

# Central limit theorem for toric Kähler manifolds

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**Abstract:** Associated to the Bergman kernels of a polarized toric Kähler manifold  $(M, \omega, L, h)$  are sequences of measures  $\{\mu_k^z\}_{k=1}^\infty$  parametrized by the points  $z \in M$ . For each  $z$  in the open orbit, we prove a central limit theorem for  $\mu_k^z$ . The center of mass of  $\mu_k^z$  is the image of  $z$  under the moment map upto  $O(1/k)$ ; after re-centering at 0 and dilating by  $\sqrt{k}$ , the re-normalized measures tend to a centered Gaussian whose variance is the Hessian of the Kähler potential at  $z$ . We further give a remainder estimate of Berry-Esseen type. The sequence  $\{\mu_k^z\}$  is generally not a sequence of convolution powers and the proofs only involve Kähler analysis.

**Keywords:** Bergman kernel, holomorphic line bundle, measures on moment polytope.

## 1. Introduction

Let  $(L, h, M, \omega)$  be a polarized toric Kähler manifold of complex dimension  $m$ , with an ample toric line bundle  $L \rightarrow M$ . Thus, there exists a Hamiltonian torus action  $\Phi^{\mathbf{T}}(z) : \mathbf{T}^m \times M \rightarrow M$  on  $M$  which extends holomorphically to a  $(\mathbb{C}^*)^m$  action, and  $M$  is the closure of the open orbit  $M^o = (\mathbb{C}^*)^m \{z_0\}$ . Let  $h$  denote a  $\mathbf{T}^m$ -invariant Hermitian metric on  $L$  with curvature form  $\omega$ . The moment map

$$(1) \quad \mu_h := \mu : M \rightarrow P \subset \mathbb{R}^m,$$

associated to this data defines a torus bundle on the open orbit over a convex lattice polytope  $P$  known as a Delzant polytope. As reviewed in Section 2.1, there is a natural basis  $\{s_\alpha\}_{\alpha \in kP \cap \mathbb{Z}^m}$  of the space  $H^0(M, L^k)$  of holomorphic sections of the  $k$ -th power of  $L$  by eigensections  $s_\alpha$  of the  $\mathbf{T}^m$  action. In a standard frame  $e_L$  of  $L$  over  $M^o$ , they correspond to monomials  $z^\alpha$  on  $(\mathbb{C}^*)^m$ .

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For any  $z \in M^\circ$  and  $k \in \mathbb{N}$ , we define the probability measure,

$$(2) \quad \mu_k^z = \frac{1}{\Pi_{h^k}(z, z)} \sum_{\alpha \in kP \cap \mathbb{Z}^m} \frac{|s_\alpha(z)|_{h^k}^2}{\|s_\alpha\|_{h^k}^2} \delta_{\frac{\alpha}{k}} \in \mathcal{M}_1(\mathbb{R}^m),$$

on  $\mathbb{R}^m$ . Here,  $\|s_\alpha\|_{h^k}$  is the  $L^2$  norm of  $s_\alpha$  with respect to the natural inner product  $\text{Hilb}_k(h)$  induced by the Hermitian metric on  $H^0(M, L^k)$  and  $\Pi_{h^k}(z, z)$  is the contracted Szegő kernel on the diagonal (or density of states); see §2.3 for background. The measures are discrete measures supported on  $P \cap \frac{1}{k}\mathbb{Z}^m$ , and were previously studied in [SoZ10, SoZ12]. The main result of this article is that for each  $z$ , the sequence  $\{\mu_k^z\}_{k=1}^\infty$  satisfies a CLT (central limit theorem) with a Berry-Esseen type remainder estimate.

To state the results precisely we need to introduce some notation and background. A Gaussian measure on  $\mathbb{R}^m$  with mean  $\vec{m}$  and covariance matrix  $\Sigma$  is a measure of the form,

$$\gamma_{\vec{m}, \Sigma}(\vec{x}) := (2\pi \det \Sigma)^{-m/2} e^{-\frac{1}{2}\langle \Sigma^{-1}(\vec{x} - \vec{m}), (\vec{x} - \vec{m}) \rangle}.$$

Our aim is to prove that in the sense of weak convergence, dilations of (2) tend to a certain Gaussian measure,

$$(3) \quad \frac{1}{\Pi_{h^k}(z, z)} \sum_{\alpha \in kP \cap \mathbb{Z}^m} \frac{|s_\alpha(z)|_{h^k}^2}{\|s_\alpha\|_{h^k}^2} \delta_{\sqrt{k}(\frac{\alpha}{k} - \mu_h(z))} \rightarrow \gamma_{0, \text{Hess}\varphi(z)},$$

whose mean is 0 and whose covariance matrix is the Hessian  $\text{Hess}\varphi(z)$  of the Kähler potential. In Section 2, we review the fact that the Kähler potential  $\varphi(z)$  of a toric variety is a convex function of  $(\rho_1, \dots, \rho_m) = (\log |z_1|^2, \dots, \log |z_m|^2)$  on  $\mathbb{R}^m$ . Here, we use orbit coordinates  $(\rho, \theta)$  where  $e^{\rho_i} = |z_i|^2$ . Then  $\nabla_\rho \varphi$  is the gradient, resp.  $\text{Hess} \varphi = \partial_{\rho_i \rho_j}^2 \varphi(e^{\rho/2})$  is the Hessian in the  $\rho$  variables. We refer to Section 2.2 for definitions and details.

### 1.1. Mean and covariance

To determine the appropriate Gaussian measure we need to determine the asymptotics as  $k \rightarrow \infty$  of the mean, resp. covariance matrix

$$(4) \quad \vec{m}_k(z) = \int_P \vec{x} d\mu_k^z(x), \text{ resp. } [\Sigma_k]_{ij}(z) = \int_P (x_i - m_{k,i}(z))(x_j - m_{k,j}(z)) d\mu_k^z.$$

**LEMMA 1.1.** *Let  $\mu_h : M \rightarrow P$  be the moment map (1). Then,*

$$\vec{m}_k(z) = \mu_h(z) + O(1/k), \quad \Sigma_k(z) = \frac{1}{k} \text{Hess} \varphi(z) + O\left(\frac{1}{k^2}\right)$$

The proof is reviewed in Section 2.5 (from [Z09, Proposition 6.3]). It implies the law of large numbers for the sequence  $\{\mu_k^z\}$ : In the weak topology of measures on  $C(\bar{P})$ ,  $\mu_k^z \rightarrow \delta_{\mu_h(z)}$ . We therefore center the measures (2) at  $\mu(z)$ , i.e. put

$$(5) \quad \tilde{\mu}_k^z = \mu_k^z(x - \mu_h(z)),$$

and then dilate by  $\sqrt{k}$  to obtain the normalized sequence,

$$(6) \quad D_{\sqrt{k}}\tilde{\mu}_k^z = \frac{1}{\Pi_{h^k}(z, z)} \sum_{\alpha \in kP \cap \mathbb{Z}^m} \frac{|s_\alpha(z)|_{h^k}^2}{\|s_\alpha\|_{h^k}^2} \delta_{\sqrt{k}(\frac{\alpha}{k} - \mu_h(z))}.$$

Equivalently, if  $f \in C_b(\mathbb{R}^m)$ , then,

$$(7) \quad \langle f, D_{\sqrt{k}}\tilde{\mu}_k^z \rangle = \frac{1}{\Pi_{h^k}(z, z)} \sum_{\alpha \in kP \cap \mathbb{Z}^m} \frac{|s_\alpha(z)|_{h^k}^2}{\|s_\alpha\|_{h^k}^2} f(\sqrt{k}(\frac{\alpha}{k} - \mu_h(z))).$$

Here,  $C_b(\mathbb{R}^m)$  denotes the space of bounded continuous functions on  $\mathbb{R}^m$ .

