COMPLETE MODULES AND TORSION MODULES

By W. G. DWYER and J. P. C. GREENLEES

Abstract. Suppose that R is a ring and that A is a chain complex over R. Inside the derived category of differential graded R-modules there are naturally defined subcategories of A-torsion objects and of A-complete objects. Under a finiteness condition on A, we develop a Morita theory for these subcategories, find conceptual interpretations for some associated algebraic functors, and, in appropriate commutative situations, identify the associated functors as local homology or local cohomology. Some of the results are suprising even in the case $R = \mathbb{Z}$ and $A = \mathbb{Z}/p$.

1. Introduction. Let R be a ring and R-mod the derived category of chain complexes of left R-modules (see Section 1.2). We choose a fixed complex A which is *perfect*, in other words, isomorphic in R-mod to a complex of finite length in which the entries are finitely generated projective R-modules. We declare another complex N to be *A-trivial* if $\mathrm{Hom}_R(A, N) \cong 0$, where $\mathrm{Hom}_R(\cdot, \cdot)$ denotes the chain complex of morphisms in R-mod. Going further, we say that X is *A-torsion* if $\mathrm{Hom}_R(X, N) \cong 0$ for all A-trivial N, and that X is *A-complete* if $\mathrm{Hom}_R(N, X) \cong 0$ for all A-trivial N. We then study the category $\mathbf{A}_{\mathrm{tors}}$ of A-torsion complexes and the category $\mathbf{A}_{\mathrm{comp}}$ of A-complete complexes (both of these are triangulated full subcategories of R-mod). It turns out that these categories are equivalent to one another, and also equivalent to the derived category of differential graded modules over the endomorphism complex of A. We construct approximation functors Cell_A: R-mod $\to \mathbf{A}_{\mathrm{tors}}$ and $(-)^{\hat{}}_A$: R-mod $\to \mathbf{A}_{\mathrm{comp}}$. For an object M of R-mod, the complex $\mathrm{Cell}_A(M)$ is a kind of A-cellular approximation to M, in the sense that it is the best approximation to M which can be cobbled together from A and its suspensions; the complex $M^{\hat{}}_A$ is the Bousfield localization of M with respect to a homology theory on R-mod derived from A. We provide algebraic formulas for the functors, and find that the functors are related in interesting ways, one of which involves an arithmetic square. We also show that if $\mathrm{Cell}_A(R)$ has certain finite-dimensionality properties, then an object M of R-mod is A-torsion or A-complete if and only if the homology groups H_iM individually satisfy appropriate torsion or completeness conditions.

Suppose now that R is a commutative ring, that $I \subset R$ is a finitely generated ideal, that $A = R/I$, and that K is the associated Koszul complex. The complex A

Manuscript received August 4, 2000; revised April 5, 2001.

Research of the first author supported in part by the National Science Foundation.

American Journal of Mathematics 124 (2002), 199–220

is not necessarily perfect (i.e., R/I does not necessarily have a finite projective resolution over R), but K is, and it turns out that an object of R-mod is A-trivial if and only if it is K-trivial, and hence A-torsion (resp. A-complete) if and only if it is K-torsion (resp. K-complete). We can therefore use our techniques to study functors $\mathrm{Cell}_A = \mathrm{Cell}_K$ and $(-)^{\wedge}_A = (-)^{\wedge}_K$, and we identify these functors respectively in terms of local cohomology and local homology. One remarkable aspect of the theory we describe is how much can be said in general. In fact, the general case seems to shed some light on local homology and cohomology, and on the meaning of torsion and completeness: the local cohomology of M is a universal R/I-torsion object mapping *to* M, and the local homology of M is a universal R/I-complete object accepting a map *from* M (see Section 6). The first is a cellular approximation, the second a Bousfield localization. Moreover, the question of whether a chain complex is R/I-torsion or R/I-complete can be settled by examining its homology groups one at at time; for instance, a chain complex is R/I-torsion if and only if each element of its homology is annihilated by some power of I.

We emphasize that we assume almost everywhere that A is *perfect*; the one exception is Section 6, where in any case $A = R/I$ is immediately replaced by the Koszul complex K. It is easy to see that a complex A is perfect if and only if it is *small* in the sense that $\mathrm{Hom}_R(A, -)$ commutes with arbitrary coproducts. Complexes like this could just as well be called *finite* R-complexes, since they represent the elements of R-mod which can be built in a finite number of steps from R itself by taking suspensions, cofibration sequences, and retracts. They are the analogs in R-mod of finite complexes in the stable homotopy category.

Acknowledgments. The authors are grateful to MIT for its hospitality in May 1999 when this work was begun, and the second author thanks J. D. Christensen for useful conversations.

1.1. Organization of the paper. In Section 2 we develop a Morita theory which shows that $\mathbf{A}_{\mathrm{tors}}$ and $\mathbf{A}_{\mathrm{comp}}$ are both equivalent to the derived category of modules over the endomorphism algebra of A; in particular, $\mathbf{A}_{\mathrm{tors}}$ and $\mathbf{A}_{\mathrm{comp}}$ are equivalent to one another. Section 3 describes some special cases of this Morita theory, and in particular a striking one with $R = \mathbb{Z}$ and $A = \mathbb{Z}/p$. Section 4 establishes new notation for some of the functors from Section 2 and interprets these functors in terms of standard constructions: cellular approximations, homology localizations, and periodicizations. The functors fit into a homotopy fibre square (Proposition 4.13) which generalizes the arithmetic square of abelian group theory (4.2). The next section establishes conditions under which the question of whether or not a chain complex belongs to $\mathbf{A}_{\mathrm{tors}}$ or $\mathbf{A}_{\mathrm{comp}}$ can be answered by examining its homology groups. Finally, Section 6 specializes to the case in which R is commutative, $I \subset R$ is a finitely generated ideal, and $A = R/I$. As explained above, a device involving the Koszul complex allows the previous theory to be applied, even though R/I is not necessarily perfect. The functors from Section 4 in this case turn out to be classical local cohomology, or its dual local homology, at the ideal I; a chain complex M belongs to $\mathbf{A}_{\text{tors}}$ or $\mathbf{A}_{\text{comp}}$ if and only if its homology groups are I-torsion or I-complete in an appropriate sense.

1.2. Notation and terminology. The derived category R-mod is obtained from the category of unbounded chain complexes of R-modules by formally inverting the maps which induce isomorphisms on homology. See [18], [2], or [19, §10] for algebraic ways to look at this, and [14] for a topological approach. Note that the differentials in our chain complexes always lower degree by one. The statements in this paper are expressed almost exclusively in terms of such derived categories. In particular, Hom is the derived homomorphism complex (sometimes written **R**Hom) considered as an object of the appropriate derived category, and $\otimes$ refers to the left derived tensor product (sometimes written $\otimes^{\mathbf{L}}$). The convention of working in the derived category has some startling effects and should not be forgotten.

There is one exception to our convention. If A is an object of R-mod, then $\text{End}_R(A)$ denotes the actual differential graded algebra obtained by taking a cofibrant (projective) model for A and forming the usual DGA of endomorphisms of this model (see [19, 2.7.4], but re-index so that all of the differentials reduce degree by one). We will use this construction only when A is perfect, in which case picking a cofibrant model amounts to chosing a finite projective resolution of A; in a more general situation, it would be necessary to find a "K-projective" resolution in the sense of [18]. Up to multiplicative homology equivalence, $\text{End}_R(A)$ does not depend upon the choice of cofibrant model.

The significance of the above exception can be explained by a topological analogy. The category R-mod is like the homotopy category Ho(**Sp**) of spectra; the derived homomorphism complex $\text{Hom}_R(M,N)$ is then like the derived mapping spectrum $\text{Map}\,(X,Y)$, which assigns to two spectra X and Y the object of Ho(**Sp**) obtained by taking a cofibrant model for X, a fibrant model for Y and forming a mapping spectrum. In particular $\text{Map}\,(X,X)$, since it belongs to Ho(**Sp**), is a ring spectrum up to homotopy; there is no good theory of modules over such an object. To improve matters one could find a model X' for X which is both fibrant and cofibrant, form the (structured, strict, A_∞, ...) endomorphism ring spectrum $\text{End}(X')$, and call it $\text{End}(X)$ for convenience. This ring spectrum does have a good module theory associated to it. The spectrum $\text{Map}\,(X,X)$ then represents an object in the homotopy category of $\text{End}(X)$-module spectra (in fact, in the homotopy category of $(\text{End}(X),\text{End}(X))$-bimodule spectra) which is isomorphic in this homotopy category to the object derived from the strict action of $\text{End}(X)$ on itself. In the same way, $\text{End}_R(A)$ is a (strict) DGA, and $\text{Hom}_R(A,A)$ represents an object in the derived category of $(\text{End}_R(A),\text{End}_R(A))$-bimodules which is isomorphic in this derived category to the object derived from the multiplicative action of $\text{End}_R(A)$ on itself.

