. We shall identify the set of generators of one-dimensional cones $|\Sigma_\sigma|$ with their inverse images $\alpha \in |\Sigma|$; $\langle \alpha, \sigma \rangle \in \Sigma$.

Consider an equivariant bundle $\mathscr{E}$ on $X$ defined by a family of filtrations $E^\alpha$, $\alpha \in |\Sigma|$. The restriction $\mathscr{E}|X_\sigma$ has a canonical $T$-structure but has no canonical toral structure with respect to the action of the torus $T^\sigma = T/T_\sigma$. It can be defined by choosing a splitting $\pi\colon T \to T_\sigma$ of the inclusion $\iota\colon T_\sigma \to T$, $\pi \cdot \iota = 1$, and identifying $T_\sigma$ with $\ker \pi$. Let $\mathscr{E}_\pi$ denote the restriction $\mathscr{E}|X_\sigma$ with the given toral structure.

Define on the group of characters $\widehat{T}_\sigma$ the partial ordering

$$\chi \leq \psi \Leftrightarrow \langle \chi, \alpha \rangle \leq \langle \psi, \alpha \rangle, \qquad \forall \alpha \in |\sigma|, \tag{6.7}$$

and put

$$E^{[\sigma]}(\chi) = E^\sigma(\chi) \Big/ \sum_{\psi > \chi} E^\sigma(\psi), \qquad \chi \in \widehat{T}_\sigma. \tag{6.8}$$

The space $E^{[\sigma]}$ is canonically identified with the isotypical component of the character $\chi$ of the fiber $\mathscr{E}(x_\sigma)$ via the map

$$f^\sigma(\chi)\colon E^\sigma(\chi) \to \mathscr{E}(x_\sigma)_\chi, \qquad e \to \lim_{tx_0 \to x_\sigma} \chi^{-1}(t)(te), \qquad t \in T_\sigma$$

(see Remark 2.1.2 and the comment after Theorem 2.2.1). From its definition it is obvious that $f^\sigma(\chi)$ annihilates all the subspaces $E^\sigma(\psi)$, $\psi > \chi$, and induces an isomorphism $f^{[\sigma]}(\chi)\colon E^{[\sigma]}(\chi) \to \mathscr{E}(x_\sigma)_\chi$. In this way the fiber $\mathscr{E}(x_\sigma)$ is identified with the graded space $E^{[\sigma]} = \bigoplus_\chi E^{[\sigma]}(\chi)$.

In the following theorem $V_\chi \subset E^{[\sigma]}(\chi)$ will denote the projection of a subspace $V \subset E$ onto the factor $E^{[\sigma]}(\chi)$:

$$V_\chi = \left(V + \sum_{\psi>\chi} E^\sigma(\psi)\right) \cap E^\sigma(\chi) \Big/ \sum_{\psi>\chi} E^\sigma(\psi).$$

6.3.1. THEOREM. *The restriction* $\mathscr{E}|X_\sigma = \mathscr{E}_\pi$ *is determined by the collection of filtrations of the graded space* $E^{[\sigma]} = \bigoplus_\chi E^{[\sigma]}(\chi)$, (6.8), *induced by the filtrations* $E^\alpha$, $\langle\alpha, \sigma\rangle \in \Sigma$, *of the space* $E$ *translated by* $n_{\alpha,\chi} = \langle\chi, \hat{\pi}^0(\alpha)\rangle$: $E^{[\sigma],\alpha}(i) = \bigoplus_\chi E^\alpha(i + n_{\alpha,\chi})_\chi$.

Here the map $\hat{\pi}^0\colon \widehat{T}^0 \to \widehat{T}^0_\sigma$ is constructed from the splitting $\pi\colon T \to T_\sigma$ of the canonical embedding $\imath\colon T_\sigma \to T$.

