20. LECTURE 20

20.1. **Colimits, directed colimits, filtered colimits.** Recall colimits of modules (over a category). Special case: filtered colimits (over filtered categories). Special case of this: direct colimits are colimits over directed sets.

Examples of colimits include finite and infinite direct sums, cokernels, and unions.

There is a subtle difference between colimits and filtered colimits. In practice filtered and direct colimits are quite similar.

Example 20.1. Q is a direct limit (union) of copies of $\mathbb{Z}$. Localization of a ring is a direct limit. A cokernel or a direct sum or pushout is a colimit but not a filtered colimit.

We will show that filtered or directed colimits of flat modules are flat, but arbitrary colimits need not be.

Example. A colimit of flat modules need not be flat (examples: the cokenel of multiplication by 2 for $\mathbb{Z} \to \mathbb{Z}$ is a non-filtered colimit. For non-directed posets use $\mathbb{Z} \leftarrow \mathbb{Z} \to \mathbb{Z}$ with both morphisms being multiplication by 2.) However a FILTERED (or directed) colimit of flat modules is flat. (Recall that a category is called filtered if any two objects can both be mapped to a third, and any two morphisms between objects can be made equal by composing them with another morphism. The special case of a filtered poset is a directed set.)

We have several sort of colimits: they can be filtered, and they can be taken over a poset rather than a category. Examples: cokernels, direct sums, pushouts, unions.

The key point is that a filtered colimit of modules M_i can be described as the disjoint union of the M_i modulo the following equivalence relation: $m = n$ if their images are the same in some M_i. The properties of a filtered category are used to show that this is an equivalence relation (and in particular is transitive).

When is a colimit of exact sequences exact?

The colimit of injective maps need not be injective. (Example: Z is a submodule of $\mathbb{Q}$, but the cokernel (colimit) of $\mathbb{Z} \mapsto^2 \mathbb{Z}$ is not a submodule of the cokernel of $\mathbb{Q} \mapsto^2 \mathbb{Q}$.) However it nearly is: taking cokernels is right exact (exercise) so the only question is when it preserves the condition of being injective. However a FILTERED colimit of injective maps is injective. The point is that any element of a filtered colimit of modules M_i is represented by an element of some M_i. Suppose that we have injective maps $M_i \mapsto N_i$. If x is in colim M_i then x is represented by some y in some M_i, so if its image in colim N_i is 0 then y is 0.

Another way of putting this is that the colimit functor from (modules indexed by a category) to modules is right exact so has a left derived functor $colim_1$. We have shown that this derived functor vanishes for filtered colimits. (On the other hand, the derived functor of filtered limits need not vanish! See later.)

Now we can show that a filtered colimit of flat modules M_i is flat. If $0 \to A \to B \to C \to 0$ is exact, it remains exact when tensored with M_i as M_i is exact. Then $0 \to colim(A \otimes M_i) \to colim(B \otimes M_i) \to colim(C \otimes M_i) \to 0$ is exact as filtered colimits preserve exactness. FInally tensor products commute with colimits, so $0 \to A \times (colimm M_i) \to ...$ is exact. This is what we needed to prove.

In particular any union of an increasing sequence of flat modules is flat.

We have proved half of Lazard's theorem: a module is flat if and only if it is a filtered colimit of finitely generated free modules. (Note that any module is a possibly unfiltered colimit of fg free modules.)

20.2. **Completions.** We define a 10adic number to have an infinite number of digits before the decimal point but only a finite number after. The 10-adic integers are those with no digits after the decimal point. Two nontrivial solutions of $x^2 = x$ in 10-adics: $76^2 = 5776, 625^2 = 390625$, etc. ...1787109376...8212890625 (second comes by starting with 5 and repeated squaring). These are the numbers congruent to 1 or 0 mod 2^n and 0 or 1 mod 5^n. Examples: $-1/7 = ...142857$ in the 10adics and $-1 = ...999999$.

There are 2 ways to define the 10-adics more rigorously. One way is to define them as the completion of the rationals under a strange metric where $d(x, y)$ is 10^{-n} if the last n digits of x, y are the same. Then we can copy the construction of the reals via Cauchy sequences. This works but is needlessly complicated. However it does suggest that completions are rather like the real numbers: for example we can often define things like exponential functions, Gamma functions, and so on for complete rings.

An easier way to to construct the Ring of 10-adics integers as an inverse limit of $\mathbb{Z}/10^n\mathbb{Z}$. (Recall definition of (inverse) limits.) The 10-adics integers are not an integral domain. By the chinese remainder theorem it is the product of 2-adics, 5 adics. These are integral domains.

The same construction works for any ideal of any ring: the I-adic numbers are the completion of R for the topology defined by the ideal I, which is just the inverse limit of the rings R/I^n.

Example. The ring of formal power series is the completion of the ring of polynomials for the ideal $(x, y, z, ...)$.