Remark 1.3. This paper is intended to establish a framework and introduce some terminology in a simple but relatively general setting. With little change, our key arguments can be extended to cover a number of important categorically similar cases. For instance, R could be replaced by a graded ring (this comes up in some examples from Section 3), a differential graded algebra, or a ring spectrum; with some adjustments in terminology R could even be replaced by a ring with many objects, or a ring spectrum with many objects [17]. These extensions are important in constructing an algebraic model for rational equivariant cohomology theories. The framework provided here is also the starting point for our study with Iyengar [6] of various duality properties in algebra and topology.

Some of the results in this paper, especially in Section 6, are related to results in [8], but here we take a different point of view. In the setting of commutative rings (or even schemes), the authors of [1] have already shown that $\mathbf{A}_{\mathrm{tors}}$ and $\mathbf{A}_{\mathrm{comp}}$ are equivalent categories; their approach does not involve the category of modules over End(A). In the commutative ring case this result appears in [12] as well. In [1] there is an interpretation of local homology as a Bousfield localization functor; in a sense this way of looking at local homology goes back to [10].

2. Morita theory. Recall that A is a perfect object of R-mod. In this section we show that the categories $\mathbf{A}_{\mathrm{tors}}$ and $\mathbf{A}_{\mathrm{comp}}$ are equivalent, by relating them to a third category which is at least as interesting as the other two. Let $\mathcal{E} = \mathrm{End}_R(A)$. This is a differential graded ring (Section 1.2), and there is a derived category mod-$\mathcal{E}$ of *right* $\mathcal{E}$-modules formed as usual by taking differential graded $\mathcal{E}$-modules and inverting homology isomorphisms. We may define a functor

$$E\colon\ R\text{-}\mathrm{mod} \longrightarrow \mathrm{mod}\text{-}\mathcal{E} \qquad E(M) = \mathrm{Hom}_R(A, M).$$

Note here that A is naturally a *left* $\mathcal{E}$-module; this left module structure commutes with the action of R on A and passes to a right $\mathcal{E}$-module structure on $\mathrm{Hom}_R(A, M)$. Let

$$(\cdot)^\sharp\colon\ R\text{-mod} \longrightarrow \text{mod-}R$$

be the duality functor defined by $M^\sharp = \mathrm{Hom}_R(M, R)$, and note that $A^\sharp$ is an object of mod-$\mathcal{E}$, as well as an object of mod-$\mathcal{R}$. Define functors

$$T\colon\ \text{mod-}\mathcal{E} \longrightarrow R\text{-mod} \qquad T(X) = X \otimes_{\mathcal{E}} A,$$
$$C\colon\ \text{mod-}\mathcal{E} \longrightarrow R\text{-mod} \qquad C(X) = \mathrm{Hom}_{\mathcal{E}}(A^\sharp, X).$$

Here the left R-structure on $T(X)$ is obtained from the left R-structure on A, and the left R-structure on $C(X)$ from the right R-structure on $A^\sharp$. The main result of this section is the following theorem.

THEOREM 2.1. *Let A be a perfect complex of R-modules, and let $\mathcal{E} = \mathrm{End}_R(A)$. Then the above functors E, T, and C give two pairs of adjoint equivalences of categories*

$$\mathbf{A}_{\mathrm{tors}} \underset{E}{\overset{T}{\rightleftarrows}} \mathit{mod}\text{-}\mathcal{E} \underset{C}{\overset{E}{\rightleftarrows}} \mathbf{A}_{\mathrm{comp}}$$

(where the left adjoints are displayed above the right ones).

Remark 2.2. If R is a commutative ring, then in the situation of Theorem 2.1 it has already been shown by Hovey, Palmieri, and Strickland [12, 3.3.5] that $\mathbf{A}_{\mathrm{tors}}$ and $\mathbf{A}_{\mathrm{comp}}$ are equivalent categories. See also [1].

Remark 2.3. Theorem 2.1 is a variant of Morita theory. Since A is perfect it is *small*, in the sense that $\mathrm{Hom}_R(A, \cdot)$ commutes with arbitrary coproducts. Ordinary Morita theory implies that if A is a small generator of R-mod (e.g., $A = R^n$ for some $n > 0$) then the category of modules over $\mathcal{E} = \mathrm{End}_R(\mathrm{A})$ is equivalent to R-mod itself. Theorem 2.1 states that if A is small but *not* necessarily a generator, then the category of modules over $\mathcal{E}$ is equivalent both to the subcategory $\mathbf{A}_{\mathrm{tors}}$ and the subcategory $\mathbf{A}_{\mathrm{comp}}$ of R-mod. This is particularly plausible in the case of $\mathbf{A}_{\mathrm{tors}}$, since this is just the category of chain complexes which can be *built from* A (Section 4.5).

Before beginning with the proof of Theorem 2.1, it is useful to point out a few simple facts.

2.4. A-equivalences. It is convenient to say that a map $M \longrightarrow N$ in R-mod is *an A-equivalence* if its cofibre is A-trivial, or equivalently if $E(M) \cong E(N)$. We leave it to the reader to check that an A-equivalence between A-torsion objects of R-mod is an isomorphism, and that an A-equivalence between A-complete objects of R-mod is an isomorphism. This is formal: for instance, the cofibre of an A-equivalence between A-torsion objects is both A-torsion and A-trivial, and hence is trivial.

2.5. Adjunctions. Note that for each left R-module M and right $\mathcal{E}$-module X there is an adjunction isomorphism

$$\mathrm{Hom}_{\mathcal{E}}(X, \mathrm{Hom}_R(A, M)) \cong \mathrm{Hom}_R(X \otimes_{\mathcal{E}} A, M).$$

The unit map

$$X \longrightarrow \mathrm{Hom}_R(A, X \otimes_{\mathcal{E}} A) \tag{2.6}$$

is obtained by applying the functor $\cdot \otimes_{\mathcal{E}} A$ to pass from $X \cong \mathrm{Hom}_{\mathcal{E}}(\mathcal{E}, X)$ to $\mathrm{Hom}_R(A, X \otimes_{\mathcal{E}} A)$. The counit map

$$\mathrm{Hom}_R(A, M) \otimes_{\mathcal{E}} A \longrightarrow M \tag{2.7}$$

is obtained from evaluation. Similarly, for any left R-module M and right $\mathcal{E}$-module X there is an adjunction isomorphism

$$\mathrm{Hom}_R(M, \mathrm{Hom}_{\mathcal{E}}(A^\sharp, X)) \cong \mathrm{Hom}_{\mathcal{E}}(A^\sharp \otimes_R M, X). \tag{2.8}$$

2.9. Maps and tensors. Finally, note that since A is perfect there are isomorphisms

$$\mathrm{Hom}_R(A, N) \cong A^\sharp \otimes_R N \quad \text{and in particular} \quad \mathcal{E} \cong A^\sharp \otimes_R A.$$

This last is an isomorphism inside the derived category of either left or right $\mathcal{E}$-modules (see Section 1.2), with $\mathcal{E}$ acting on the left on $A^\sharp \otimes_R A$ via its action on A, and on the right via its action on $A^\sharp$.

2.10. The left side of Theorem 2.1. We begin by observing that for any right $\mathcal{E}$-module X, the module $T(X) = X \otimes_{\mathcal{E}} A$ is in fact an object of $\mathbf{A}_{\mathrm{tors}}$. In fact, one can use 2.5 to calculate that for any A-trivial N,

$$\mathrm{Hom}_R(T(X), N) = \mathrm{Hom}_R(X \otimes_{\mathcal{E}} A, N) = \mathrm{Hom}_{\mathcal{E}}(X, \mathrm{Hom}_R(A, N)) \cong 0.$$

Note that this is an isomorphism in the derived category $\mathbb{Z}$-mod. This adjunction shows that T is left adjoint to E.