PROOF. We use the identification $\mathscr{E}(x_\sigma) \simeq E^{[\sigma]}$ described above and put $\forall\psi \in \widehat{T}$

$$\begin{aligned}\mathscr{E}(x_\sigma)^\alpha(\psi) &= \{e \in \mathscr{E}(x_\sigma) | \exists \lim_{tx_\sigma \to x_{\langle\sigma,\alpha\rangle}} \psi^{-1}(t)(te)\}\\ &= (\text{projections of the space } E^\alpha(\psi) \text{ to } \mathscr{E}(x_\sigma)_{\hat{\imath}(\psi)}) = E^\alpha(\psi_{\hat{\imath}(\psi)}).\end{aligned}$$

Then by decomposing the character $\psi = \psi^\sigma + \psi_\sigma$, $\psi^\sigma \in \ker\hat{\imath}$, $\psi_\sigma = \hat{\pi}\hat{\imath}(\psi) \in \operatorname{Im}\hat{\pi}$, we have

$$\begin{aligned}\mathscr{E}(x_\sigma)^\alpha(i) &= \bigoplus_{\langle\psi^\sigma,\alpha\rangle=i} \mathscr{E}(x_\sigma)^\alpha(\psi) = \bigoplus_{\langle\psi^\sigma,\alpha\rangle=i} E^\alpha(\langle\psi, \alpha\rangle)_{\hat{\imath}(\psi)}\\ &= \bigoplus_{\langle\psi^\sigma,\alpha\rangle=i} E^\alpha(i + \langle\hat{\pi}\hat{\imath}\psi, \alpha\rangle)_{\hat{\imath}\psi}\\ &= \bigoplus_{\chi\in\widehat{T}_\sigma} E^\alpha(i + \langle\hat{\pi}\chi, \alpha\rangle)_\chi = \bigoplus_{\chi\in\widehat{T}_\sigma} E^\alpha(i + \langle\chi, \hat{\pi}^0(\alpha)\rangle)_\chi.\end{aligned}$$

As an application of the results of this section we shall prove the following fact.

6.3.2. PROPOSITION. *Let* $\mathscr{E}$ *be a toral bundle on* $\mathbf{P}^n$. *Assume that for some point* $x \in \mathbf{P}^n$ *the representation of the stabilizer* $T_x$ *in the fiber* $\mathscr{E}(x)$ *satisfies one of the following conditions*:

i) *the dimensions of isotypical components are smaller than* $\operatorname{codim} T_x$;

ii) *the number of isotypical components is at most* $\dim T_x$.

*Then the bundle* $\mathscr{E}$ *splits.*

PROOF. i) Consider the restriction of the bundle $\mathscr{E}$ to the closure of the orbit $\overline{O}_x = \mathbf{P}^m$, $m = \operatorname{codim} T_x$. By the previous theorem $\mathscr{E}|\mathbf{P}^m$ decomposes into a sum of bundles whose dimensions equal the multiplicities of the irreducible characters of the representation $T_x\colon \mathscr{E}(x)$. By the assumption these multiplicities are smaller than $m$ and therefore the restriction $\mathscr{E}|\mathbf{P}^m$ splits (Corollary 6.1.4 or 6.1.5). By Horrocks' theorem ([12], Chapter 1, 2.3.2), this implies that the bundle $\mathscr{E}$ itself splits if $m \geq 2$. The case $m = 1$ is possible only for the zero bundle.

ii) Let $x \in O_\sigma$, $T_x = T_\sigma$, and $\sigma \in \Sigma(\mathbf{P}^n)$. If the number of components of the representation $T_\sigma\colon \mathscr{E}(x)$ is at most $\dim T_\sigma = \dim\sigma$, then for two different faces $\tau \subset \delta \subset \sigma$ the representations $T_\tau\colon \mathscr{E}(x)$ and $T_\delta\colon \mathscr{E}(x)$ will have the same number of isotypical components. Then isotypical decomposition of these representations coincide. In the language of filtrations this means that the spaces $E^\beta(i)$, $\beta \in |\delta|$, can be expressed by sums of intersections of the spaces $E^\alpha(j)$, $\alpha \in |\tau|$. Since any proper subfamily of the family of filtrations $E^\alpha$, $\alpha \in |\Sigma(\mathbf{P}^n)|$, is splittable, it follows that the whole family $E^\alpha$, $\alpha \in |\Sigma|$, and the bundle $\mathscr{E}$ are splittable.