### 1.2. Weak\* convergence on $C_b(\mathbb{R}^m)$

Our first main result is the following

**THEOREM 1.2.** *In the topology of weak\* convergence on  $C_b(\mathbb{R}^m)$ ,*

$$D_{\sqrt{k}}\tilde{\mu}_k^z \xrightarrow{w*} \gamma_{0, \text{Hess } \varphi(z)}.$$

That is, for any  $f \in C_b(\mathbb{R}^m)$ ,

$$\int_{\mathbb{R}^m} f(x) D_{\sqrt{k}} d\tilde{\mu}_k^z(x) \rightarrow \int_{\mathbb{R}^m} f(x) d\gamma_{0, \text{Hess } \varphi(z)}(x).$$

The role of the parameter  $z$  is similar to that of the parameter  $p$  in the Bernoulli measures  $\mu_p = p\delta_0 + (1 - p)\delta_1$  and their convolution powers on the unit interval  $[0, 1]$ . In very special cases, such as the Fubini-Study metric  $h$  of  $M = \mathbb{C}\mathbb{P}^m$ ,  $\mu_k^z$  is itself a sequence of dilated convolution powers,  $\mu_k^z = (\mu_1^z)^{*k} = \mu_1^z * \mu_1^z \cdots * \mu_1^z$  ( $k$  times). It has been pointed out in [D08, STZ03] that such a situation occurs for the Fubini-Study metric on the dual hyperplane line bundle  $\mathcal{O}(1) \rightarrow \mathbb{C}\mathbb{P}^m$  and for the Bargmann-Fock case. This is equivalent to the condition that  $\Pi_{h^k} = (\Pi_{h^1})^k$ . In [STZ03] the relation between  $\Pi_{h^k}$  and  $(\Pi_{h^1})^k$  on a toric Kähler manifold has been given in terms of partition functions of lattice random walks and a certain pseudo-

differential operator. To the author's knowledge, conditions on  $h$  which are necessary and/or sufficient that  $\Pi_{h^k} = (\Pi_{h^1})^k$  are not known even for a toric variety. Donaldson points out that it holds for sequences of metrics defined by Veronese embeddings.<sup>1</sup> The proof of Theorem 1.2 does not appeal to any prior results on central limit theorems or probability theory but is purely a result of toric Kähler analysis and is based on the quantum dynamics of the torus action (see (27)).

### 1.3. Berry-Esseen type remainder

The classical Berry-Esseen theorem gives a quantitative remainder estimate for the CLT for sums  $S_N = X_1 + \cdots + X_N$  of i.i.d. real-valued random variables with finite third moment. With no loss of generality, assume that  $\mathbb{E}X_j = 0$ ,  $\text{Var}(X_j) = 1$  and let  $m_3 = \mathbb{E}|X_j|^3$ . Let  $\mu$  denote the common distribution of the  $X_j$ . Then the Berry-Esseen remainder bound states that if  $f$  is a " $\gamma_{0,1}$ -continuous bounded function" then

$$(8) \quad \int_{\mathbb{R}^m} f(x) D_{\sqrt{k}} d\mu^{*k}(x) = \int_{\mathbb{R}^m} f(x) d\gamma_{0,1}(x) + O\left(\frac{m_3}{\sqrt{k}}\right).$$

Such functions include characteristic functions of sets whose boundaries have Lebesgue measure zero. The Berry-Esseen bound was extended to the multivariate CLT by Bergstrom, Bhattacharya, Rotar, Sazonov and von Bahr around 1970; see [Bhat] for background and references. The measures  $\mu_k^z$  of this article would be referred to as distributions of lattice random variables  $\vec{X}_k$ , i.e. random variables whose values are almost surely located on lattice points of  $\frac{1}{k}\mathbb{Z}^m \cap P$ . Special techniques are available for lattice random variables (see [Bhat, Chapter 5]) but we do not use them here. The following is a simple analogue of the remainder estimate of [ZZ19b], and is stated for certain continuous test functions rather than for characteristic functions of sets.

**THEOREM 1.3.** *If  $f \in C_0(\mathbb{R}^m)$ , with  $\hat{f} \in L^1(\mathbb{R}^m)$  bounded by a radially decreasing  $L^1$  function, then*

$$\int_{\mathbb{R}^m} f(x) D_{\sqrt{k}} d\tilde{\mu}_k^z(x) = \int_{\mathbb{R}^m} f(x) d\gamma_{0, \text{Hess } \varphi(z)}(x) + O_f\left(\frac{1}{\sqrt{k}}\right).$$

The analogous remainder estimate is proved for  $S^1$  actions with scalar moment map in [ZZ19, Theorem 2] and we generalize the periodization argument from that article.

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<sup>1</sup>What Donaldson calls the CLT in [D08, (9)] is a local limit law of the kind proved in [SoZ07] (see Section 5). The Poisson limit law alluded to in [D08] was proved in [SoZ10, F12].

### 1.4. Local limit theorem

In this section, we tie together some results of [SoZ10, SoZ12] to Theorem 1.2. We show that the former imply a “local limit law” for the dilated measures  $\mu_k^{z,1}(p) := \mu_k^z(p/k)$  for  $p \in \mathbb{R}^m$ .

Classically, a local limit theorem for lattice random variables pertains to a triangular array  $\{X_{N,k}\}_{k=1}^N$  of independent random variables with values in  $\mathbb{Z}^m$ . Let  $S_N = \sum_{k=1}^N X_{N,k}$ . Assuming that  $\frac{S_N - \mathbb{E}S_N}{\sqrt{N}} \rightarrow \gamma_{0,\Sigma}$  in the weak-\* sense, the local limit theorem states that

$$\mathbb{P}_N(\alpha) := \mathbb{P}\{S_N = \alpha\} = N^{-m/2} \gamma_{0,\Sigma}\left(\frac{\alpha - \mathbb{E}S_N}{\sqrt{N}}\right) + o(N^{-m/2}).$$

For instance, in the model case of Bernoulli random variables with  $\mathbb{P}(X_1 = 1) = \frac{1}{2} = \mathbb{P}(X_1 = 0)$ ,  $\mathbb{P}(S_N = k) = \binom{N}{k} 2^{-N}$  and

$$P(S_N = k) \simeq \frac{\sqrt{2}}{\sqrt{\pi N}} e^{-\frac{(k-N/2)^2}{N/2}}.$$

We refer to [GK, Chapter 9] and [Muk91] for discussion of local limit theorems for lattice distributions.

These results do not apply to the measures (2). However, we prove that they satisfy the following local limit theorem:

**THEOREM 1.4.** *for  $\alpha \in \mathbb{Z}^m$ ,*

$$(9) \quad \mu_k^{z,1}(\alpha) = k^{-m/2} \gamma_{0,\text{Hess}\varphi(z)}\left(\sqrt{k}\left(\frac{\alpha}{k} - \mu_h(z)\right)\right) (1 + O(1/k)).$$

All of the necessary calculations and estimates were proved in [SoZ10, SoZ12], but the conclusion was not drawn there.

In the classical case of independent lattice variables, such as the de Moivre-Laplace theorem, the CLT can be derived from the local limit law by integrating (i.e. summing) the latter. The localization formulae in [SoZ10] could probably be used to prove Theorem 1.2 from Theorem 1.4 in this way. But the proof we give of Theorem 1.2 seems simpler as well as giving a sharper remainder estimate.

### 1.5. Related results

Theorem 1.2 has some resemblance in both its statement and proof to the CLT proved in [ZZ19b] for Hamiltonian flows and in [ZZ19] for  $S^1$  actions. See also [PS, RS] for prior articles with related results. But these articles involve sequences of probability measures on  $\mathbb{R}$ , while the CLT in this article

is about the sequence  $\mu_k^z$  of probability measures on  $P \subset \mathbb{R}^m$ . Moreover, those articles gave Erf asymptotics for scaled partial Bergman kernels around the interface  $\partial\mathcal{A}$  in  $M$  between an allowed region  $\mathcal{A}$  and its complement. This article gives a vector-valued refinement of the CLT of [ZZ19] in which  $\mu_h^{-1}(z)$  is a single torus instead of a hypersurface  $\partial\mathcal{A}$ , and Gaussian asymptotics hold in all normal directions to the torus.

In Theorem 1.2, we assume that  $\mu_h(z) \in P^\circ$ , the interior of  $P$ , and show that the limit is uniform on compact subsets of  $M^\circ$ . If we allow varying points  $z_k \rightarrow \partial P$ , then as in the model binomial case, the measures  $\mu_k^{z_k}$  tend to some kind of Poisson limit law. Results of this kind are proved in [SoZ10, F12] in the toric setting. It would be interesting to investigate such Poisson limit laws on general Kähler manifolds and for general Hamiltonian, where  $\mathcal{D}$  is replaced by the set of critical points of  $H$ . Critical levels were excluded in [ZZ19, ZZ19b].

An intriguing question is whether Theorem 1.2 admits a generalization to non-toric Kähler manifolds. One possibility is to try to adapt it to the other Kähler manifolds of large symmetry discussed in [D08]. Another is to try to define analogues of  $\mu_k^z$  on Okounkov bodies of polarized Kähler manifolds. In the latter case, even the law of large numbers does not seem to have been formulated.