We now need to show that the unit map (2.6) is always an equivalence, and that the counit (2.7) is an equivalence if M is an object of $\mathbf{A}_{\mathrm{tors}}$. The unit is a natural map and preserves both cofibre sequences and coproducts (since A is small (Section 2.3)) and therefore it suffices to check the result for $X = \mathcal{E}$, where it is clear. The counit is a natural map and its domain is A-torsion. In order to prove that it is an isomorphism if its range is A-torsion, it is enough (Section 2.4) to show that the counit map is always an A-equivalence. To prove this, we calculate

$$\begin{aligned} \mathrm{Hom}_R(A, TE(M)) &= A^\sharp \otimes_R (\mathrm{Hom}_R(A, M) \otimes_{\mathcal{E}} A) \\ &\cong \mathrm{Hom}_R(A, M) \otimes_{\mathcal{E}} (A^\sharp \otimes_R A) \\ &\cong \mathrm{Hom}_R(A, M). \end{aligned}$$

The first equivalence comes from the fact that the action of R on $\mathrm{Hom}_R(A, M) \otimes_{\mathcal{E}} A$ comes from an action of R on A which commutes with the action of $\mathcal{E}$; the second equivalence from Section 2.9. □

2.11. The right side of Theorem 2.1. First we show that $C(X) = \mathrm{Hom}_{\mathcal{E}}(A^\sharp, X)$ belongs in fact to $\mathbf{A}_{\mathrm{comp}}$. Suppose that N is A-trivial. We may use (2.8) and 2.9 to calculate

$$\begin{aligned}\mathrm{Hom}_R(N, C(X)) &= \mathrm{Hom}_R(N, \mathrm{Hom}_{\mathcal{E}}(A^\sharp, X)) \\ &\cong \mathrm{Hom}_{\mathcal{E}}(A^\sharp \otimes_R N, X) \\ &\cong \mathrm{Hom}_{\mathcal{E}}(\mathrm{Hom}_R(A, N), X) \cong 0.\end{aligned}$$

This reasoning also shows that C is right adjoint to E.

We now have to show that the counit map $EC(X) \to X$ is an isomorphism for all right $\mathcal{E}$-modules X. This follows from 2.5 and 2.9:

$$EC(X) = \mathrm{Hom}_R(A, \mathrm{Hom}_{\mathcal{E}}(A^\sharp, X)) \cong \mathrm{Hom}_{\mathcal{E}}(A^\sharp \otimes_R A, X) \cong X.$$

Finally, we have to check that the unit map $M \to CE(M)$ is an isomorphism for each A-complete left R-module M. Since the target of this map is A-complete, it is enough by Section 2.4 to verify that the map itself is always an A-equivalence, i.e., becomes an isomorphism when the functor E is applied. But as above, $EC(X) \cong X$, so $ECE(M) \cong E(M)$. □

Remark 2.12. It is useful to note that even when A is not perfect, the functor $C\colon \text{mod-}\mathcal{E} \to R\text{-mod}$ given by $C(X) = \mathrm{Hom}_{\mathcal{E}}(A^\sharp, X)$ is right adjoint to the functor E': $R\text{-mod} \to \text{mod-}\mathcal{E}$ given by $E'(N) = A^\sharp \otimes_R N$.

3. Sample applications of the Morita theory. We describe three situations in which Theorem 2.1 holds.

3.1. The paradoxical case of $\mathbb{Z}/p$. This simplest nontrivial application of Theorem 2.1 is already very striking. Let $R = \mathbb{Z}$ and $A = \mathbb{Z}/p$, this last considered as a chain complex concentrated in degree 0. The complex A is perfect because it is isomorphic in $\mathbb{Z}$-mod to the resolution $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$. An object N of $\mathbb{Z}$-mod is A-trivial if and only if all of its homology groups are uniquely p-divisible, or equivalently if and only if the natural map $N \to \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} N$ is an isomorphism. From this and the isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} Y) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[1/p] \otimes_{\mathbb{Z}} X, \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} Y)$$

it is easy to see that $\mathbf{A}_{\mathrm{tors}}$ is the subcategory of $\mathbb{Z}$-mod consisting of objects X with $\mathbb{Z}[1/p] \otimes_{\mathbb{Z}} X \cong 0$, i.e., objects X which have p-primary torsion homology groups. By inspection $\mathcal{E}$ is a DG algebra whose homology algebra is isomorphic to $\mathrm{Ext}^*_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/p)$; from a multiplicative point of view this homology is an exterior algebra over $\mathbb{Z}/p$ on one generator of dimension (-1). (Recall that all of our chain complexes have lower indices, so that the differentials decrease degree by one; the group $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/p)$ corresponds to $H_{-1}\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/p)$.) Theorem 2.1 says that the functor E gives an equivalence between the category $\mathbf{A}_{\mathrm{tors}}$ and the category of $\mathcal{E}$-modules.

For instance, $E(\mathbb{Z}/p^\infty) = \mathbb{Z}/p$. Accordingly, it follows that

$$H_0\mathrm{Hom}_{\mathcal{E}}(\mathbb{Z}/p, \mathbb{Z}/p) = H_0\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, \mathbb{Z}/p^\infty) = \mathbb{Z}_{\hat{p}}.$$

This seems very difficult to believe, since the identity map of $\mathbb{Z}/p$ has additive order p.

The issue, though, revolves around what "additive order" means. Although p times the identity map of $\mathbb{Z}/p$ is null-homotopic as a $\mathbb{Z}$-map, it is *not* null-homotopic as an $\mathcal{E}$-map. To calculate $\mathcal{E} = \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}/p)$ as a strict DGA we have to replace the abelian group $\mathbb{Z}/p$ by the above resolution $M = (\mathbb{Z} \xrightarrow{p} \mathbb{Z})$. Viewing an element of M as a column vector with the top entry recording the copy of $\mathbb{Z}$ in homological degree 0, we may view $\mathcal{E} = \mathrm{End}_{\mathbb{Z}}(M)$ as the algebra of 2×2 matrices (appropriately graded). One then calculates

$$d\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}_0 = \begin{pmatrix}0 & p\\ 0 & 0\end{pmatrix}_{-1} \qquad d\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}_{-1} = 0$$
$$d\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}_1 = \begin{pmatrix}p & 0\\ 0 & p\end{pmatrix}_0 \qquad d\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}_0 = \begin{pmatrix}0 & -p\\ 0 & 0\end{pmatrix}_{-1}.$$

The elements of $M^\sharp$ are row vectors, with $\mathcal{E}$ acting on the right. The map $p\colon M^\sharp \longrightarrow M^\sharp$ is indeed the boundary of right multiplication by $\left(\begin{smallmatrix}0 & 0\\ 1 & 0\end{smallmatrix}\right)$. However this matrix is not a central element of $\mathcal{E}$ and therefore right multiplication by it does not represent an $\mathcal{E}$-map. In fact, the identity map of M has infinite order as an $\mathcal{E}$-map.

We may also give a constructive interpretation. Observe that the usual construction of an Adams spectral sequence gives a conditionally convergent spectral sequence

$$\mathrm{Ext}^{s,t}_{H_*\mathcal{E}}(H_*(\mathbb{Z}/p), H_*(\mathbb{Z}/p)) \Rightarrow H_{t-s}\mathrm{End}_{\mathcal{E}}(\mathbb{Z}/p).$$

This may be identified as an unravelled Bockstein spectral sequence. The generator of $H_{-1}\mathcal{E}$ is a Bockstein operator, and the multiplication-by-p map described above is represented by an element of bidegree $(1, 1)$ which a stable homotopy theorist would denote h_0. In effect we may regard the category mod-$\mathcal{E}$ as encoding Bockstein spectral sequences for both p-torsion and p-complete modules, and Theorem 2.1 as stating that the Bockstein spectral sequence determines a p-torsion or a p-complete module.

Example 3.2. Let T be the circle group ("T" stands for "torus"; in the following discussion T is not related to the functor of the same name from Section 2). This second example suggests an approach to studying T-equivariant rational cohomology theories. We content ourselves here with sketching an analogy which is refined elsewhere to a theorem and proved in greater generality.

We take $R = k[c]$, where k is a field, c is of degree -2, $I = (c)$, and $A = R/I$. If $k = \mathbb{Q}$ it is natural to think of $R/I = k$ as analogous to the free T-cell T_+. Thus $\mathcal{E}$ is analogous to the DGA of self-maps of T_+, and mod-$\mathcal{E}$ to the category of free rational T-spectra. The left-hand equivalence of Theorem 2.1 is then analogous to the theorem of [7] stating

$$(c)\text{-torsion-}\mathbb{Q}[c]\text{-mod} \simeq \text{free rational } T\text{-spectra.}$$

From this point of view, the great attraction of the equivalence is that the category of torsion $\mathbb{Q}[c]$-modules is of injective dimension 1, whilst that of $\mathcal{E}$-modules is of infinite homological dimension. The analogue of the Adams spectral sequence in Example 3.1 is the descent spectral sequence. Generally speaking this is much less useful than the Adams spectral sequence based on the torsion module $k[c, c^{-1}]/k[c]$, which collapses to a short exact sequence.

