

GKZ DISCRIMINANT AND MULTIPLICITIES

JESSE HUANG AND PENG ZHOU

ABSTRACT. Let $T = (\mathbb{C}^*)^k$ act on $V = \mathbb{C}^N$ faithfully and preserving the volume form, i.e. $(\mathbb{C}^*)^k \hookrightarrow \mathrm{SL}(V)$. On the B-side, we have toric stacks Z_W labelled by walls W in the GKZ fan, and toric stacks $Z_{/F}$ labelled by faces of a polytope corresponding to minimal semi-orthogonal decomposition (SOD) components. The B-side multiplicity $n_{W,F}^B$ is the number of times $\mathrm{Coh}(Z_{/F})$ appears in the SOD of $\mathrm{Coh}(Z_W)$. On the A-side, we have the GKZ discriminant loci components $\nabla_F \subset (\mathbb{C}^*)^k$, and its tropicalization $\nabla_F^{\mathrm{trop}} \subset \mathbb{R}^k$. The A-side multiplicity $n_{W,F}^A$ is defined as the multiplicity of the tropical complex ∇_F^{trop} on the wall W . We prove that $n_{W,F}^A = n_{W,F}^B$, confirming a conjecture in Kite-Segal [KS22] inspired by [APW17]. Our proof is based on a lemma about the B-side SOD multiplicities, which allows us to reduce to lower dimensions just as in the A-side [GKZ][Ch 11].

1. INTRODUCTION

Homological mirror symmetry (HMS) for toric varieties is a well-studied subject, yet it can still offer new insights to classical problems. Our paper concerns a numerical conjecture that is a shadow of a full categorical conjecture [APW17, KS22, HK22]. We first sketch the categorical conjecture. On the B-side, we have a toric Calabi-Yau GIT problem and we study the derived equivalences and semi-orthogonal decompositions that arises from wall-crossing. On the A-side we have a fiberwise partially wrapped Fukaya category [AA21] associated to a fibration $\pi : Y \rightarrow B$ and a superpotential $W : Y \rightarrow \mathbb{C}$, and we want to study the 'pushforward' of $\mathrm{Fuk}(Y, W)$ along π to get a Fukaya category on B with categorical coefficient. For $b \in B$, let Y_b be the fiber over b and W_b the restriction of W , then there is a discriminant loci $\nabla \subset B$ where the fiberwise wrapped Fukaya category $\mathrm{Fuk}(Y_b, W_b)$ become 'degenerate'. The full HMS predicts that, for each wall crossing W (corresponding to certain asymptotic region of ∇), there is a B-side perverse schober coming from SOD of $\mathrm{Coh}(Z_W)$ where Z_W is certain toric stack associated to W , and there is an A-side analog coming from a transversal curve (annuli) intersecting the discriminant loci ∇ in the asymptotic region of the wall. Our main theorem is a verification that the two schobers has the same number of singularities of each type.

The full conjecture is explained beautifully in the Kite-Segal paper [KS22], which is based on the physicists conjecture [APW17]. In the remaining part of the introduction, we will focus on the numerical conjecture and state our main results.

1.1. Main Result. Our input data is a collection of lattice points q_1, \dots, q_N in \mathbb{Z}^k , where $q_i = (q_{i1}, \dots, q_{ik})$, called weights. We assume that q_i generate the linear space \mathbb{R}^k , and they satisfy the balanced condition $\sum_i q_i = 0$. Equivalently, the input data is a full rank linear map

$$Q : \mathbb{Z}^N \rightarrow \mathbb{Z}^k, \quad e_i \mapsto q_i,$$

such that $Q(1, \dots, 1) = 0$. From this input data, we can setup two problems (called A-side and B-side) as follows.

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First, we consider the dual of Q , $Q^\vee : \mathbb{Z}^k \rightarrow \mathbb{Z}^N$, and let A denote the cokernel map

$$A : \mathbb{Z}^N \rightarrow \mathbb{N}$$

where \mathbb{N} may have torsion if Q is not surjective.

For the sake of the introduction, we assume that Q is surjective and thus \mathbb{N} is a lattice. We further assume that for any $i \in [N] = \{1, \dots, N\}$, $a_i = A(e_i)$ are distinct. This is for easy quotation of results in [GKZ]. We also abuse notation and let A denote the set $\{a_i\}$. Both of the assumptions can be easily removed, as described in the appendix.