6.3.3. EXAMPLE. It follows from the proposition that toral bundles of rank $< n$ over $\mathbf{P}^n$ split (see also §6.1.5). Consider a bundle $\mathscr{E}$ of rank $n$ determined by a family of filtrations $E^\alpha$, $\alpha \in 0, \dots, n$. The proposition implies that if it does not split then one of the spaces $E^{[\alpha]}(i) = E^\alpha(i)/E^\alpha(i+1)$ should have dimension $\geq n-1$. It follows from this that nontrivial spaces of filtrations $E^\alpha(i)$ have dimension 1 or $n-1$ and form a configuration of $n+1$ straight lines or $n+1$ hypersurfaces in general position which pass through the origin. With a suitable numbering of the members of the filtration the tangent and cotangent bundles of $\mathbf{P}^n$ correspond to these configurations. These configurations also play a central role in the classification theorem of the following section.

**6.4.** Let $E^\alpha$, $\alpha \in A$, be a family of subspaces of a vector space $E$. We say that this family is *indecomposable* if it cannot be represented in the form $E = E_1 \oplus E_2$, $E^\alpha = E_1^\alpha \oplus E_2^\alpha$, with $\dim E_i > 0$. Any system of subspaces decomposes into a direct sum of indecomposable systems which are uniquely determined up to isomorphism and order.

Indecomposable systems containing no more than four subspaces were described by Gel′fand and Ponomarev [13]. In the general case the classification of indecomposable systems is a complex problem.

In this subsection we describe all indecomposable families of subspaces any proper subfamily of which splits. As an example of such a family there is the configuration of $n+1$ hypersurfaces in general position in an $n$-dimensional space which pass through the origin. We shall denote this configuration by $\Omega_n^1 = (E; E^\alpha, \alpha \in 0, \dots, n)$, and we shall set $\Omega_n^k = (\Lambda^k E; \Lambda^k E^\alpha, \alpha = 0, \dots, n)$. The configuration $\Omega_n^k$ is associated to the bundle of $k$-forms on $\mathbf{P}^n$.

6.4.1. THEOREM. *An indecomposable family of subspaces $E^\alpha \subset E$, $\alpha \in 0, \dots, n$, all proper subfamilies of which are splittable, either has rank one* ($= \dim E$) *or is isomorphic to one of the systems $\Omega_n^k$, $k = 0, \dots, n$.*

6.4.2. COMMENT. Here is a typical situation in which the configurations in Theorem 6.4.1 arise. Consider a toral bundle $\mathscr{E}$ over $\mathbf{P}^n$ determined by a family of filtrations $E^\alpha(i)$, $\alpha \in \{0, \dots, n\}$. Then by Theorem 2.2.1, for any set of indices $i_\alpha \in \mathbf{Z}$, the family of subspaces $E^\alpha(i_\alpha)$, $\alpha \in \{0, \dots, n\}$, satisfies the conditions of Theorem 6.4.1. In particular, for any character $\chi \in \hat{T}$ these conditions are satisfied for the family of subspaces

$$E(\chi) = (E^\alpha(\chi), \alpha = 0, \dots, n).$$

By Theorem 6.4.1 it decomposes into the sum

$$E(\chi) = \left(\bigoplus_k m_k \Omega_n^k\right) \oplus F, \tag{6.9}$$

where $F$ is a sum of rank 1 systems not including $\Omega_n^0 = (L; L, \dots, L)$ or $\Omega_n^n = (L; 0, \dots, 0)$, $\dim L = 1$. The multiplicities $m_k$ can be expressed in terms of the $\chi$-components of the cohomology groups $H^k(\mathbf{P}^n, \mathscr{E})_\chi$:

$$m_k = \dim H^k(\mathbf{P}^n, \mathscr{E})_\chi, \tag{6.10}$$

which, as we know from §4.1.3, is completely determined by the configuration $E(\chi)$. Furthermore, the splittable component $F$ does not contribute to the cohomology groups, and therefore (6.10) follows from the classical equality $\dim H^p(\mathbf{P}^n, \Omega^q) = \delta_{pq}$.

From (6.9) and (6.10) we get

6.4.3. Corollary. *For a toral bundle $\mathscr{E}$ over $\mathbf{P}^n$ and any character $\chi$,*

$$\sum_k \binom{n}{k} \dim H^k(\mathbf{P}^n, \mathscr{E})_\chi \leq \operatorname{rk}\mathscr{E}.$$

*In particular, if* $\operatorname{rk}\mathscr{E} < \binom{n}{k}$ *then* $H^k(\mathbf{P}^n, \mathscr{E})_\chi = 0$.