Example 3.3. The third example is connected with chromatic stable homotopy theory [16]. One might take $R = \mathbb{Z}[v_1, v_2, \ldots, v_{n-1}, v_n, v_n^{-1}]$, $I = (p, v_1, \ldots, v_{n-1})$, and $A = R/I$. The category of I-primary torsion modules is analogous to the nth monochromatic category, whilst the category of I-complete modules is analogous to the category of Bousfield $K(n)$-complete modules. For topological purposes it is better to take A to be a small $L_n(S^0)$-module Bousfield equivalent to $K(n)$. The proof of Theorem 2.1 then gives the equivalence [13, 6.19]

$$n\text{th monochromatic category} \simeq K(n)\text{-complete spectra,}$$

with the intermediate category of $\mathcal{E}$-modules being the category of modules over the ring spectrum of self-maps of A. We plan to investigate this example in more detail in [6].

4. Cellular approximations and homology localizations. We work in the setting of Section 2, but in order to organize the results more clearly we will introduce some new notation. For an object M of R-mod, let $\mathrm{Cell}_A(M)$ stand for $TE(M)$ and $M_A^{\wedge}$ for $CE(M)$. There is a natural A-equivalence (Section 2.4) $M \longrightarrow M_A^{\wedge}$. There is also a natural A-equivalence $\mathrm{Cell}_A(M) \longrightarrow M$, and we will denote the cofibre (which is A-trivial) by $M_{1/A}$.

It follows easily from the arguments in Section 2 that the functor $\mathrm{Cell}_A(\cdot)$ is idempotent and is right adjoint to the inclusion $\mathbf{A}_{\mathrm{tors}} \longrightarrow R$-mod. Similarly, the completion functor $(\cdot)_A^{\wedge}$ is idempotent and is left adjoint to the inclusion $\mathbf{A}_{\mathrm{comp}} \longrightarrow R$-mod.

4.1. Another look at $\mathbb{Z}/p$. One way to get some insight into these constructions is to consider the special case $R = \mathbb{Z}$, $A = \mathbb{Z}/p$; this can be analyzed either by direct calculation (Section 3.1) or by applying the results of Section 6. We describe the situation in this case, using terminology and results which will be explained below. For simplicity, write $\mathrm{Cell}_p(\cdot)$, $(\cdot)_{1/p}$, and $(\cdot)_{p}^{\wedge}$, for $\mathrm{Cell}_A(\cdot)$, $(\cdot)_{1/A}$, and $(\cdot)_{A}^{\wedge}$ when $A = \mathbb{Z}/p$. For any object M of $\mathbb{Z}$-mod, there are isomorphisms

$$\begin{aligned} \mathrm{Cell}_p(M) &\cong \Sigma^{-1}\mathbb{Z}/p^{\infty} \otimes_{\mathbb{Z}} M \\ M_{1/p} &\cong \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} M \\ M_{p}^{\wedge} &\cong \mathrm{Hom}_{\mathbb{Z}}(\Sigma^{-1}\mathbb{Z}/p^{\infty}, M). \end{aligned}$$

The map $\mathrm{Cell}_p(M) \to M$ is cellular approximation (Section 4.5) with respect to $\mathbb{Z}/p$ (i.e., gives the universal p-torsion approximation to M). The map $M \to M_{p}^{\wedge}$ is Bousfield localization (Section 4.7) with respect to the homology theory on $\mathbb{Z}$-mod given by $M \mapsto H_*(\mathbb{Z}/p \otimes_{\mathbb{Z}} M)$ (the complex $M_{p}^{\wedge}$ is sometimes called the *Ext-p-completion* of M.) . The map $M \to M_{1/p}$ is nullification (Section 4.10) with respect to $\mathbb{Z}/p$. Since $\mathbb{Z}/p$ is perfect, this is a smashing localization [15], and so $M \to M_{1/p}$ is also Bousfield localization with respect to the homology theory $M \mapsto H_*(\mathbb{Z}[1/p] \otimes_{\mathbb{Z}} M)$. Finally, there is a homotopy fibre square

$$\begin{array}{ccc} M & \longrightarrow & M_{p}^{\wedge} \\ \downarrow & & \downarrow \\ M_{1/p} & \longrightarrow & (M_{p}^{\wedge})_{1/p} \end{array} \quad . \tag{4.2}$$

The square is obtained by applying the natural map $X \to X_{1/p}$ to the upper row.

The goal of this section is to obtain the above results in the general case. We assume as usual that A is a perfect object of R-mod.

PROPOSITION 4.3. *For any object M of R-mod, there are natural isomorphisms*

$$\begin{aligned} \mathit{Cell}_A(M) &\cong \mathit{Cell}_A(R) \otimes_R M \\ M_{1/A} &\cong R_{1/A} \otimes_R M \\ M_{A}^{\wedge} &\cong \mathit{Hom}_R(\mathit{Cell}_A(R), M). \end{aligned}$$

Remark 4.4. Implicit in the above formulas is the fact that $\mathrm{Cell}_A(R)$ and $R_{1/A}$ can be constructed as objects in the derived category of R-bimodules. This is a consequence of the fact that R itself is an R-bimodule. It is easier to explain this in a slightly more general setting. Suppose that M is a left module over R and a right module over S, in such a way that the two actions commute. Recall the formula

$$\mathrm{Cell}_A(M) = \mathrm{Hom}_R(A, M) \otimes_{\mathcal{E}} A.$$

The extra right action of S on M persists to a right action of S on $\mathrm{Hom}_R(A, M)$. This S-action commutes with the action of $\mathcal{E}$ on $\mathrm{Hom}_R(A, M)$ (because the action of $\mathcal{E}$ works on the A variable) and so passes through the tensor product to give a right action of S on $\mathrm{Cell}_A(M)$ which commutes with the normal left action of R. Applying this in the special case $M = R$ and $S = R$ gives the R-bimodule structure on $\mathrm{Cell}_A(R)$. The bimodule structure on $R_{1/A}$ is obtained similarly.

If R is commutative, these bimodules are obtained from ordinary modules by taking the action on one side to be the same as the action on the other.

Proof. For the first isomorphism we use the chain

$$\begin{aligned}\mathrm{Cell}_A(M) &= \mathrm{Hom}_R(A, M) \otimes_{\mathcal{E}} A \\ &\cong (A^\sharp \otimes_R M) \otimes_{\mathcal{E}} A \\ &\cong (A^\sharp \otimes_{\mathcal{E}} A) \otimes_R M,\end{aligned}$$

where the last isomorphism comes from the fact that the action of $\mathcal{E}$ on $A^\sharp \otimes_R M$ is induced by an action of $\mathcal{E}$ on $A^\sharp$ which commutes with the right action of R. Now note that $A^\sharp \otimes_{\mathcal{E}} A$ is $\mathrm{Cell}_A(R)$. For the second, use a naturality argument and tensor the exact triangle

$$\mathrm{Cell}_A(R) \to R \to R_{1/A}$$

over R with M. For the third, use the adjunction argument

$$\begin{aligned}M_A^{\wedge} &= \mathrm{Hom}_{\mathcal{E}}(A^\sharp, \mathrm{Hom}_R(A, M)) \\ &\cong \mathrm{Hom}_R(A^\sharp \otimes_{\mathcal{E}} A, M) = \mathrm{Hom}_R(\mathrm{Cell}_A(R), M).\end{aligned}$$

□

4.5. Cellular approximation. An object M of R-mod is said to be *A-cellular* if M is *built from A* in the sense that it belongs to the smallest triangulated subcategory of R-mod which contains A and is closed under arbitrary coproducts (i.e., it belongs to the localizing subcategory of R-mod generated by A). If M is A-cellular then any A-equivalence (Section 2.4) $X \to Y$ induces an isomorphism $\mathrm{Hom}_R(M, X) \to \mathrm{Hom}_R(M, Y)$ (in particular, M is A-torsion). A map $M' \to M$ is said to be an *A-cellular approximation map* if M' is A-cellular and the map $M' \to M$ is an A-equivalence. A formal argument shows that A-cellular approximations are unique, if they exist; the arguments of Dror-Farjoun show that they do exist [5].