1.1.1. *The A-side setup.* The A-side problem concerns with a holomorphic function $W : Y \rightarrow \mathbb{C}$ and a fibration $\pi : Y \rightarrow B$. Here $Y = (\mathbb{C}^*)^N$, $B = (\mathbb{C}^*)^k$, and

$$W : (\mathbb{C}^*)^N \rightarrow \mathbb{C}, \quad W(z) = z_1 + \dots + z_N,$$

$$\pi = Q_{\mathbb{C}^*} : (\mathbb{C}^*)^N \rightarrow (\mathbb{C}^*)^k, \quad Q_{\mathbb{C}^*}(z) = \left(\prod_{i=1}^N z_i^{q_{i1}}, \dots, \prod_{i=1}^N z_i^{q_{ik}} \right).$$

Gelfand-Kapranov-Zelevinsky [GKZ] defined a polynomial E_A of N variables, called principal A -determinant, whose vanishing loci is a variety $\tilde{\nabla}_{GKZ} \subset (\mathbb{C}^*)^N$. This variety descend along π to a variety $\nabla_{GKZ} \subset B$.

Let Π be the convex hull of a_i . By the Calabi-Yau condition $\sum_i q_i = 0$, Π is a $k-1$ dimensional polytope in an affine \mathbb{Z}^{k-1} . Let \mathcal{F}_0 denote the faces of Π (Π itself is a face), and let $\mathcal{F} \subset \mathcal{F}_0$ denote the set of 'minimal faces', where a F is minimal if one remove any point in $F \cap A$ then the remaining points still generate the affine span of F .

For each minimal face F , there is a irreducible variety $\nabla_F \subset B$, and the divisor ∇_{GKZ} can be decomposed as

$$\nabla_{GKZ} = \sum_{F \in \mathcal{F}} m_F \cdot \nabla_F,$$

where m_F is some explicit integer (not to be confused with our A-side multiplicity). In [APW17], the multiplicity m_F is interpreted as the rank of K_0 of certain Higgs problem (Section 2.4). Even better, if $F \neq \Pi$, then ∇_F is the pullback along certain torus quotient $\pi_F : B \rightarrow B_F$ of a discriminant $\nabla'_F \subset B_F$.

The tropicalization $\nabla_{GKZ}^{trop} \subset \mathbb{R}^k$ of $\nabla_{GKZ} \subset (\mathbb{C}^*)^k$ is the codimension-1 part of the GKZ-fan. Given a subvariety $V \subset (\mathbb{C}^*)^k$, we define its tropicalization as follows. Consider the map $\text{Log} : (\mathbb{C}^*)^k \rightarrow \mathbb{R}^k$ by componentwise $z \mapsto \log |z|$. Then, we define $V^{trop} = \lim_{t \rightarrow 0^+} t \cdot \text{Log}(V)$. As shown in [GKZ], the Newton polytope for the defining polynomial of ∇_{GKZ} is the secondary polytope, whose exterior normal fan is the GKZ fan $\Sigma_{GKZ} \subset \mathbb{R}^k$.

The tropicalization method [Mik04] output not just a set, but a polyhedral complex with multiplicities. It allows for easy computation of intersection multiplicities.

Example 1.1. See Figure 1. The tropicalization of $x+y=1$ in $(\mathbb{C}^*)^2$ is a tropical curve with weight 1 on each leg, where the three legs represent the approximate equations $x=1, y=1, x+y=0$. The tropicalization of $x^2+y^3=1$ is a tropical curve consist of three legs, representing the three region $x^2 \approx 1, y^3 \approx 1$ and $x^2+y^3 \approx 0$. The weight 2 over the leg $x^2 \approx 1$ means there are two branches of complex solutions $x = \pm 1$ above the tropical 'shadow'.

△

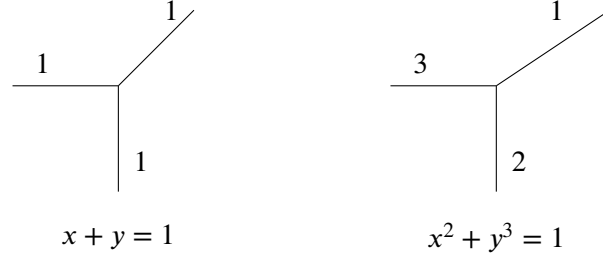


FIGURE 1. Tropicalization gives balanced polyhehral complex. The integers labels multiplicities on each cell.

