1) Recall bar resolution for derived tensor product:

Let A be an algebra (over field k).
Let $M \in \text{Mod-}A$ (right A-mod)
$N \in A\text{-Mod}$ (left A-mod)
Then we have the naive tensor product

$$M \otimes_A N := \operatorname{coker}\left(M \otimes_k A \otimes_k N \longrightarrow \underset{\sim\sim}{M \otimes N} \right)$$
$$m \otimes a \otimes n \mapsto ma \otimes n - m \otimes an$$

The derived tensor product is the ~~total~~ complex:

$$M \overset{\mathbb{L}}{\underset{A}{\otimes}} N \cong \tilde{M} \otimes_A N$$ where $\tilde{M} \simeq M$ is a flat resolution of M.

or $\simeq M \otimes_A \tilde{N}$, (resolve N to $\tilde{N}$).

One canonical, but redundant way to get free resolution of M is to use bar resolution

$$\cdots \longrightarrow M \otimes_k A \otimes_k A \xrightarrow{d_2} M \otimes_k A \xrightarrow{d_1} M$$
$$m \otimes a \overset{d_1}{\mapsto} ma$$

$$m \otimes a_1 \otimes a_2 \overset{d_2}{\mapsto} ma_1 \otimes a_2 - m \otimes a_1 a_2$$

Plug in this resolution of M to $M \overset{\mathbb{L}}{\underset{A}{\otimes}} N$, we get the bar complex expression of $M \overset{\mathbb{L}}{\underset{A}{\otimes}} N$:

$$\cdots\cdots \longrightarrow M \otimes A \otimes A \otimes N \xrightarrow{d_2} M \otimes A \otimes N \longrightarrow M \otimes_k N$$

$$M \otimes A^{n} \otimes N \longrightarrow M \otimes A^{n-1} \otimes N$$

$$m \otimes a_1 \otimes \cdots \otimes a_n \otimes N \longmapsto \sum_{i=0}^{n} (-1)^{i} \, m \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes n$$

(there are $n+1$ $\otimes$ sign, we consider all possible way to merge away them.)

On the level of category, we cannot take "alternative sum" and get chain complex of categories, but we can still consider a "simplicial" diagram of categories, and consider colimit of the diagram of categories.

- Let A denote a monoidal category, where we have functor.

$$m_A : \quad A \times A \longrightarrow A$$

- Let M be a right A-module, which is a category, that A can acts from right:

$$a_M : \quad M \times A \longrightarrow M$$

such that there is a natural equivalence

$$\begin{tikzcd}
M \times A \times A \arrow[r, "\alpha_M \otimes id_A"] \arrow[d, "id_M \otimes m_A"'] \arrow[dr, phantom, "\Swarrow"] & M \times A \arrow[d, "\alpha_M"] \\
M \times A \arrow[r, "\alpha_M"'] & M
\end{tikzcd}$$

- We can define the derived tensor product of a pair of A-modules,

$$M \overset{\mathbb{L}}{\underset{A}{\otimes}} N := \operatorname{colim}\left(M \otimes N \leftleftarrows M \otimes A \otimes N \leftleftarrows M \otimes A \otimes A \otimes N \cdots \right)$$

In our case:

A, M, N are $\mathbb{Z}_{\geq 0}$ - graded.

$A = \bigoplus_{k \geq 0} A_k, \qquad A_k = \mathrm{Fuk}\ \mathrm{Sym}^k($  $)$

$M_k = \mathrm{Fuk}\ \mathrm{Sym}^k($  $)$ $\qquad \Sigma_i$: Riemann surface with one boundary stop.

Σ_1

$N_k = \mathrm{Fuk}\ \mathrm{Sym}^k($  $)$

Σ_2

Let $A_+ = \bigoplus_{k > 0} A_k$ be the positive ideal of A.
we should have a reduced diagram.

$$M \overset{\mathbb{L}}{\underset{A}{\otimes}} N = \operatorname{colim}\left(M \otimes N \leftleftarrows M \otimes A_+ \otimes N \leftleftarrows M \otimes A_+ \otimes A_+ \otimes N \cdots \right)$$

$\boxed{\bullet\ k = 1,}$ we have

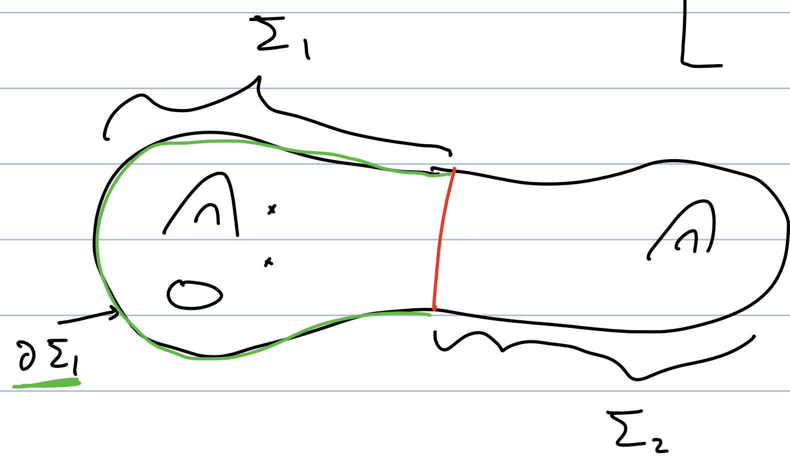
$$\left(M \overset{\mathbb{L}}{\underset{A}{\otimes}} N \right)_1 = \operatorname{colim}\left(\begin{array}{c} M_1 \otimes N_0 \quad \oplus \quad M_0 \otimes N_1 \\ \uparrow \\ M_0 \otimes A_1 \otimes N_0 \end{array} \right)$$

$M_0, N_0, A_0 = \mathrm{Vect}$

$$= \operatorname{colim}\left(\begin{tikzcd} M_1 & & N_1 \\ & A_1 \arrow[ul] \arrow[ur] & \end{tikzcd} \right)$$

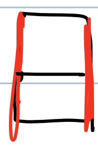
This is saying : for wrapped Fukaya categories, we have.