PROPOSITION 4.6. *The objects of* $\mathbf{A}_{\text{tors}}$ *are exactly the A-cellular objects of R-mod. For any object M of R-mod, the natural map* $\mathrm{Cell}_A(M) \to M$ *is an A-cellular approximation map.*

Proof. We noted above that any A-cellular object of R-mod belongs to $\mathbf{A}_{\text{tors}}$. If M is A-torsion, we can choose an A-cellular approximation map $M' \to M$. This is an A-equivalence between A-torsion complexes, and so it is an isomorphism. Therefore, M is A-cellular.

The second statement is proved by observing that $\mathrm{Cell}_A(M)$ is A-torsion, hence A-cellular, and that the map $\mathrm{Cell}_A(M) \to M$ is an A-equivalence (2.10). □

4.7. Homology localization. For our purposes a *homology theory* on R-mod is a functor S_* from R-mod to graded abelian groups determined by the recipe

$$S_*(M) = H_*(S \otimes_R M)$$

for some object S in the derived category of right R-modules. An object M is said to be S_**-acyclic* if $S_*(M) = 0$. A map $M \to M'$ is *an* S_**-equivalence* if it induces an isomorphism $S_*(M) \cong S(M')$, or equivalently if its cofibre is S_*-acyclic. An object M is S_**-local* if $\mathrm{Hom}_R(N, M) \cong 0$ for each S_*-acyclic N. *An* S_**-localization* of M is an S_*-equivalence $M \to M'$ with the property that M' is S_*-local. A formal argument shows that S_*-localizations are unique, if they exist; Bousfield's arguments show that in fact they do exist [3] [4] [5].

PROPOSITION 4.8. *For any object M of R-mod, the natural map* $M \to M_A^{\wedge}$ *is an* S_**-localization map for* $S = A^\sharp$.

Proof. It is necessary to show that $M_A^{\wedge}$ is S_*-local and that $M \to M_A^{\wedge}$ is a S_*-equivalence. Given that $A^\sharp \otimes_R M \cong \mathrm{Hom}_R(A, M)$, it is clear than an object of R-mod is S_*-acyclic if and only if it is A-trivial in the sense of Section 1. Thus the first statement follows from the fact that $M_A^{\wedge}$ is A-complete, and the second from the arguments of Section 2.11. □

PROPOSITION 4.9. *For any object M of R-mod, the natural map* $M \to M_{1/A}$ *is an* S_**-localization map for* $S = R_{1/A}$.

Proof. In this case $S \otimes_R M \cong M_{1/A}$ (Proposition 4.3), so it is enough to show that $(M_{1/A})_{1/A} \cong M_{1/A}$ (i.e., that $M \to M_{1/A}$ is an S_*-equivalence), and that if $N_{1/A} \cong 0$, then $\mathrm{Hom}_R(N, M_{1/A}) \cong 0$ (i.e., that $M_{1/A}$ is S_*-local). The first statement follows from the fact that $\mathrm{Cell}_A(\mathrm{Cell}_A M) \cong \mathrm{Cell}_A(M)$, so that $(\mathrm{Cell}_A M)_{1/A} \cong 0$. The second follows from the fact that if $N_{1/A} \cong 0$ then $N \cong \mathrm{Cell}_A N$, i.e., N is A-torsion, so that $\mathrm{Hom}_R(N, K) \cong 0$ for any A-trivial object K, in particular, for $K = M_{1/A}$. □

4.10. Nullification. Suppose that W is an object of R-mod. An object N of R-mod is said to be W*-null* if $\mathrm{Hom}_R(W, N) \cong 0$. A map $M \to M'$ is a W*-null* *equivalence* if it induces an isomorphism $\mathrm{Hom}_R(M', N) \to \mathrm{Hom}_R(M, N)$ for each W-null object N. A map $f\colon M \to M'$ is a *W-nullification* of M if M' is W-null and f is a W-null equivalence. A formal argument shows that W-nullifications are unique up to isomorphism if they exist; the arguments of Bousfield and Dror-Farjoun show that in fact they do exist [5].

PROPOSITION 4.11. *For any object M of R-mod, the map $M \to M_{1/A}$ is a W-nullification of M for $W = A$.*

Proof. Note that for $W = A$, "W-null" is the same as "A-trivial." The object $M_{1/A}$ is A-trivial because $\mathrm{Cell}_A(M) \to M$ is an A-equivalence. The map $M \to M_{1/A}$ is a W-null equivalence because its fibre $\mathrm{Cell}_A(M)$ is A-torsion, and so $\mathrm{Hom}_R(\mathrm{Cell}_A(M), N) \cong 0$ for any A-trivial N. □

4.12. An arithmetic square. For our purposes, the term "homotopy fibre square" means a commutative square which can be completed in such a way as to induce an isomorphism between the fibre of the right vertical map and the fibre of the left vertical map.

PROPOSITION 4.13. *For any object M of R-mod, there is a homotopy fibre square*

$$\begin{array}{ccc} M & \longrightarrow & M^{\wedge}_{A} \\ \downarrow & & \downarrow \\ M_{1/A} & \longrightarrow & (M^{\wedge}_{A})_{1/A} \end{array} .$$

Proof. Since $M \to M^{\wedge}_{A}$ is an A-equivalence, it induces an isomorphism $\mathrm{Cell}_A(M) \to \mathrm{Cell}_A(M^{\wedge}_{A})$. This is the required isomorphism between the fibres. □

5. Homology groups. In this section we identify conditions on R under which it is possible to determine whether an object of R-mod is A-torsion or A-complete by examining its homology groups one by one. As always, we assume that A is perfect.

We emphasize that in this section an *R-module*, as opposed to an object of R-mod, is an ordinary classical left R-module. Of course, an R-module M can be viewed as a chain complex concentrated in degree 0, and thus treated as an object of R-mod.

Definition 5.1. An R-module M is *homotopically A-complete* if the natural map $M \to M^{\wedge}_{A}$ is an isomorphism in R-mod. An R-module M is *homotopically A-torsion* if the natural map $\mathrm{Cell}_A(M) \to M$ is an isomorphism in R-mod.

The main results of this section are the following: Recall that $\mathrm{Cell}_A(R) = A^{\sharp} \otimes_{\mathcal{E}} A$ can be considered to be an object of the derived category of R-bimodules

(Remark 4.4). For the statements, we will say that a chain complex is *essentially concentrated between dimensions i and j* ($i \leq j$) if it is isomorphic in the appropriate derived category to a chain complex of projective modules which vanishes except between dimension i and dimension j.

PROPOSITION 5.2. *Suppose there exists an $n \geq 0$ such that as an object of R-mod the complex $\mathrm{Cell}_A(R)$ is essentially concentrated between dimensions $-n$ and 0. Then an object X of R-mod is A-complete if and only if each homology group $H_i(X)$ is homotopically A-complete.*

PROPOSITION 5.3. *Suppose there exists an $n \geq 0$ such that as an object of mod-R the complex $\mathrm{Cell}_A(R)$ is essentially concentrated between dimensions $-n$ and 0. Then an object X of R-mod is A-torsion if and only if each homology group $H_i(X)$ is homotopically A-torsion.*

Remark 5.4. The hypotheses in the previous two propositions may seem unmotivated. Of course, the hypotheses are needed for the proofs. But we are particularly interested in these conditions because they apply when R is a commutative ring, $I \subset R$ is a finitely generated ideal, and A is the associated Koszul complex (the perfect surrogate for R/I).

5.5. Some initial observations. Somewhat surprisingly, an R-module M is homotopically A-complete if and only if $H_i(M_A^\wedge) = 0$ for $i < 0$ and the natural map $M \to H_0(M_A^\wedge)$ is an isomorphism. The surprise is that under these conditions the groups $H_i(M_A^\wedge)$ vanish for $i > 0$. For suppose the conditions are satisfied. Let X be the quotient of $M_A^\wedge$ obtained by dividing out by the cycles in dimension 1 and by all elements in dimensions > 1. Then $X \cong M$, and the composite

$$M \to M_A^\wedge \to X$$

exhibits M as a retract of the A-complete complex $M_A^\wedge$. It follows that M is A-complete, so that $M_A^\wedge \cong M$ and M is homotopically A-complete. It is useful to note that (by Proposition 4.3) the condition $H_i(M_A^\wedge) = 0$ for $i < 0$ is automatically satisfied under the assumptions of Proposition 5.2.