There is one more method for picking out the components in the decomposition (6.9):

$$m_k\Omega_n^k = \Omega_n^k \otimes H^k(\mathbf{P}^n, \mathscr{E})_\chi = \bigcap_{|I|=k+1} \sum_{\alpha\in I} E^\alpha(\chi) \Big/ \sum_{|J|=n-k+1} \bigcap_{\alpha\in J} E^\alpha(\chi),$$

which also gives a symmetrical formula for calculating cohomology groups (cf. §§4.3.1 and 4.3.2). The proof reduces to checking this relation for splittable bundles and for $\Omega^k$.

Proof of the theorem. It will be convenient for us to convert to the language of bundles. We call a filtration $E^\alpha(i)$ of the space $E$ *short* if it contains no more than one subspace different from 0 and $E$. Let $\mathscr{E}$ be a toral bundle over $\mathbf{P}^n$ defined by a family of short filtrations $E^\alpha$, $\alpha = 0, \dots, n$. We say that the bundle $\mathscr{E}$ is *standard* if it decomposes into a direct sum of line bundles and twisted bundles of $p$-forms $\Omega^p \otimes \mathscr{O}(f)$. We need to prove that every bundle defined by short filtrations is standard.

6.4.5. Assertion. *A toral bundle $\mathscr{E}$ over $\mathbf{P}^n$ determined by short filtrations admits a decompositon $\mathscr{E} = \mathscr{E}^0 \oplus \mathscr{E}^1$, where $\mathscr{E}^0$ is standard and* $\operatorname{Ext}^1(\mathscr{F}, \mathscr{E}^1) = H^1(\mathbf{P}^n, \mathscr{E}^1 \otimes \mathscr{F}^*) = 0$ $0 < p < n$, *for $i > 0$ it follows that* $H^{n-1}(\mathbf{P}^n, \mathscr{E}_{i+1} \otimes \mathscr{F}) = 0$ *for any splittable bundle $\mathscr{F}$.*

We shall prove the theorem from this proposition.

Step 1. i). *For any bundle $\mathscr{E}$ over $\mathbf{P}^n$ there exists an exact sequence of bundles*

$$0 \to \mathscr{E}' \to \mathscr{F} \to \mathscr{E} \to 0,$$

*in which $\mathscr{F}$ is splittable and* $\operatorname{Ext}^1(\mathscr{F}', \mathscr{E}') = 0$ *for any splittable bundle $\mathscr{F}'$.*

ii) The bundle $\mathscr{E}'$ is uniquely determined by $\mathscr{E}$ up to splittable components.

iii) If $\mathscr{E}$ is standard, so is $\mathscr{E}'$.

Proof. i) Let $s_i : \mathscr{O} \to \mathscr{E}(m_i)$ generate the module $\bigoplus_m H^0(\mathbf{P}^n, \mathscr{E}(m))$ over the ring $\bigoplus_m H^0(\mathbf{P}^n, \mathscr{O}(m))$. Then the corresponding sequence $0 \to \mathscr{E}' \to \bigoplus_i \mathscr{O}(-m_i) \to \mathscr{E} \to 0$ has the necessary properties.

ii) Suppose we have two exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{E}' & \to & \mathscr{F} & \to & \mathscr{E} & \to & 0 \\ & & \uparrow\downarrow & & \uparrow\downarrow & & \| & & \\ 0 & \to & \mathscr{E}'' & \to & \mathscr{F}' & \to & \mathscr{E} & \to & 0 \end{array} \tag{6.11}$$

The condition $\operatorname{Ext}^1(\mathscr{F}, \mathscr{E}'') = 0 = \operatorname{Ext}^1(\mathscr{F}', \mathscr{E}')$ implies the existence of the vertical morphisms making diagram (6.11) commute. Then standard arguments from Schanuel's lemma show that $\mathscr{E}' \oplus \mathscr{F}' \simeq \mathscr{E}'' \oplus \mathscr{F}$.

iii) One can restrict to the case $\mathscr{E} = \Omega^p$, $0 < p < n$. Then there is an exact sequence ([12], Chapter 1, §1)

$$0 \to \Omega^{p+1} \to \binom{n+1}{p+1}\mathscr{O}(-p-1) \to \Omega^p \to 0,$$

from which it follows that $\mathscr{E}' = \Omega^{p+1}$.