## 2. Background on toric varieties

We employ the same notation and terminology as in [Z09, SoZ10, SoZ12]. We recall that a toric Kähler manifold is a Kähler manifold  $(M, J, \omega)$  on which the complex torus  $(\mathbb{C}^*)^m$  acts holomorphically with an open orbit  $M^\circ$ . We choose a basepoint  $z_0$  on the orbit open and identify  $M^\circ \equiv (\mathbb{C}^*)^m \{z_0\}$ . The underlying real torus is denoted  $\mathbf{T}^m$  so that  $(\mathbb{C}^*)^m = \mathbf{T}^m \times \mathbb{R}_+^m$ , which we write in coordinates as  $z = e^{\rho/2 + i\theta}$  in a multi-index notation.

We assume that  $M$  is a smooth projective toric Kähler manifold, hence that  $P$  is a Delzant polytope, i.e. that  $P$  is defined by a set of linear inequalities

$$l_r(x) := \langle x, v_r \rangle - \alpha_r \geq 0, \quad r = 1, \dots, d,$$

where  $v_r$  is a primitive element of the lattice and inward-pointing normal to the  $r$ -th  $(n-1)$ -dimensional face of  $P$ . We denote by  $P^\circ$  the interior of  $P$  and by  $\partial P$  its boundary;  $P = P^\circ \cup \partial P$ .

### 2.1. Monomial basis of $H^0(M, L^k)$ , norms and Szegő kernels

A natural basis of the space of holomorphic sections  $H^0(M, L^k)$  associated to the  $k$ th power of  $L \rightarrow M$  is defined by the monomials  $z^\alpha$  where  $\alpha$  is a lattice

point in the  $k$ th dilate of the polytope,  $\alpha \in kP \cap \mathbb{Z}^m$ . That is, there exists an invariant frame  $e_L$  over the open orbit so that  $s_\alpha(z) = z^\alpha e_L$ . We equip  $L$  with a toric Hermitian metric  $h$  whose curvature  $(1, 1)$ -form  $\omega = i\partial\bar{\partial} \log \|e\|_h^2$  is positive. We often express the norm in terms of a local Kähler potential,  $\|e\|_h^2 = e^{-\varphi}$ , so that  $|s_\alpha(z)|_{h^k}^2 = |z^\alpha|^2 e^{-k\varphi(z)}$  for  $s_\alpha \in H^0(M, L^k)$ .

Any hermitian metric  $h$  on  $L$  induces inner products  $\text{Hilb}_k(h)$  on  $H^0(M, L^k)$ , defined by

$$(10) \quad \langle s_1, s_2 \rangle_{\text{Hilb}_k(h)} = \int_M (s_1(z), s_2(z))_{h^k} \frac{\omega_h^m}{m!}.$$

The monomials are orthogonal with respect to any such toric inner product and have the norm-squares

$$(11) \quad Q_{h^k}(\alpha) = \int_{\mathbb{C}^m} |z^\alpha|^2 e^{-k\varphi(z)} dV_\varphi(z),$$

where  $dV_\varphi = (i\partial\bar{\partial}\varphi)^m/m!$ . We denote the dimension of  $H^0(M, L^k)$  by  $N_k$ .

### 2.2. Kähler potential, moment map and symplectic potential

Recall that we use log coordinate  $(\rho, \theta)$  on  $M^\circ \cong (\mathbb{C}^*)^m$  by setting  $z_i = e^{\rho_i/2 + \sqrt{-1}\theta_i}$ . Since the Kähler potential  $\varphi$  is  $\mathbf{T}^m$ -invariant,  $\varphi(z)$  only depends on the  $\rho$  variables, hence we may write it as  $\varphi(\rho)$ .

The moment map  $\mu_h$  is defined as the gradient of the Kähler potential  $\varphi : \mathbb{R}^m \rightarrow \mathbb{R}$ . Let  $\mathbb{R}_\rho^m$  be the dual space of  $\mathbb{R}_\rho^m$ , where we use coordinates  $p = (p_1, \dots, p_m)$  and  $\rho = (\rho_1, \dots, \rho_m)$  respectively. The gradient map induced by  $\varphi$  is defined by

$$\Phi_\varphi : \mathbb{R}_\rho^m \rightarrow \mathbb{R}_p^m, \quad \rho \mapsto p(\rho) := (\partial_{\rho_1}\varphi, \dots, \partial_{\rho_m}\varphi).$$

The moment map is then defined by,

$$(12) \quad \mu_h(z) = \Phi_\varphi(\rho).$$

The moment map  $\mu_h : M \rightarrow \mathbb{R}^m$  is only well-defined up to an additive constant. The equivariant toric line bundle  $L$  fixes this degree of freedom as follows: Let  $I_k \subset \mathbb{Z}^m$  be the subset consisting of weight  $H^0(M, L^k)$  under the action of  $(\mathbb{C}^*)^m$ , and let  $P_k$  be the convex hull of  $I_k$ . Then  $P_k = kP'$  for a fixed convex polytope  $P'$ . We normalize  $\mu_h$  by requiring that  $\mu_h(M) = P'$ . For background, see [Fu].

### 2.3. The Szegő kernel and the Bergman kernel

The Szegő (or Bergman) kernels of a positive Hermitian line bundle  $(L, h) \rightarrow (M, \omega)$  are the kernels of the orthogonal projections  $\Pi_{h^k} : L^2(M, L^k) \rightarrow H^0(M, L^k)$  onto the spaces of holomorphic sections with respect to the inner product  $\text{Hilb}_k(h)$ ,

$$(13) \quad \Pi_{h^k} s(z) = \int_M \Pi_{h^k}(z, w) \cdot s(w) \frac{\omega_h^m}{m!},$$

where the  $\cdot$  denotes the  $h$ -hermitian inner product at  $w$ . In terms of a local frame  $e$  for  $L \rightarrow M$  over an open set  $U \subset M$ , we may write sections as  $s = fe$ . If  $\{s_j^k = f_j e_L^{\otimes k} : j = 1, \dots, N_k\}$  is an orthonormal basis for  $H^0(M, L^k)$ , then the Szegő kernel can be written in the form

$$(14) \quad \Pi_{h^k}(z, w) := F_{h^k}(z, w) e_L^{\otimes k}(z) \otimes \overline{e_L^{\otimes k}(w)},$$

where

$$(15) \quad F_{h^k}(z, w) = \sum_{j=1}^{N_k} f_j(z) \overline{f_j(w)}, \quad N_k = \dim H^0(M, L^k).$$

We also introduce the local kernel  $B_{h^k}(z, w)$ , defined with respect to the unitary frame:

$$(16) \quad \Pi_{h^k}(z, w) = B_{h^k}(z, w) \frac{e_L^k(z)}{\|e_L^k(z)\|_h} \otimes \overline{\frac{e_L^k(w)}{\|e_L^k(w)\|_h}}$$

The *density of states*  $\Pi_{h^k}(z)$  is the contraction of  $\Pi_{h^k}(z, w)$  with the hermitian metric on the diagonal,

$$\Pi_{h^k}(z) := \sum_{i=0}^{N_k} \|s_i^k(z)\|_{h_k}^2 = F_{h^k}(z, z) |e(z)|_h^{2k} = B_{h^k}(z, z),$$

where in the first equality we record a standard abuse of notation in which the diagonal of the Szegő kernel is identified with its contraction. On the diagonal, we have the following asymptotic expansion the density of states,

$$(17) \quad \Pi_{h^k}(z) = k^m + \frac{1}{2} S(z) k^{m-1} + a_2(z) k^{m-2} + \dots$$

where  $S(z)$  is the scalar curvature of  $\omega$ .