A similar argument shows that M is homotopically A-torsion if and only if $H_i\mathrm{Cell}_A(M)$ vanishes for $i > 0$ and the map $H_0\mathrm{Cell}_A(M) \to M$ is an isomorphism. Again, it is useful to note that (by Proposition 4.3) the condition $H_i\mathrm{Cell}_A(M) = 0$ for $i > 0$ is automatically satisfied under the assumptions of Proposition 5.3.

Proof of Proposition 5.2. Let $C = \mathrm{Cell}_A(R)$, which can be taken to be a projective chain complex supported between dimension $(-n)$ and 0. For any object X of R-mod, $X_A^\wedge$ is isomorphic to $\mathrm{Hom}_R(C, X)$ (see Proposition 4.3). Suppose that the homology groups of X are homotopically A-complete. Let $X\langle i,j\rangle$ ($i \leq j$) be the subquotient of X which agrees with X between dimensions i and j, has

X_{j+1}/cycles in dimension $j+1$, has $\partial(X_i)$ in dimension $i-1$, and has zero elsewhere. In particular, the homology groups of $X\langle i,j\rangle$ agree with the homology groups of X between dimensions i and j and are otherwise zero. The cofibre sequences

$$X\langle j+1,j+1\rangle \to X\langle i,j+1\rangle \to X\langle i,j\rangle$$

allow $X\langle i,j\rangle$ to be pieced together inductively out of complexes with only one nonzero homology group. By assumption, each of these complexes is A-complete. It follows that $X\langle i,j\rangle$ is A-complete, and hence that the (evident) complexes $X\langle i,\infty\rangle$ are A-complete, since

$$\begin{aligned}\mathrm{Hom}_R(C, X\langle i,\infty\rangle) &\cong \mathrm{Hom}_R(C, \mathrm{holim}_j\, X\langle i,j\rangle)\\ &\cong \mathrm{holim}_j\, \mathrm{Hom}_R(C, X\langle i,j\rangle)\\ &\cong \mathrm{holim}_j\, X\langle i,j\rangle \cong X\langle i,\infty\rangle.\end{aligned}$$

The identification $X\langle i,\infty\rangle \cong \mathrm{holim}_j\, X\langle i,j\rangle$ is made by noting that before passing to the derived category the tower $\{X\langle i,j\rangle\}_{j\geq i}$ of chain complexes is a tower of epimorphisms with inverse limit $X\langle i,\infty\rangle$. Since C is concentrated in a finite number of dimensions, an easy connectivity argument shows that

$$\mathrm{Hom}_R(C,X) \cong \mathrm{Hom}_R(C, \mathrm{colim}_i\, X\langle i,\infty\rangle) \cong \mathrm{colim}_i\, \mathrm{Hom}_R(C, X\langle i,\infty\rangle).$$

The point is that the direct system $\{X\langle i,\infty\rangle\}$ is convergent in the sense that the homology groups above any given dimension stabilize after a certain point. It follows that X is A-complete.

Suppose on the other hand that X is A-complete. We need to show that for any k, the R-module $H_k(X)$ is homotopically A-complete. By the dimensional assumption on C there are isomorphisms

$$H_k\mathrm{Hom}_R(C, X\langle k-n-1,\infty\rangle) \cong H_k\mathrm{Hom}_R(C,X) = H_k(X)$$

and so after replacing X by the complex on the left, it is possible to assume that the homology groups of X vanish below a certain point. We now show by ascending induction on k that $H_i(X)$ is homotopically A-complete for $i \leq k$. This is true for $k \ll 0$, because in that case the groups all vanish. Suppose that the claim is true for $k-1$; we must show that the module $M = H_k(X)$ is homotopically A-complete. By the inductive assumption, $X\langle -\infty, k-1\rangle$ is a complex with homology groups that are homotopically A-complete, and so, as above, this complex is A-complete. The cofibre sequence

$$X\langle k,\infty\rangle \to X \to X\langle -\infty, k-1\rangle$$

shows that $X\langle k,\infty\rangle$ is A-complete. Since C vanishes in positive dimensions and the map $X\langle k,\infty\rangle \to X\langle k,k\rangle \cong \Sigma^k M$ is an isomorphism on homology up through dimension k, the induced map

$$M \cong H_k\mathrm{Hom}_R(C, X\langle k,\infty\rangle) \to H_0\mathrm{Hom}_R(C, M)$$

is also an isomorphism. By 5.5, M is homotopically A-complete. □

The proof of Proposition 5.3 is exactly parallel to the one above, with the opposite orientation (upward induction replaced by downward induction). It depends on the tensor product formula for $\mathrm{Cell}_A(X)$ from Proposition 4.3.

6. Commutative rings. In this section, we assume that R is commutative and that $I \subset R$ is a finitely generated ideal. We wish to study the A-complete and A-torsion objects of R-mod in the special case $A = R/I$, but the theory from the rest of the paper does not immediately apply, because R/I need not be perfect, i.e., R/I need not have a finite length resolution by finitely generated projective R-modules. To get around this problem we construct an associated perfect complex K with the property that an object M of R-mod is K-torsion, K-complete, or K-trivial if and only if M is R/I-torsion, R/I-complete, or R/I-trivial. Elaborating a little on the construction of Kleads to explicit formulas for the torsion functor $\mathrm{Cell}_K(\cdot)$ and the completion functor $(\cdot)^{\wedge}_K$; these turn out to be identical to the usual local cohomology and local homology functors with respect to the ideal I. This, for instance, gives an interpretation of local homology with respect to I as the Bousfield localization functor on the category R-mod associated to the homology theory $M \mapsto H_*(R/I \otimes_R M)$.

The Koszul complex. For an element r in R, let $K^\bullet(r)$ denote the chain complex r: $R \to R$, with the two copies of R in dimensions 0 and (-1), respectively. For a sequence $\mathbf{r} = (\mathrm{r}_1, \ldots, \mathrm{r}_n)$, let $K^\bullet(\mathbf{r})$ be the tensor product $K^\bullet(\mathrm{r}_1) \otimes_R \cdots \otimes_R K^\bullet(\mathrm{r}_n)$. This is the *Koszul complex* associated to $\mathbf{r}$.

Recall from Section 4.5 that if A and B are two objects of R-mod, then B is said to be *built from* A if B is in the smallest localizing subcategory of R-mod which contains A, i.e., B is in the smallest full subcategory of R-mod which contains A and is closed under isomorphisms, desuspensions, coproducts, and cofibre sequences. If B is built from A then the class of B-trivial objects in R-mod contains the class of A-trivial objects. If A and B can each be built from one another, then the classes of A-trivial and B-trivial objects coincide, and it follows immediately that the classes of A-torsion and B-torsion objects also coincide, as do the classes of A-complete and B-complete objects.

The following is the property of $K^\bullet(\mathbf{r})$ which interests us.

PROPOSITION 6.1. *Suppose that R is a commutative ring and that $I \subset R$ is an ideal generated by the sequence* $\mathbf{r} = (r_1, \ldots, r_n)$. *Then the two objects R/I and* $K^\bullet(\mathbf{r})$

of R-mod can each be built from one another.

Remark 6.2. Given Proposition 6.1 the results in the previous sections can be applied with $A = K^{\bullet}(\mathbf{r})$ to give conclusions involving R/I. It follows from Theorem 2.1, for instance, that the category of R/I-torsion objects of R-mod is equivalent to mod-$\mathcal{E}$, where $\mathcal{E} = \mathrm{End}_R(K^{\bullet}(\mathbf{r}))$. The same is true of the category of R/I-complete objects of R-mod. Note that $\mathcal{E}$ is a matrix algebra of rank 2^n over R, graded so as to lie between dimensions $-n$ and n, and provided with a suitable differential.

Remark 6.3. Suppose that R is a noetherian ring. The argument below easily shows that if R/I is a field (or more generally R/I is a regular ring) then $K^{\bullet}(\mathbf{r})$ can be built from R/I and its suspensions by a finite number of cofibration sequences, i.e., $K^{\bullet}(\mathbf{r})$ is in the thick subcategory of R-mod generated by R/I. On the other hand, R/I is in the thick subcategory of R-mod generated by $K^{\bullet}(\mathbf{r})$ only if R itself is regular.