Consider now the canonical resolution (3.1) of the bundle $\mathscr{E}$

$$0 \to \mathscr{E} \to \mathscr{F}_0 \to \mathscr{F}_1 \to \cdots \to \mathscr{F}_n \to 0 \tag{6.12}$$

and split it into short exact sequences

$$0 \to \mathscr{E}_i \to \mathscr{F}_i \to \mathscr{E}_{i+1} \to 0. \tag{6.13}$$

It follows from the definition of the canonical resolution that, along with $\mathscr{E}$, all the bundles $\mathscr{F}_i$ and $\mathscr{E}_i$ are also determined by short filtrations.

STEP 2. *All the bundles* $\mathscr{E}_i$, $i > 0$, *are standard.*

The proof proceeds by inverse induction on $i$. Suppose that the bundle $\mathscr{E}_{i+1}$ is standard. We write the sequence (6.13) in the form

$$0 \to \mathscr{E}_i^0 \oplus \mathscr{E}_i^1 \to \mathscr{F}_i \to \mathscr{E}_{i+1} \to 0, \tag{6.14}$$

where $\mathscr{E}_i = \mathscr{E}_i^0 \oplus \mathscr{E}_i^1$ is the decomposition in Proposition 6.4.5.

From the canonical resolution (6.12) and the acyclicity of splittable bundles $H^p(\mathbf{P}^n, \mathscr{F}) = 0$, $0 < p < n$, for $i > 0$ it follows that $H^{n-1}(\mathbf{P}, \mathscr{E}_{i+1} \otimes \mathscr{F}) = 0$ for any splittable bundle $\mathscr{F}$. Therefore, from part ii) of Step 1 applied to the sequence dual to (6.14) it follows that the bundle $\mathscr{E}_{i+1}$ is uniquely determined by $\mathscr{E}_i$ up to splittable components. Therefore, one can replace the resolution (6.14) by a direct sum of resolutions

$$0 \to \mathscr{E}_i^0 \to \mathscr{F}_i^0 \to \mathscr{E}_{i+1}^0 \to 0,$$
$$0 \to \mathscr{E}_i^1 \to \mathscr{F}_i^1 \to \mathscr{E}_{i+1}^1 \to 0.$$

Moreover, $\mathscr{E}_{i+1}$ differs from $\mathscr{E}_{i+1}^0 \oplus \mathscr{E}_{i+1}^1$ by a splittable component. Since $\mathscr{E}_{i+1}$ is standard by the induction hypothesis, $\mathscr{E}_{i+1}^1$ is also standard. Then, from (iii) of Step 1, $\mathscr{E}_i^1$ is also standard; consequently the sum $\mathscr{E}_i^0 \oplus \mathscr{E}_i^1 = \mathscr{E}_i$ is standard.

STEP 3. *The bundle* $\mathscr{E}$ *is standard.*

Indeed, according to Proposition 6.4.5 one can assume that $\mathrm{Ext}^1(\mathscr{F}, \mathscr{E}) = 0$ for any splittable bundle $\mathscr{F}$. Consider the start of the canonical resolution (6.12)

$$0 \to \mathscr{E} \to \mathscr{F}_0 \to \mathscr{E}_1 \to 0.$$

The bundle $\mathscr{E}_1$ here is splittable by Step 2. Therefore, from Step 1 it follows that $\mathscr{E}$ is standard.