### 2.4. Bergman kernels for a toric variety

In the case of a toric variety, we have

$$(18) \quad F_{h^k}(z, w) = \sum_{\alpha \in kP \cap \mathbb{Z}^m} \frac{z^\alpha \bar{w}^\alpha}{Q_{h^k}(\alpha)},$$

where  $Q_{h^k}(\alpha)$  is defined in (11). If we sift out the  $\alpha$ th term of  $\Pi_{h^k}$  by means of Fourier analysis on  $\mathbf{T}^m$ , we obtain

$$(19) \quad \mathcal{P}_{h^k}(\alpha, z) := \frac{|z^\alpha|^2 e^{-k\varphi(z)}}{Q_{h^k}(\alpha)}.$$

Let  $\tilde{\varphi}(z, w)$  denote an almost analytic extension of  $\varphi(z)$  from the diagonal, that is  $\tilde{\varphi}$  satisfying the condition  $\bar{\partial}_z^k \tilde{\varphi}(z, w)|_{z=w} = \partial_w^k \tilde{\varphi}(z, w)|_{z=w} = 0$  for all  $k \in \mathbb{N}$  and  $\tilde{\varphi}(z, w)|_{z=w} = \varphi(z)$ . The  $\mathbf{T}^m$  action is by holomorphic isometries of  $(M, \omega)$  and therefore, by averaging over the compact group  $T^m$ , we may assume

$$(20) \quad \tilde{\varphi}(\Phi^{\vec{t}}z, \Phi^{\vec{t}}w) = \tilde{\varphi}(z, w).$$

The Szegő kernel (16) admits a parametrix with complex phase  $\tilde{\varphi}$  (see e.g. [BBSj]). In the case of a toric Kähler manifold, it takes the following simple form [STZ03].

**PROPOSITION 2.1.** *For any hermitian toric positive line bundle over a toric variety, the Szegő kernel for the metrics  $h_\varphi^N$  have the asymptotic expansions in a local frame on  $M$ ,*

$$B_{h^k}(z, w) \sim e^{k(\tilde{\varphi}(z,w) - \frac{1}{2}(\varphi(z) + \varphi(w)))} A_k(z, w) \text{ mod } k^{-\infty},$$

where  $A_k(z, w) \sim k^m \left(1 + \frac{a_1(z,w)}{k} + \dots\right)$  is a semi-classical symbol of order  $m$  and where the phase satisfies (20).

### 2.5. Proof of Lemma 1.1

As mentioned above, Lemma 1.1 was proved in [Z09, SoZ10]. We briefly review the proof as preparation for the proof of Theorem 1.2.

**PROPOSITION 2.2.** *Let  $(M, L, h, \omega)$  be a polarized toric Hermitian line bundle. Then the means, resp. variances of  $\mu_k^z$  are given respectively by,*

1.  $m_k(z) = \mu_h(z) + O(k^{-1});$
2.  $\Sigma_k(z) = k^{-1} \text{Hess } \varphi + O(k^{-2}).$

*Proof.* We briefly review the proof. Recall that the Bergman density function  $\Pi_{h^k}(z)$  is  $\mathbf{T}^m$ -invariant, hence is a function of  $\rho$ , and can be written as

$$\Pi_{h^k}(\rho) = \sum_{\alpha \in kP \cap \mathbb{Z}^m} \Pi_{h^k, \alpha}(\rho) = \sum_{\alpha \in kP \cap \mathbb{Z}^m} \frac{e^{\langle \alpha, \rho \rangle - k\varphi(\rho)}}{Q_k(\alpha)},$$

Thus by explicit calculation we have

$$\begin{aligned} k^{-1} \partial_{\rho_j} \Pi_{h^k}(\rho) &= \sum_{\alpha \in kP \cap \mathbb{Z}^m} \left( \frac{\alpha_j}{k} - \partial_{\rho_j} \varphi(\rho) \right) \frac{e^{\langle \alpha, \rho \rangle - k\varphi(\rho)}}{Q_k(\alpha)} \\ &= \Pi_{h^k}(\rho) (m_k(z) - \mu_h(z))_j \end{aligned}$$

where we used  $\partial_{\rho_j} \varphi(\rho) = \mu_h(z)_j$ . Using the asymptotic expansion for  $\Pi_{h^k}(\rho)$ , we get  $m_k(z) = \mu_h(z) + O(1/k)$ .

Then for the variance, we use

$$\begin{aligned} k^{-2} \partial_{\rho_i \rho_j}^2 \Pi_{h^k}(\rho) &= \sum_{\alpha \in kP \cap \mathbb{Z}^m} \left( \frac{\alpha_i}{k} - \partial_{\rho_i} \varphi(\rho) \right) \left( \frac{\alpha_j}{k} - \partial_{\rho_j} \varphi(\rho) \right) \frac{e^{\langle \alpha, \rho \rangle - \mu_h(z)}}{Q_k(\alpha)} \\ &\quad - \sum_{\alpha \in kP \cap \mathbb{Z}^m} \left( \frac{1}{k} \partial_{\rho_i \rho_j}^2 \varphi(\rho) \right) \frac{e^{\langle \alpha, \rho \rangle - k\varphi(\rho)}}{Q_k(\alpha)} \end{aligned}$$

Then divide by  $\Pi_k(\rho)$  and use  $m_k(z) = \mu_h(z) + O(1/k)$ , we get the desired result for variance. □

### 3. Quantum dynamics: Proof of Theorem 1.2

The proof is somewhat similar to that of [ZZ19, Theorem 4] but in fact simpler because of the extra degrees of symmetry of a toric variety. A key simplifying feature is that, like the  $S^1$  action of [ZZ19],  $\mathbf{T}^m$  acts holomorphically on  $M$ . As above, denote the action by

$$(21) \quad (e^{i\vec{t}}, z) \in \mathbf{T}^m \times M \mapsto e^{i\vec{t}} \cdot z =: \Phi^{\vec{t}}(z)$$

and denote the infinitesimal generators of the action by  $\frac{\partial}{\partial \theta_j}$ . As discussed in [STZ03, SoZ10, SoZ12], the torus action can be quantized as a sequence of unitary operators  $U_k(\vec{t})$  on  $H^0(M, L^k)$ , or more precisely as a semi-classical Toeplitz Fourier integral operator. We briefly review the key ideas and refer to the articles above for details and further background.

Let  $X_h = \partial D_h^*$  where  $D_h^*$  is the unit co-disc bundle in  $L^*$  with respect to  $h$  and let  $\mathcal{H}^2(X_h) \subset \mathcal{L}^2(X_h)$  denote the Hardy space of  $L^2$  Cauchy-Riemann

functions on  $X_h$ . It is an  $S^1$  bundle  $\pi : X_h \rightarrow M$  and carries the  $S^1$  action  $r_\theta : S^1 \times X_h \rightarrow X_h$  by rotation of the fibers. The  $S^1$  action on  $X_h$  commutes with  $\bar{\partial}_b$ ; hence  $\mathcal{H}^2(X_h) = \bigoplus_{k=0}^\infty \mathcal{H}_k^2(X_h)$  where  $\mathcal{H}_k^2(X_h) = \{F \in \mathcal{H}^2(X_h) : F(r_\theta x) = e^{ik\theta} F(x)\}$ . Section  $s_k$  of  $L^k$  lift to equivariant functions  $\hat{s}_k$  on  $L^*$  by the rule

$$\hat{s}_k(\lambda) = \left( \lambda^{\otimes k}, s_k(z) \right), \quad \lambda \in L_z^*, \quad z \in M,$$

where  $\lambda^{\otimes k} = \lambda \otimes \dots \otimes \lambda$ . We henceforth restrict  $\hat{s}$  to  $X_h$  and then the equivariance property takes the form  $\hat{s}_k(r_\theta x) = e^{ik\theta} \hat{s}_k(x)$ . The map  $s \mapsto \hat{s}$  is a unitary equivalence between  $H^0(M, L^k)$  and  $\mathcal{H}_k^2(X_h)$ .

There is a natural contact 1-form  $\alpha$  on  $X_h$  defined by the Hermitian connection 1-form, which satisfies  $d\alpha = \pi^*\omega$ . The  $\mathbf{T}^m$  action lifts to  $X_h$  as an action of the torus by contact transformations. The generators  $\frac{\partial}{\partial \theta_j}$  of the  $\mathbf{T}^m$  action on  $M$  lift to contact vector fields  $\Xi_1, \dots, \Xi_m$  on  $X_h$ . The horizontal lifts of the Hamilton vector fields  $\xi_j$  are then defined by

$$\pi_* \xi_j^h = \xi_j, \quad \alpha(\xi_j^h) = 0,$$

and the contact vector fields  $\Xi_j$  are given by:

$$(22) \quad \Xi_j = \xi_j^h + 2\pi i \langle \mu \circ \pi, \xi_j^* \rangle \frac{\partial}{\partial \theta} = \xi_j^h + 2\pi i (\mu \circ \pi)_j \frac{\partial}{\partial \theta},$$

where  $\mu$  is the moment map. These vector fields act as differential operators  $\hat{I}_j : \mathcal{H}_N^2(X_h) \rightarrow \mathcal{H}_N^2(X_h)$  satisfying

$$(23) \quad (\hat{I}_j \hat{S})(\zeta) = \frac{1}{i} \frac{\partial}{\partial \varphi_j} \hat{S}(e^{i\varphi} \cdot \zeta)|_{\varphi=0}, \quad \hat{S} \in \mathcal{C}^\infty(X_h).$$

Furthermore, the generator of the  $S^1$  action acts on these spaces and

$$(24) \quad \hat{I}_{m+1} : \mathcal{H}_k^2(X_h) \rightarrow \mathcal{H}_k^2(X_h), \quad \frac{1}{i} \frac{\partial}{\partial \theta} \hat{s}_k = k \hat{s}_k \quad \text{for } \hat{s}_N \in \mathcal{H}_k^2(X_h).$$

The monomial sections  $s_\alpha$  (equal to  $z^\alpha$  on the open orbit) lift to  $\mathbf{T}^m \times S^1$  equivariant functions  $\hat{s}_\alpha$  on  $X_h$ , i.e. as joint eigenfunctions of the  $(m + 1)$  commuting operators  $\hat{I}_j$ .