Convention. Suppose that R is a commutative ring and that $I \subset R$ is an ideal generated by the finite sequence $\mathbf{r}$. For the rest of this section we will write $\mathrm{Cell}_{R/I}(\cdot)$ and $(\cdot)\hat{}_{R/I}$ instead of $\mathrm{Cell}_K(\cdot)$ and $(\cdot)\hat{}_K$ with $K = K^{\bullet}(\mathbf{r})$. These objects are defined in terms of K (Section 2) but, in view of Proposition 6.1 and the results of Section 4, easy to interpret in terms of R/I. For instance, it follows from Proposition 4.6 that the natural map $\mathrm{Cell}_{R/I}(M) \to M$ is an R/I-cellular approximation map. There is a corresponding interpretation of $(M)\hat{}_{R/I}$ in Proposition 6.14.

Proof of Proposition 6.1. Let $K = K^{\bullet}(\mathbf{r})$. Any object M of R-mod can be built out of R, and it follows immediately that $K \otimes_R M$ can be built out of K. Since $K \otimes_R (R/I)$ is a direct sum of shifted copies of R/I, we conclude that R/I can be built from K.

Conversely, note that for each i the map r_i: $K \to K$ is chain homotopic to zero, because r_i: $K^{\bullet}(\mathrm{r}_i) \to K^{\bullet}(\mathrm{r}_i)$ is chain homotopic to zero. It follows that the homology groups of K are modules over the ring R/I. Any module M over R/I can be built out of R/I as an object in the derived category of R/I, and the same recipe will build M out of R/I in the category R-mod. In the notation of the proof of Proposition 5.2, there are cofibre sequences

$$K\langle j,j\rangle \to K\langle -\infty,j\rangle \to K\langle -\infty,j-1\rangle.$$

As just noted, each one of the fibres is built from R/I. The fact that K itself is built from R/I follows from a finite induction, beginning with the fact that $K\langle -\infty, -(n+1)\rangle \cong 0$ and ending with the fact that $K\langle -\infty, 0\rangle = K$. □

The dual Koszul complex. If r$\in R$, let $K_\bullet(\mathrm{r})$ denote the chain complex r: $R \to R$, with the two copies of R in dimensions 1 and 0, respectively. Clearly $K_\bullet(\mathrm{r}) = \Sigma K^\bullet(\mathrm{r}) \cong K^\bullet(\mathrm{r})^\sharp$. For a sequence $\mathbf{r} = (\mathrm{r}_1, \ldots, \mathrm{r}_n)$, let $K_\bullet(\mathbf{r})$ denote $K_\bullet(\mathrm{r}_1) \otimes_R \cdots \otimes_R K_\bullet(\mathrm{r}_n)$. Then $K_\bullet(\mathbf{r}) = \Sigma^n K^\bullet(\mathbf{r}) \cong K^\bullet(\mathbf{r})^\sharp$. Since $K^\bullet(\mathbf{r})$ and $K_\bullet(\mathbf{r})$ are suspensions of one another, the following proposition is a consequence of Proposition 6.1.

PROPOSITION 6.4. *Suppose that R is a commutative ring and $I \subset R$ is an ideal generated by the sequence $\mathbf{r} = (r_1, \ldots, r_n)$. Then the two objects R/I and $K_\bullet(\mathbf{r})$ of R-mod can each be built from one another.*

As a consequence we obtain the following.

PROPOSITION 6.5. *Suppose that R is a commutative ring and $I \subset R$ is an ideal generated by the sequence $\mathbf{r} = (r_1, \ldots, r_n)$. Then for an object M of R-mod the following four conditions are equivalent:*

(1) $R/I \otimes_R M \cong 0$,

(2) $K^\bullet(\mathbf{r}) \otimes_R M \cong 0$,

(3) $Hom_R(K_\bullet(\mathbf{r}), M) \cong 0$, *and*

(4) $Hom_R(R/I, M) \cong 0$.

Proof. The first and second are equivalent by Proposition 6.1, the third and fourth by Proposition 6.4, the second and third because $K_\bullet(\mathbf{r}) = K^\bullet(\mathbf{r})^\sharp$. □

Remark 6.6. It is a little surprising that the first and fourth conditions of Proposition 6.5 are equivalent in such generality.

Local cohomology. For r$\in R$, the commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{=} & R \\ {\scriptstyle r^k}\downarrow & & \downarrow{\scriptstyle r^{k+1}} \\ R & \xrightarrow{r} & R \end{array} \tag{6.7}$$

gives a map $K^\bullet(\mathrm{r}^k) \to K^\bullet(\mathrm{r}^{k+1})$. Let $K^\bullet(\mathrm{r}^\infty)$ denote $\mathrm{colim}_k K^\bullet(\mathrm{r}^k)$; this is just the flat complex r: $R \to R[1/\mathrm{r}]$. It is easy to see that $K^\bullet(\mathrm{r}^\infty)$ is isomorphic in R-mod to a free chain complex over R with nonzero groups only in dimensions 0 and -1; this can be taken to be the chain complex

$$d\colon \oplus_{i\geq 0} R \to \oplus_{i \geq 0} R$$

$$d(x_0, x_1, x_2, \ldots) = (x_0 - x_1, rx_1 - x_2, rx_2 - x_3, \ldots).$$

If $\mathbf{r} = (\mathrm{r}_1, \ldots, \mathrm{r}_n)$ is a sequence of elements in R, we let $\mathbf{r}^k$ denote $(\mathrm{r}_1^k, \ldots, \mathrm{r}_n^k)$. Tensoring together the maps from (6.7) gives a map $K^\bullet(\mathbf{r}^k) \to K^\bullet(\mathbf{r}^{k+1})$. Let

$K^\bullet(\mathbf{r}^\infty)$ denote $\mathrm{colim}_k K^\bullet(\mathbf{r}^k)$, so that in R-mod there is an isomorphism

$$(6.8) \qquad K^\bullet(\mathbf{r}^\infty) \cong K^\bullet(r_1^\infty) \otimes_R \cdots \otimes_R K^\bullet(r_n^\infty).$$

The following result is clear.

LEMMA 6.9. *If* $\mathbf{r} = (r_1, \ldots, r_n)$, *then* $K^\bullet(\mathbf{r}^\infty)$ *is isomorphic in R-mod to a free chain complex over R which is concentrated between dimensions* $(-n)$ *and* 0.

Suppose that $I \subset R$ is the ideal generated by $\mathbf{r}$. If M is an (ordinary) R-module, the *local cohomology* of M at I (see [11] or [9]) is denoted $H_I^*(M)$ and defined by the formula

$$H_I^k(M) = H_{-k}(K^\bullet(\mathbf{r}^\infty) \otimes_R M).$$

In line with this, if M is an arbitrary object of R-mod we define $H_I(M) = K^\bullet(\mathbf{r}^\infty) \otimes_R M$ and call $H_I(M)$ the derived local cohomology of M at I. There is a Kunneth spectral sequence

$$E^2_{p,q} = H_I^{-p}(H_q(M)) \Rightarrow H_{p+q}(H_I(M)).$$

This converges strongly because $K^\bullet(\mathbf{r}^\infty)$ is a flat chain complex concentrated in a finite range of dimensions.

PROPOSITION 6.10. *Suppose that R is a commutative ring and that* $I \subset R$ *is the ideal generated by the sequence* $\mathbf{r} = (r_1, \ldots, r_n)$. *Then* $Cell_{R/I}(R)$ *is isomorphic as an object of R-mod to* $K^\bullet(\mathbf{r}^\infty)$.

Proof. Let $K = K^\bullet(\mathbf{r})$ and $K^\infty = K^\bullet(\mathbf{r}^\infty)$. There is a natural map $K^\infty \to R$ which amounts to taking a quotient by the elements in $K^\infty$of strictly negative dimension. We have to show that $K^\infty$is K-torsion, i.e., built from K (see Proposition 4.6) and that the map $K^\infty \to R$ becomes an isomorphism when $\mathrm{Hom}_R(K, \cdot)$ is applied.

Fix k, and consider an ordinary R-module M on which the elements $r_1^k, \ldots, r_n^k$ act trivially. Such a module is annihilated by the ideal I^{nk} and so has a finite filtration $\{I^jM\}$ such that the associated graded modules are annihilated by I. In particular, each subquotient $\{I^jM/I^{j+1}M\}$ is a module over R/I and so, as in the proof of Proposition 6.1, can be built from R/I. The cofibre sequences

$$I^jM/I^{j+1}M \to M/I^{j+1}M \to M/I^jM$$

allow for an inductive proof that the quotient modules M/I^jM, and hence also the module M itself, are built from R/I. Since each homology group of $K^\bullet(\mathbf{r}^k)$ is annihilated by $r_1^k, \ldots, r_n^k$, it now follows from the argument in the proof of

Proposition 6.1 that $K^{\bullet}(\mathbf{r}^k)$ can be built from R/I. Passing to a directed (homotopy) colimit shows that K^{∞}can be built from R/I and hence (Proposition 6.1) from K.