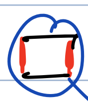
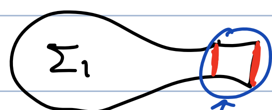
$$\mathrm{Fuk}^w(\Sigma_1 \cup \Sigma_2) \simeq \operatorname{colim}\left[\begin{tikzcd} \mathrm{Fuk}^w(\Sigma_1) & & \mathrm{Fuk}^w(\Sigma_2) \\ & \mathrm{Fuk}^w(T^*I) \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] & \end{tikzcd} \right]$$



red = "cut" = " stop"

- Σ_1 is the left half of Σ, where the " cut " become a stop for Σ_1

- T^*I =  = disk with 2 stops.

- $\mathrm{Fuk}^w($  $) \xrightarrow{\alpha} \mathrm{Fuk}^w($  $)$

  is by inclusion

We can consider (left) - module over these wrapped Fuk.

$$\mathrm{Fuk}^w(\Sigma)\text{-Mod} := \mathrm{Fun}^{ex}(\mathrm{Fuk}^w(\Sigma)^{op}, \mathrm{Vect}) =: \mathrm{Fuk}^{\diamondsuit}(\Sigma)$$

ind completion of $\mathrm{Fuk}^w(\Sigma)$.

We have Yoneda embedding

$$\mathrm{Fuk}^w(\Sigma) \hookrightarrow \mathrm{Fuk}^{\diamondsuit}(\Sigma)$$

· We also have the right-adjoint of the above sectorial inclusion.

$$\mathrm{Fuk}^{\Diamond}(\Sigma) = \lim \left(\begin{tikzcd} \mathrm{Fuk}^{\Diamond}(\Sigma_1) \arrow[dr] & & \mathrm{Fuk}^{\Diamond}(\Sigma_2) \arrow[dl] \\ & \mathrm{Fuk}^{\Diamond}(T^*I) & \end{tikzcd} \right)$$

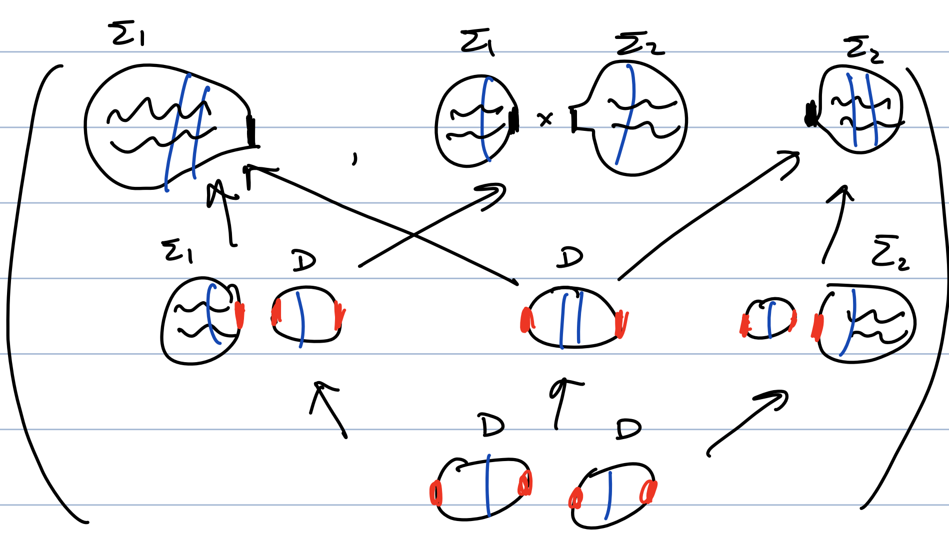
$\boxed{k=2}$, we have

$$(M \underset{A}{\otimes} N)_2 := \mathrm{colim} \left(\begin{tikzcd} (M \otimes N)_2 \\ (M \otimes A_+ \otimes N)_2 \arrow[u, shift left] \arrow[u, shift right] \\ (M \otimes A_+ \otimes A_+ \otimes N)_2 \arrow[u, shift left=2] \arrow[u] \arrow[u, shift right=2] \\ \vdots \end{tikzcd} \right)$$

(omit M_0, N_0 = vect factor)

$$= \mathrm{colim} \left[\begin{tikzcd} M_2 & & M_1 \otimes N_1 & & N_2 \\ M_1 \otimes A_1 \arrow[u] \arrow[urr] & & A_2 \arrow[ull] \arrow[urr] & & A_1 \otimes N_1 \arrow[u] \arrow[ull] \\ & & A_1 \otimes A_1 \arrow[ull] \arrow[u] \arrow[urr] & & \end{tikzcd} \right]$$

= colim



(wavy lines in Σ_1, Σ_2 means "there are features" in them, like genus, verma hole etc...)

To show the colimit computes Fuk $\mathrm{Sym}^2(\Sigma_1 \cup \Sigma_2)$, we will first create an auxillary diagram such that

- the auxillary diagram satisfies easy local-to-global property,
- the desired homotopy colimit diagram corresponds to localization of the auxillary diagram