The lifts  $\hat{\Pi}_k(x, y)$  of the Szegő kernels (13) are the (Schwarz) kernel of the orthogonal projection  $\hat{\Pi}_k : \mathcal{L}^2(X_h) \rightarrow \mathcal{H}_k^2(X_h)$ . They are Fourier components,

$$(25) \quad \hat{\Pi}_{h^k}(x, y) = \int_0^{2\pi} e^{-ik\theta} \hat{\Pi}(r_\theta x, y) \frac{d\theta}{2\pi},$$

of the full Szegő projector  $\hat{\Pi}(x, y)$ .

The quantum torus action is defined by

$$U_{h^k}(\vec{t}) := e^{\sum_{j=1}^m t_j \hat{I}_j} = \prod_{j=1}^m e^{it_j \hat{I}_j}$$

on  $\mathcal{H}_k^2(X_h)$ . Since the torus acts holomorphically, it is simply given by

$$(26) \quad \hat{U}_k(\vec{t}, x, y) = \hat{\Pi}_{h^k} \Phi^{\vec{t}} \hat{\Pi}_{h^k}(x, y) = \hat{\Pi}_{h^k}(x, \Phi^{\vec{t}} y).$$

We are most interested in the diagonal  $\hat{U}_k(\vec{t}, x, x)$ . It is  $S^1$ -invariant and depends only on  $z = \pi(x)$ , so we denote it by

$$(27) \quad U_k(\vec{t}, z, z) := B_{h^k}(z, \Phi^{\vec{t}} z) = \sum_{\alpha \in kP \cap \mathbb{Z}^m} \frac{|s_\alpha(z)|_{h^k}^2}{\|s_\alpha\|_{h^k}^2} e^{-i\langle \vec{t}, \alpha \rangle}.$$

Here and henceforth we use the identification of the base and lifted Szegő kernels and torus actions. Literally speaking, the translation of sections on the base requires parallel translation; but on the open orbit we may think of the sections as scalar functions.

### 3.1. Proof of Theorem 1.2

We prove Theorem 1.2 by the classical Fourier method, which is based on the ‘continuity theorem’ that weak convergence  $D_{\sqrt{k}} \mu_k^z \rightarrow \gamma_{0, \text{Hess } \varphi(z)}$  is equivalent to pointwise convergence of the Fourier transforms (‘characteristic functions in probability language’) as long as the pointwise limit is continuous at 0 (see e.g. [Res, Theorem 9.5.2]).

It is obvious that

$$(28) \quad \mathcal{F}_{x \rightarrow t}^{-1} D_{\sqrt{k}} \tilde{\mu}_k^z(\vec{t}) = B_{h^k}^{-1}(z) U_k\left(\frac{\vec{t}}{\sqrt{k}}, z, z\right) e^{i\sqrt{k}\langle \mu(z), t \rangle},$$

or equivalently,

$$(29) \quad \langle f, D_{\sqrt{k}} \tilde{\mu}_k^z \rangle = B_{h^k}^{-1}(z) \int_{\mathbb{R}^m} \hat{f}(\vec{t}) U_k\left(\frac{\vec{t}}{\sqrt{k}}, z, z\right) e^{i\sqrt{k}\langle \mu(z), t \rangle} dt$$

Thus, the key point is to study the pointwise scaling asymptotics of (27).

Let  $H_z = \text{Hess} \varphi(z) = \frac{\partial^2}{\partial \rho_i \partial \rho_j} \varphi|_z$  denote the Hessian of  $\varphi$ .

**PROPOSITION 3.1.**  $B_{h^k}^{-1}(z) U_k\left(\frac{\vec{t}}{\sqrt{k}}, z, z\right) e^{i\sqrt{k}\langle \mu(z), t \rangle} \rightarrow \mathcal{F}^{-1} \gamma_{0, H_z}$  pointwise.

*Proof.* We need to show that, for each  $z \in M^o$ ,

$$B_k^{-1}(z)U_k\left(\frac{\vec{t}}{\sqrt{k}}, z, z\right)e^{i\sqrt{k}\langle\vec{t}, \mu_h(z)\rangle} = B_{h^k}^{-1}(z)B_k\left(\Phi\frac{\vec{t}}{\sqrt{k}}z\right)e^{i\sqrt{k}\langle\vec{t}, \mu_h(z)\rangle} \rightarrow \mathcal{F}^{-1}\gamma_{0, H_z}(\vec{t}).$$

Substituting the Boutet-de-Monvel-Sjostrand parametrix of Proposition 2.1 gives,

$$(30) \quad U_k\left(\frac{\vec{t}}{\sqrt{k}}, z, z\right)e^{i\sqrt{k}\langle\vec{t}, \mu_h(z)\rangle} \sim e^{k(\tilde{\varphi}(z, \Phi\frac{\vec{t}}{\sqrt{k}}z) - \varphi(z))}e^{i\sqrt{k}\langle\vec{t}, \mu_h(z)\rangle}A_k\left(z, \Phi\frac{\vec{t}}{\sqrt{k}}z\right),$$

where  $\sim$  means that the difference is a function which decays rapidly in  $k$  along with its derivatives. Such a remainder may be neglected if we only consider expansions modulo rapidly decaying functions of  $k$ . Using the parametrix, this comes down to the statement that

$$(31) \quad e^{k(\tilde{\varphi}(z, \Phi\frac{\vec{t}}{\sqrt{k}}z) - \varphi(z))}e^{-i\sqrt{k}\langle\vec{t}, \mu_h(z)\rangle}A_k\left(z, \left(\Phi\frac{\vec{t}}{\sqrt{k}}z\right)\right) \rightarrow \mathcal{F}\gamma_z(\vec{t})$$

Use the  $\mathbf{T}^m$ -invariance of  $\varphi$ , we get (20), i.e.

$$\tilde{\varphi}(\Phi^c z, \Phi^c w) = \tilde{\varphi}(z, w),$$

which implies

$$\tilde{\varphi}\left(z, \Phi\frac{\vec{t}}{\sqrt{k}}z\right) = \tilde{\varphi}\left(\Phi\frac{-\vec{t}}{2\sqrt{k}}z, \Phi\frac{\vec{t}}{2\sqrt{k}}z\right).$$

The  $\mathbf{T}^m$  can be extended to a  $(\mathbb{C}^*)^m$  action, For  $\tau \in \mathbb{C}^m$ , let  $e^\tau \in (\mathbb{C}^*)^m$  acts on  $z \in M^o$  by multiplication. Then we may define the following function

$$\Psi(\tau) := \tilde{\varphi}(e^\tau z, \overline{e^\tau} z),$$

so that  $\tilde{\varphi}(z, \Phi\frac{\vec{t}}{\sqrt{k}}z) = \Psi(-it/2\sqrt{k})$ . Since  $\tilde{\varphi}(z, w)$  is almost holomorphic in  $z$  and anti-holomorphic in  $w$  when  $z = w$ ,  $\Psi(\tau)$  is almost holomorphic in  $\tau$  at  $\tau = 0$ . If  $\tau \in i\mathbb{R}^m$ , this corresponds to action of  $\mathbf{T}^m$ . We restrict to  $\tau = c \in \mathbb{R}^m$ , and then  $\overline{e^c}z = e^c z$ . Recall that  $z = e^{\rho/2+i\theta}$  and  $\varphi$  is a function of  $\rho$  only,

$$\Psi(c) = \varphi(e^c z) = \varphi(\rho) + 2c\partial_\rho\varphi(\rho) + \frac{1}{2}\langle\text{Hess}_\rho\varphi(\rho)2c, 2c\rangle + O(c^3).$$