To finish the proof, it is enough, by Proposition 6.5, to show that tensoring the above map $K^{\infty} \to R$ with R/I gives an equivalence

$$R/I \otimes_R K^{\infty} \to R/I \otimes_R R = R/I.$$

However, K^{∞} is given as a flat chain complex (6.8) and so we can compute the derived tensor product on the left as an ordinary tensor product. In this interpretation, the displayed map is an actual isomorphism of chain complexes, since $R[1/r_i] \otimes_R R/I = 0$ for $i = 1, \ldots, n$. □

The following proposition is a consequence of Propositions 6.10 and 4.3.

PROPOSITION 6.11. *Suppose that R is a commutative ring and that $I \subset R$ is a finitely generated ideal. Then local cohomology at I computes R/I-cellular approximation, in the sense that for any object M of R-mod there is a natural isomorphism $H_I(M) \cong Cell_{R/I}(M)$.*

We can now apply the results of Section 5 to give a simple characterization of the R/I-cellular objects of mod-R. An ordinary R-module M is said to be an I-power torsion module if for each $x \in M$ there is a k such that $I^k x = 0$.

PROPOSITION 6.12. *Suppose that R is a commutative ring and that $I \subset R$ is a finitely generated ideal. Then an object M of R-mod is R/I-cellular if and only if each homology group of M is an I-power torsion module.*

Proof. Let $\mathbf{r}$ be a finite sequence of generators for I, and $K = K^{\bullet}(\mathbf{r})$. It follows from Propositions 4.6 and 6.1 that an object M of R-mod is R/I-cellular if and only if M is K-cellular, or, equivalently, K-torsion.

It is easy to argue from the definitions that if M is built from K then the homology groups of M are I-power torsion modules. Suppose that the homology groups of M are I-power torsion modules. Let $K^{\infty} = K^{\bullet}(\mathbf{r}^{\infty})$. The argument in the proof of Proposition 6.10 shows immediately that $K^{\infty} \otimes_R H_i(M) \cong H_i(M)$ for all i, so by Proposition 6.10 $\mathrm{Cell}_K(H_i M) \cong H_i M$. In other words, all of the homology groups of M are homotopically K-torsion. In view of Lemma 6.9 and Proposition 5.3, M is K-torsion. □

Local homology. Suppose that $I \subset R$ is an ideal generated by the finite sequence $\mathbf{r}$. If M is an (ordinary) R-module, the *local homology* of M at I (see [9]) is denoted $H^I_*(M)$ and defined by the formula

$$H^I_k(M) = H_k \mathrm{Hom}_R(K^{\bullet}(\mathbf{r}^{\infty}), M).$$

In line with this, if M is an arbitrary object of R-mod we define $H^I(M) = \mathrm{Hom}_R(K^\bullet(\mathbf{r}^\infty), M)$ and call $H^I(M)$ the derived local homology of M at I. It is easy to construct a spectral sequence

$$E^2_{p,q} = H^I_p(H_q(M)) \Rightarrow H_{p+q}(H^I(M)).$$

This converges strongly because $K^\bullet(\mathbf{r}^\infty)$ is equivalent to a projective chain complex of finite length.

The following proposition is a consequence of Propositions 6.10 and 4.3.

PROPOSITION 6.13. *Suppose that R is a commutative ring and that $I \subset R$ is a finitely generated ideal. Then local homology at I computes R/I-completion, in the sense that for any object M of R-mod there is a natural isomorphism $H^I(M) \cong M^\wedge_{R/I}$.*

Note that under the isomorphism of Proposition 6.13, the natural completion map $M \to M^\wedge_{R/I}$ is obtained by applying $\mathrm{Hom}_R(-, M)$ to the map $K^\bullet(\mathbf{r}^\infty) \to R$ mentioned at the beginning of the proof of Proposition 6.10.

According to Proposition 4.8, the natural map $M \to M^\wedge_{\mathcal{K}}$ is a Bousfield localization map for the homology theory on R-mod given by $M \mapsto H_*(K^\sharp \otimes_R M)$. By Proposition 6.4, this homology theory has the same acyclic objects as the theory given by $M \mapsto H_*(R/I \otimes_R M)$, and thus the same notion of localization. We have obtained the following interpretation of $(\cdot)^\wedge_{R/I}$, and hence of local homology.

PROPOSITION 6.14. *Suppose that R is a commutative ring and that $I \subset R$ is a finitely generated ideal. Then the natural map $M \to M^\wedge_{R/I}$ is a Bousfield localization map for the homology theory on R-mod given by $M \mapsto H_*(R/I \otimes_R M)$. In particular, since $H^I(M) \cong M^\wedge_{R/I}$, local homology at I computes Bousfield localization with respect to the homology theory determined by R/I.*

Finally, along the lines of Proposition 6.12 we obtain the following characterization of the objects of R-mod which are local with respect to the above homology theory. It depends on combining Proposition 5.2 and Section 5.5 with the above observation that these local objects are exactly the objects which are K-complete.

PROPOSITION 6.15. *Suppose that R is a commutative ring and that $I \subset R$ is a finitely generated ideal. Then an object M of R-mod is local with respect to the homology theory on R-mod given by $M \mapsto H_*(R/I \otimes_R M)$ if and only if for each integer k the natural map $H_k(M) \to H^I_0(H_k M)$ is an isomorphism.*

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, IN 46556
E-mail: dwyer.1@nd.edu

SCHOOL OF MATHEMATICS AND STATISTICS, HICKS BUILDING, SHEFFIELD S3 7RH, UK
E-mail: j.greenlees@sheffield.ac.uk

REFERENCES

[1] L. Alonso Tarrío, A. Jeremías López, and J. Lipman, *Studies in Duality on Noetherian Formal Schemes and non-Noetherian Ordinary Schemes*, American Mathematical Society, Providence, RI, 1999.

[2] L. L. Avramov and H.-B. Foxby, Homological dimensions of unbounded complexes, *J. Pure Appl. Algebra* **71** (1991), 129–155.

[3] A. K. Bousfield, The localization of spaces with respect to homology, *Topology* **14** (1975), 133–150.

[4] ———, The localization of spectra with respect to homology, *Topology* **18** (1979), 257–281.

[5] E. Dror-Farjoun, *Cellular Spaces, Null Spaces and Homotopy Localization*, Springer-Verlag, Berlin, 1996.

[6] W. G. Dwyer, J. P. C. Greenlees, and S. B. Iyengar, Duality in algebra and topology, in preparation.

[7] J. P. C. Greenlees, Rational S^1-equivariant stable homotopy theory, *Mem. Amer. Math. Soc.* **138** (1999), xii+289.

[8] ———, Tate cohomology in axiomatic stable homotopy theory, *Proceedings of the 1998 Barcelona conference on Algebraic Topology* (to appear).

[9] J. P. C. Greenlees and J. P. May, Derived functors of I-adic completion and local homology, *J. Algebra* **149** (1992), 438–453.

[10] ———, Completions in algebra and topology, *Handbook of Algebraic Topology*, North-Holland, Amsterdam, 1995, pp. 255–276.

[11] A. Grothendieck, *Local Cohomology*, Springer-Verlag, Berlin, 1967; Course notes taken by R. Hartshorne, Harvard University, Fall, 1961.

[12] M. Hovey, J. H. Palmieri, and N. P. Strickland, Axiomatic stable homotopy theory, *Mem. Amer. Math. Soc.* **128** (1997), x+114.

[13] M. Hovey and N. P. Strickland, Morava K-theories and localisation, *Mem. Amer. Math. Soc.* **139** (1999), viii+100.

[14] I. Kříž and J. P. May, Operads, algebras, modules and motives, *Astérisque* (1995), iv+145pp.

[15] H. Miller, Finite localizations, *Bol. Soc. Mat. Mexicana (2)* **37** (1992), 383–389 (papers in honor of José Adem (Spanish)).

[16] D. C. Ravenel, *Nilpotence and Periodicity in Stable Homotopy Theory*, Princeton University Press, Princeton, NJ, 1992, Appendix C by Jeff Smith.

[17] S. Schwede and B. E. Shipley, A Gabriel equivalence for model categories enriched over spectra, in preparation.

[18] N. Spaltenstein, Resolutions of unbounded complexes, *Compositio Math.* **65** (1988), 121–154.

[19] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge University Press, Cambridge, 1994.