PROOF OF PROPOSITION 6.4.5. This requires a quite different technique, for which the language of filtrations is more suited. In this language the acyclicity of the bundles $\mathscr{E} \otimes \mathscr{F}$ ($\mathscr{F}$ splittable) in dimension one or codimension one is equivalent to verifying the conditions

$$E^\alpha + \bigcap_{\beta \neq \alpha} E^\beta = \bigcap_{\beta ; \beta \neq \alpha} (E^\alpha + E^\beta) \tag{6.15}$$

or

$$E^\alpha \cap \sum_{\beta \neq \alpha} E^\beta = \sum_{\beta ; \beta \neq \alpha} E^\alpha \cap E^\beta, \tag{6.16}$$

where we let $E^\gamma$, $\gamma = 0, \dots, n$, denote an arbitrary term of the filtration $E^\gamma(i)$ defining the bundle $\mathcal{E}$ (see §4.3.1).

Now let $E^\alpha$, $\alpha = 0, \dots, n$, be a family of subspaces of which any proper subfamily is splittable. Put

$$H = \sum_{\alpha\neq\beta} E^\alpha \cap E^\beta, \qquad H^\alpha = H \cap E^\alpha.$$

STEP 1. *The system* $(E; E^\alpha)$ *decomposes into the direct sum*

$$(E; E^\alpha) \simeq (H; H^\alpha) \oplus m\Omega_n^{n-1} \oplus F,$$

*where* $F$ *is a splittable system of subspaces.*

This is precisely the decomposition of the system $(E; E^\alpha)$ by using the top cube of Gel′fand and Ponomarev $B^+(2)$ [13]. Note that under our assumptions on the spaces $E^\alpha$, the higher cubes $B^+(m)$, $m > 2$, do not give any new components.

We shall show that the system $(H; H^\alpha)$ is acyclic in codimension 1.

STEP 2.

$$H^\gamma = \sum_{\beta;\beta\neq\gamma} E^\gamma \cap E^\beta. \tag{6.17}$$

Indeed,

$$H^\gamma = E^\gamma \cap \sum_{\alpha\neq\beta} E^\alpha \cap E^\beta = \sum_{\beta;\beta\neq\gamma} E^\gamma \cap E^\beta + E^\gamma \cap \sum_{\alpha\neq\beta;\alpha,\beta\neq\gamma} E^\alpha \cap E^\beta.$$

We shall check that the second summand of this sum is contained in the first one. It is obvious that, for all $\delta = 0, \dots, n$,

$$\sum_{\alpha\neq\beta;\alpha,\beta\neq\gamma} E^\alpha \cap E^\beta \subset \sum_{\beta\neq\gamma,\delta} E^\beta,$$

since amongst pairs of indices $\alpha \neq \beta$ one differs from $\delta$. As the system $E^\alpha$, $\alpha \neq \delta$, is splittable, we get

$$E^\gamma \cap \sum_{\alpha\neq\beta;\alpha,\beta\neq\gamma} E^\alpha \cap E^\beta \subset E^\gamma \cap \sum_{\beta\neq\gamma,\delta} E^\beta = \sum_{\beta;\beta\neq\gamma,\delta} E^\gamma \cap E^\beta \subset \sum_{\beta;\beta\neq\gamma} E^\gamma \cap E^\beta,$$

which proves (6.17).

STEP 3.

$$H^\alpha \cap \sum_{\beta\neq\alpha} H^\beta = H^\alpha = \sum_{\beta;\beta\neq\alpha} H^\alpha \cap H^\beta. \tag{6.18}$$

The first of these equalities follows from the fact that

$$\sum_{\beta\neq\alpha} H^\beta \underset{(6.17)}{=} \sum_{\beta\neq\gamma} E^\beta \cap E^\gamma = H \supset H^\alpha.$$

The second can be obtained by using the formula

$$H^\alpha \cap H^\beta = E^\alpha \cap E^\beta \cap \sum_{\gamma\neq\delta} E^\gamma \cap E^\delta = E^\alpha \cap E^\beta,$$

from which we get

$$\sum_{\beta;\beta\neq\alpha} H^\alpha \cap H^\beta = \sum_{\beta;\beta\neq\alpha} E^\alpha \cap E^\beta \underset{(6.17)}{=} H^\alpha.$$

Formula (6.18) means that the bundle $\mathscr{H}$ defined by the system of subspaces $H^\alpha \subset H$, $\alpha = 0, \dots, n$, is acyclic in codimension 1 (see (6.16)). Together with Step 1 this proves the proposition dual to 6.4.5. Proposition 6.4.5 itself is obtained by passing to the dual bundles.

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