Finally, using  $\Psi$  is analytic at  $c = \vec{0}$ , we have

$$\Psi(-it/2\sqrt{k}) = \varphi(\rho) - (it/\sqrt{k})\partial_\rho\varphi(\rho) - \frac{1}{2}k^{-1}\langle H_z\vec{t}, \vec{t}\rangle + O(k^{-3/2})$$

Also note that  $\mu_h(z) = \partial_\rho(\varphi(\rho))$  from (12), we then have

$$k\tilde{\varphi}(z, \Phi^{\frac{\vec{t}}{\sqrt{k}}}z) - k\varphi(z) + i\sqrt{k}\langle \vec{t}, \mu_h(z) \rangle = -\frac{1}{2}\langle H_z \vec{t}, \vec{t} \rangle + O(k^{-1/2})$$

The amplitude  $\tilde{A}_k$  has an expansion of the form,

$$\tilde{A}_k(z, e^{i\frac{\vec{t}}{k}z}, 0, N) = k^m + k^{m-1}a_1 + O(k^{m-1}),$$

for various smooth coefficients  $a_j(z)$ ; the first one is a universal constant. We conclude that

$$(32) \quad k^{-m}U_k\left(\frac{\vec{t}}{\sqrt{k}}, x, x\right)e^{i\sqrt{k}\langle \vec{t}, \mu_h(z) \rangle} \rightarrow e^{-\frac{1}{2}\langle H_z \vec{t}, \vec{t} \rangle}.$$

□

The proof of Proposition 3.1 actually shows that there is a pointwise expansion asymptotic expansion to all orders, with remainders of polynomial growth in  $\vec{t}$ . For this it suffices to carry out the Taylor expansions of the phase and amplitudes to higher order. We can then integrate the result against suitable test functions to obtain the following result, analogous to [ZZ19, Proposition 8.1]:

**PROPOSITION 3.2.** *Let  $z \in M_0$ . For  $f \in \mathcal{S}(\mathbb{R}^m)$  with  $\hat{f} \in C_0^\infty(\mathbb{R}^m)$ ,*

$$\int_{\mathbb{R}^m} \hat{f}(\vec{t})B_{h^k}^{-1}(z)U_k\left(\frac{\vec{t}}{\sqrt{k}}, z, z\right)edt = \int_{\mathbb{R}^m} \hat{f}(\vec{t})e^{-\frac{1}{2}\langle H_z \vec{t}, \vec{t} \rangle}dt + O(k^{-\frac{1}{2}}),$$

where  $H_z = \text{Hess } \varphi(\rho)$  is the Hessian of the toric Kähler potential. In fact, there exists a complete asymptotic expansion of the integral in powers of  $k^{-\frac{1}{2}}$ , and the asymptotics are uniform on compact subsets of  $M^o$ .

*Proof.* For  $f \in \mathcal{S}(\mathbb{R}^m)$  with  $\hat{f} \in C_0^\infty(\mathbb{R}^m)$ ,

$$\begin{aligned} \langle f, D_{\sqrt{k}}\tilde{\mu}_k^z \rangle &= \frac{1}{\Pi_{h^k}(z, z)} \sum_{\alpha \in kP \cap \mathbb{Z}^m} \frac{|s_\alpha(z)|_{h^k}^2}{\|s_\alpha\|_{h^k}^2} f\left(\sqrt{k}\left(\frac{\alpha}{k} - \mu_h(z)\right)\right) \\ (33) \quad &= B_{h^k}^{-1}(z) \int_{\mathbb{R}^m} \hat{f}(\vec{t})U_k\left(\frac{\vec{t}}{\sqrt{k}}, z, z\right)e^{i\sqrt{k}\langle \vec{t}, \mu_h(z) \rangle}dt \\ &= B_{h^k}^{-1}(z) \int_{\mathbb{R}^m} \hat{f}(\vec{t})B_{h^k}(z, \Phi^{\frac{\vec{t}}{\sqrt{k}}}z)e^{i\sqrt{k}\langle \vec{t}, \mu_h(z) \rangle}dt \end{aligned}$$

Since the integrand is compactly supported, we may apply the pointwise limit of Proposition 3.1 to obtain the principal term. By Taylor expanding

the factor  $e^{k^{-\frac{1}{2}}R_3(k,z)}$  one obtains an oscillatory integral with the same phase and a remainder of order  $k^{-\frac{1}{2}}$ .  $\square$

Further details will be given in the proof of Proposition 4.1.

### 3.2. Completion of the proof of Theorem 1.2

Since  $\mathcal{S}(\mathbb{R}^m)$  is dense in  $C_0(\mathbb{R}^m)$  (continuous functions vanishing at infinity, equipped with the sup norm), Proposition 3.2 implies weak\* convergence of  $D_{\sqrt{k}}\tilde{\mu}_k^z \rightarrow \gamma_{0,\text{Hess } \varphi(z)}$  on  $C_0(\mathbb{R}^m)$  (continuous functions vanishing at infinity). For weak\* convergence on  $C_b(\mathbb{R}^m)$  one needs tightness of the sequence  $\mu_k^z$ . The so-called Levy continuity theorem on  $\mathbb{R}^m$  says that if  $\mu_k$  is a sequence of probability measures on  $\mathbb{R}^m$  and  $\hat{\mu}_k(t) \rightarrow \varphi(t)$  pointwise and  $\varphi(t)$  is continuous at 0, then  $\varphi(t) = \hat{\mu}(t)$  for some probability measure  $\mu$  and  $\mu_k \xrightarrow{w*} \mu$  on  $C_b$ . The continuity of  $\varphi$  at  $t = 0$  implies tightness of the sequence  $\mu_k$ . We refer to [Res, Theorem 9.5.2] for background. Hence,  $D_{\sqrt{k}}\mu_k^z \xrightarrow{w*} \gamma_{0,\text{Hess } \varphi(z)}$  in the sense of weak\* convergence on  $C_b(\mathbb{R}^n)$  as long as  $\mathcal{F}_{x \rightarrow t} D_{\sqrt{k}}\mu_k^z \rightarrow \mathcal{F}\gamma_{0,\text{Hess } \varphi(z)}$  pointwise. Proposition 3.1 thus implies that the sequence is tight, and further implies weak\* convergence on  $C_b(\mathbb{R})$ . Hence Theorem 1.2 is proved.

It follows from Theorem 1.2 and the Portmanteau theorem that

$$\lim_{k \rightarrow \infty} D_{\sqrt{k}}\tilde{\mu}_k^z(K) = \gamma_{0,\text{Hess } \varphi(z)}(K),$$

for any convex subset of  $\mathbb{R}^m$ , or more generally for any ‘continuity set’ such that  $\gamma_{0,\text{Hess } \varphi(z)}(\partial K) = 0$ .

### 4. Berry-Esseen remainder estimate

The purpose of this section is to improve the limit formula of Theorem 1.2 and the expansion in Proposition 3.2 by giving the remainder estimate of Theorem 1.3. We aim to give a representative result rather than the most general one possible, and therefore restrict to a reasonably general class of continuous functions rather than indicator functions.

For any bounded continuous function  $f \in C_b(\mathbb{R})$ , we define a slight simplification of (33),

$$(34) \quad I_{k,f}(z) := k^{-m} \sum_{\alpha \in kP \cap \mathbb{Z}^m} \frac{|s_\alpha(z)|_{h^k}^2}{\|s_\alpha\|_{h^k}^2} f(\sqrt{k}(\frac{\alpha}{k} - \mu_h(z))).$$

Since  $\Pi_{h^k}(z, z) = k^m(1 + O(k^{-1/2}))$ , we have

$$\langle f, D_{\sqrt{k}}\tilde{\mu}_k^z \rangle = I_{k,f}(z)(1 + O(k^{-\frac{1}{2}})),$$

and it suffices to prove the desired bound for (34).

We now prove the Berry-Esseen remainder bound for integrals of  $\mu_k^z$  against certain types of  $f \in C_0(\mathbb{R}^m)$  (functions vanishing as  $|x| \rightarrow \infty$ ).

**PROPOSITION 4.1.** *Let  $f \in C_0(\mathbb{R}^m)$  have the properties that  $\hat{f} \in L^1$  and that  $|\hat{f}(\vec{t})| \leq Cg(|\vec{t}|)$  where  $g(|\vec{t}|) \in L^1$  and is monotonically decreasing as a function of  $|\vec{t}|$ . Then,*

$$I_{k,f}(z) = \int_{\mathbb{R}^m} f(x) d\gamma_{0, \text{Hess } \varphi(z)}(x) + O_f(k^{-1/2}).$$

*Proof.* We start again from the last formula of (33). We note that  $\vec{t} \rightarrow \hat{\Pi}_{h_k}(\Phi_{\sqrt{k}}^{\vec{t}} x, r_\theta x)$  is periodic with respect to the lattice  $2\pi\sqrt{k}\mathbb{Z}^m$  (similarly for the parametrix and remainder terms), so the integrals converge when  $\hat{f} \in \mathcal{S}(\mathbb{R})$ . We periodize  $g(\vec{t}) = \hat{f}(\vec{t})e^{-i\sqrt{k}\langle \vec{t}, \mu_h(z) \rangle}$  with respect to the lattice  $2\pi\sqrt{k}\mathbb{Z}^m$  by means of the  $\sqrt{k}$ -periodization operator

$$\mathcal{P}_{\sqrt{k}}g(\vec{t}) := \sum_{\ell \in \mathbb{Z}^m} g(\vec{t} + 2\pi\sqrt{k}\ell), \quad g \in \mathcal{S}(\mathbb{R}^m).$$

The sum converges as long as  $|g(\vec{t})| \in L^1(\mathbb{R}^m)$  is bounded by a decreasing positive  $L^1$  function. Hence, as long as  $\hat{f}$  has this property,

$$\begin{aligned} \mathcal{P}_{\sqrt{k}}(\hat{f}e^{-i\sqrt{k}\langle \vec{t}, \mu_h(z) \rangle}) &= \sum_{\ell \in \mathbb{Z}^m} \hat{f}(\vec{t} + 2\pi\sqrt{k}\ell)e^{-i\sqrt{k}\langle \vec{t}, \mu_h(z) \rangle} e^{-2\pi i k \langle \ell, \mu_h(z) \rangle} \\ &=: e^{-i\sqrt{k}\langle \vec{t}, \mu_h(z) \rangle} \hat{F}_k(\vec{t}), \end{aligned}$$

with  $\hat{F}_k(\vec{t}) = \sum_{\ell \in \mathbb{Z}^m} \hat{f}(\vec{t} + 2\pi\sqrt{k}\ell)e^{-2\pi i k \langle \ell, \mu_h(z) \rangle}$ . Then,

$$I_{k,f}(z) = k^{-m} \int_{\sqrt{k}[-\pi, \pi]^m} \hat{F}_k(\vec{t}) e^{-i\sqrt{k}\langle \vec{t}, \mu_h(z) \rangle} B_{h^k}(\Phi_{\sqrt{k}}^{\vec{t}} z, z) d\vec{t}.$$

We then localize the last integral using a smooth cutoff  $\chi(\frac{\vec{t}}{(\log k)^2})$ , where  $\chi \in C_0^\infty(\mathbb{R}^m)$  is supported in  $(-1, 1)^m$  and equals to 1 in  $(-1/2, 1/2)^m$ . When  $\pi\sqrt{k} \geq |\vec{t}| \geq (\log k)^2$ , the off-diagonal Bergman kernel  $B_{h^k}(\Phi_{\sqrt{k}}^{\vec{t}} z, r_\theta z)$  is rapidly decaying at the rate  $O(e^{-(\log k)^2})$ . Here, we use the standard off-diagonal estimate,  $|B_{h^k}(z, w)| \leq Ck^m e^{-\beta\sqrt{k}d(z,w)}$  for certain  $\beta, C > 0$  (see [ZZ19b] for background). Hence,

$$I_{k,f}(z) = k^{-m} \int_{\mathbb{R}^m} \chi\left(\frac{\vec{t}}{(\log k)^2}\right) \hat{F}_k(\vec{t}) e^{-i\sqrt{k}\langle \vec{t}, \mu_h(z) \rangle} B_{h^k}(\Phi_{\sqrt{k}}^{\vec{t}} z, z) d\vec{t} + O_f(k^{-\infty}),$$



where the constant in  $O_f(k^{-\infty})$  depends on  $\|\hat{F}_k\|_{L^1(-\sqrt{k},\sqrt{k})^m} = \|\hat{f}\|_{L^1}$ .

We then introduce the Boutet-de-Monvel-Sjöstrand parametrix (2.1) to get,

$$I_{k,f}(z) = \int_{-\infty}^{\infty} \chi\left(\frac{t}{(\log k)^2}\right) \hat{F}_k(t) e^{-i\sqrt{k}\langle \vec{t}, \mu_h(z) \rangle} e^{k\tilde{\varphi}(e^{it/\sqrt{k}}z, z) - k\varphi(z)} A_k(e^{it/\sqrt{k}}z, z) dt + \int_{-\infty}^{\infty} \chi\left(\frac{t}{(\log k)^2}\right) \hat{F}_k(t) e^{-i\sqrt{k}\langle \vec{t}, \mu_h(z) \rangle} R_k(e^{it/\sqrt{k}}z, z) dt + O_f(k^{-\infty}).$$

By the parametrix construction,  $R_k \in k^{-\infty}C^\infty(M \times M)^2$  hence the second term is  $O(k^{-\infty})$  and may be absorbed into the remainder estimate.

As in the proof of Proposition 3.1, the phase function of  $I_{k,f}$  has the Taylor expansion (or asymptotic expansion),

$$(35) \quad \begin{aligned} \Psi(it, z) &= -i\sqrt{k}\langle \vec{t}, \mu_h(z) \rangle + k\tilde{\varphi}(e^{it/\sqrt{k}}z, z) - k\varphi(z) \\ &= -\frac{1}{2}\langle H_z \vec{t}, \vec{t} \rangle + g_1(it, z), \end{aligned}$$

where

$$(36) \quad g_1 = O(k^{-1/2}|t|^3).$$

We substitute the Taylor expansion into the phase of the first term of  $I_{k,f}(z)$ , and also Taylor expand  $e^{g_1}$  to order 1. Let  $e_1(x) = 1 - e^x$ . Since  $|\vec{t}| \leq (\log k)^2$  on the support of the integrand,  $|g_1| \leq C(\frac{(\log k)^6}{\sqrt{k}})$  on  $|\vec{t}| \leq (\log k)^2$ . Since  $e^x = 1 + e_1(x)$  where  $e_1(x) \leq 2x$  on  $[0, C(\frac{(\log k)^6}{\sqrt{k}})]$ ,  $e^{g_1} = 1 + \tilde{g}_1$  where  $\tilde{g}_1(k, t) \leq 2g_1 \leq C_0k^{-\frac{1}{2}}(1 + |\vec{t}|^3)$  on  $[0, (\log k)^2]$ .

We get

$$\begin{aligned} I_{k,f}(z) &= \int_{\mathbb{R}^m} \chi\left(\frac{\vec{t}}{(\log k)^2}\right) \hat{F}_k(\vec{t}) e^{-\frac{1}{2}\langle H_z \vec{t}, \vec{t} \rangle} (1 + \tilde{g}_1) dt + O_f(k^{-1/2}) \\ &= \int_{\mathbb{R}^m} \chi\left(\frac{\vec{t}}{(\log k)^2}\right) \hat{F}_k(\vec{t}) e^{-\frac{1}{2}\langle H_z \vec{t}, \vec{t} \rangle} dt + O_f(k^{-1/2}) \end{aligned}$$

where  $\chi(\frac{t}{(\log k)^2})|\tilde{g}_1| \leq C_0k^{-1/2}(1 + |\vec{t}|^3)$  after integration against the Gaussian factor is of size  $O(k^{-1/2})$ .

---

<sup>2</sup>To be clear, for any integer  $m \geq 0$ , we have an  $m$ -term asymptotic expansion  $A_k^{(m)}(x, y) = \sum_{j=0}^m A_{k,j}(x, y)k^{-j}$ , and  $|R_k^{(m)}(x, y)| \leq k^{-m-1}r(x, y)$  for some  $r(x, y) \in C^\infty(M \times M)$ .

Finally, we unravel the periodization  $\hat{F}_k$  to evaluate the first term.

$$\begin{aligned} & \int_{\mathbb{R}^m} \chi\left(\frac{t}{(\log k)^2}\right) \hat{F}_k(t) e^{-\frac{1}{2}\langle H_z \vec{t}, \vec{t} \rangle} dt \\ &= \int_{\mathbb{R}^m} \chi\left(\frac{t}{(\log k)^2}\right) \hat{f}(\vec{t}) e^{-\frac{1}{2}\langle H_z \vec{t}, \vec{t} \rangle} dt \\ &+ \sum_{\ell \in \mathbb{Z}^m \setminus \{0\}} \int_{\mathbb{R}} \chi\left(\frac{\vec{t}}{(\log k)^2}\right) \hat{f}(\vec{t} + 2\pi\sqrt{k}\ell) e^{2\pi i k \langle \ell, \mu_h(z) \rangle - \frac{1}{2}\langle H_z \vec{t}, \vec{t} \rangle} dt \\ &= \int_{\mathbb{R}^m} \chi\left(\frac{\vec{t}}{(\log k)^2}\right) \hat{f}(\vec{t}) e^{-\frac{1}{2}\langle H_z \vec{t}, \vec{t} \rangle} dt + O(k^{-\frac{1}{2}} \|\hat{f}\|). \end{aligned}$$

In the  $\ell \neq 0$  sum, we use that

$$\begin{aligned} \sum_{\ell \in \mathbb{Z}^m \setminus \{0\}} |\hat{f}(\vec{t} + 2\pi\sqrt{k}\ell)| &\leq \sum_{\ell \in \mathbb{Z}^m \setminus \{0\}} g(|\vec{t} + 2\pi\sqrt{k}\ell|) \\ &\leq C \int_{|y| \geq 1} g(|\vec{t} + 2\pi\sqrt{k}y|) dy \\ &= \frac{C}{\sqrt{k}} \int_{|y| \geq \sqrt{k}} g(|\vec{t} + 2\pi y|) dy \\ &= C k^{-\frac{1}{2}} \|g\|_{L^1}, \end{aligned}$$

so that

$$\sum_{\ell \in \mathbb{Z}^m \setminus \{0\}} \int_{\mathbb{R}} \chi\left(\frac{\vec{t}}{(\log k)^2}\right) \hat{f}(\vec{t} + 2\pi\sqrt{k}\ell) e^{2\pi i k \langle \ell, \mu_h(z) \rangle - \frac{1}{2}\langle H_z \vec{t}, \vec{t} \rangle} dt$$

is bounded by

$$[C' k^{-\frac{1}{2}} \|g\|_{L^1}] \int_{\mathbb{R}} \chi\left(\frac{\vec{t}}{(\log k)^2}\right) e^{-\frac{1}{2}\langle H_z \vec{t}, \vec{t} \rangle} dt = O(k^{-\frac{1}{2}} \|g\|_{L^1}).$$

Finally, removing the cut-off  $\chi(t/(\log k)^2)$  from the  $\ell = 0$  term introduces an error of order  $\int_{(\log k)^2}^{\infty} e^{-ax^2} dx = O(k^{-\infty})$ . We have

$$\begin{aligned} I_{k,f}(z) &= \int_{\mathbb{R}^m} \hat{f}(t) e^{-\frac{1}{2}\langle H_z \vec{t}, \vec{t} \rangle} dt + O_f(k^{-1/2}) \\ &= \frac{1}{(2\pi)^{m/2} \sqrt{\det(H_z)}} \int_{\mathbb{R}^m} f(x) e^{-\frac{1}{2}\langle H_z^{-1} x, x \rangle} dx + O_f(k^{-1/2}) \end{aligned}$$

by the Plancherel theorem. This completes the proof of Theorem 1.3. □

**REMARK 1.** *The result can be generalized to indicator functions  $\mathbf{1}_K$  of convex sets  $K$ . It would suffice to smooth  $\mathbf{1}_K$  and to measure the error in the smoothing. The terms contributing to the latter are sums of (19) over lattice points close to  $\partial K$ . The size of the remainder thus depends on the position of  $\mu_h(z)$  relative to  $K$ .*

### 5. Local limit law: Proof of Theorem 1.4

To prove Theorem 1.4 we need to review some further background on toric Kähler manifolds.

Let  $h = e^{-\varphi}$  be a toric Hermitian metric on  $L$ . Recall that the *symplectic potential*  $u_\varphi$  associated to  $\varphi$  is its Legendre transform: for  $x \in P$  there is a unique  $\rho(x)$  such that  $\mu_\varphi(e^{\rho(x)/2}) = d\varphi(\rho(x)) = x$ . If  $z = e^{\rho/2+i\theta}$  then we write  $\rho_z = \rho = \log |z|^2$ . Then the Legendre transform is defined to be the convex function

$$(37) \quad u_\varphi(x) = \langle x, \rho(x) \rangle - \varphi(\rho(x)).$$

Also define

$$(38) \quad I^z(x) = u_\varphi(x) - \langle x, \rho_z \rangle + \varphi(\rho_z).$$

Then  $I^z(x)$  is a convex function on  $P$  with a minimum of value 0 at  $x = \mu_h(z)$  and with Hessian that of  $u_\varphi$ .

The weights  $\mathcal{P}_{h^k}(\alpha, z)$  (19) of the dilate  $\mu_k^{z,1}$  admit pointwise asymptotic expansions.

**LEMMA 5.1.**  $\mathcal{P}_{h^k}(\alpha, z) = k^{m/2}(2\pi)^{-m/2} |\det \text{Hess}(u_\varphi(\mu_h(z)))|^{\frac{1}{2}} e^{-kI^z(\frac{\alpha}{k})} (1 + O(1/k))$ , where  $O(1/k)$  is uniform in  $z, \alpha$ .

*Proof.* For the sake of completeness, we briefly review the elements of the proof. In [SoZ07, SoZ10], the norming constants (11) were evaluated in terms of the symplectic potential:

$$(39) \quad \mathcal{Q}_{h^k}(\alpha) = \int_P e^{k(u_\varphi(x) + \langle \frac{\alpha}{k} - x, \nabla u_\varphi(x) \rangle)} d\text{Vol}(x).$$

By applying a steepest descent method, it was shown in [SoZ07, Proposition 3.1] that for interior  $\alpha \in kP$ ,

$$(40) \quad \mathcal{Q}_{h^k}(\alpha) = k^{-m/2} \frac{(2\pi)^{m/2}}{|\det \text{Hess } u_\varphi|^{\frac{1}{2}}} e^{ku_\varphi(\alpha/k)} (1 + O(1/k)).$$

Hence, in the coordinates  $z = e^{\rho/2+i\theta}$ ,

$$\begin{aligned} \mathcal{P}_{h^k}(\alpha, z) &= \frac{e^{\langle \alpha, \rho z \rangle} e^{-k\varphi(z)}}{Q_{h^k}(\alpha)} \\ &\sim k^{m/2} (2\pi)^{-m/2} |\det \text{Hess } u_\varphi|^{\frac{1}{2}} e^{-ku_\varphi(\alpha)} e^{\langle \alpha, \rho z \rangle} e^{-k\varphi(z)}, \end{aligned}$$

as stated in the Lemma. □

We now assume that  $\frac{\alpha}{k} = \mu_h(z) + O(1/k)$  and have

$$\begin{aligned} I^z\left(\frac{\alpha}{k}\right) &\simeq I^z(\mu_h(z)) + \nabla_x I^z(\mu_h(z)) \cdot \left(\frac{\alpha}{k} - \mu_h(z)\right) \\ &\quad + \langle \text{Hess } I^z(\mu_h(z)) \left(\frac{\alpha}{k} - \mu_h(z)\right), \frac{\alpha}{k} - \mu_h(z) \rangle + O(k^{-3}). \end{aligned}$$

As mentioned above,

$$\begin{aligned} I^z(\mu_h(z)) &= 0, \quad \nabla_x I^z|_{x=\mu_h(z)} = 0, \\ \text{Hess } I_z(\mu_h(z)) &= \text{Hess } u_\varphi(\mu_h(z)) = [\text{Hess } \varphi(\rho_z)]^{-1} = H_z^{-1}. \end{aligned}$$

By Lemma 5.1, and by normalizing the weight,

$$(41) \quad k^{-m} \mathcal{P}_{h^k}(\alpha, z) = k^{-m/2} |\det H_z|^{-\frac{1}{2}} e^{-k\langle H_z^{-1}(\frac{\alpha}{k} - \mu_h(z)), \frac{\alpha}{k} - \mu_h(z) \rangle} (1 + O(1/k)),$$

where  $O(1/k)$  is uniform in  $z, \alpha$ . Distributing the  $k$  in the exponent as  $\sqrt{k}$  in each argument of the bilinear form completes the proof.

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