Let W be a wall in the GKZ fan. We can ask for the multiplicity of a tropical complex ∇_{GKZ}^{trop} or ∇_F^{trop} along W , and denote them by $n_{W,GKZ}^A, n_{W,F}^A$. We get

$$n_{W,GKZ}^A = \sum_{F \in \mathcal{F}} m_F \cdot n_{W,F}^A.$$

It is not too hard to compute $n_{W,GKZ}^A$ in terms of volumes of some polytope, but to compute $n_{W,F}^A$, some extra work is needed. The key observation, made in [GKZ][Ch 11], is that ∇_F actually comes from a lower dimensional problem, and one can solve for $n_{W,F}^A$ recursively (or express the result as an alternating sum, if $m_F = 1$). We borrow this idea to the B-side and show such recursive relation also holds for SOD multiplicity, as predicted by mirror symmetry.

1.1.2. *The B-side setup.* Let $(\mathbb{C}^*)^k$ acts on \mathbb{C}^N with weight q_1, \dots, q_N . There is a GKZ fan in \mathbb{R}^k that labels all possible GIT quotients. The GKZ fan is the 'shadow' of the $\mathbb{R}_{\geq 0}^N$ (the moment polytope of \mathbb{C}^N for the $(\mathbb{C}^*)^N$ -action) under the map $Q_{\mathbb{R}} : \mathbb{R}^N \rightarrow \mathbb{R}^k$.

The GIT quotient stacks X_C for different chambers $C \subset \mathbb{R}^r$ are derived equivalent. For two adjacent chambers C_1, C_2 separated by a wall W , we have \mathbb{Z} many equivalence functors

$$\phi_i = \phi_{i,W,C_1,C_2} : \text{Coh}(X_{C_1}) \rightarrow \text{Coh}(X_{C_2}), i \in \mathbb{Z}$$

where i labels of a choice of window subcategory [HHP08, Seg11, HL15, BFK19] in the GIT quotient $X_W = [V//_{\theta}T]$, $\theta \in W$.

It is interesting to study the autoequivalence induced by window shift $\mu = \phi_1^{-1}\phi_0$ (for simplicity, we fix a window shift here). They comes from a spherical functors

$$S : \Phi = \text{Coh}(Z_W) \rightarrow \Psi = \text{Coh}(X_{C_1})$$

in that $\mu = \text{cone}(1 \rightarrow SS^j)[-1]$, where Φ is called the vanishing cycle category, and Ψ the nearby cycle category. Hence, this defines B-model perverse Schober over a disk.

Halpern-Leistner and Shipman [HLS16] showed that if Φ admits a semi-orthogonal decomposition $\Phi = \langle T_1, \dots, T_m \rangle$, then we have several fractional spherical twists $S_i : T_i \rightarrow \Psi$, and the monodromy $\mu = \mu_1 \circ \dots \circ \mu_m$.

Kite and Segal [KS22] identified the possible SOD factors $Z_{/F}$ for $\text{Coh}(Z_W)$, where F runs through minimal faces of the polytope Π . They have shown that different SOD has Jordan-Hölder property, namely the multiplicity $n_{W,F}^B$ of $\text{Coh}(Z_{/F})$ in $\text{Coh}(Z_W)$ is well-defined. They conjectured that $n_{W,F}^A = n_{W,F}^B$ and proved it when rank k of the action torus is ≤ 2 .

1.1.3. *Statement of Results.* Our first result is about toric GIT problem, possibly non-CY. Let $Q : \mathbb{Z}^N \rightarrow \mathbb{Z}^k$ be a toric GIT problem, C a chamber in the GKZ fan and let $H \subset \mathbb{R}^k$ be a relevant subspace i.e. $H = \text{cone}(\{q_i : q_i \in H\})$ (Definition 2.12). Let X_C be the GIT quotient corresponding to chamber C , and let Z_H be the irreducible SOD component for relevant subspace H . We are interested in the SOD multiplicity of $\text{Coh}(Z_H)$ in $\text{Coh}(X_C)$, denoted as $[X_C : Z_H]$. Let $[N]_H = \{i \in [N] : q_i \in H\}$ be the index set of the weights lying on H . Then we have a Coulomb GIT problem (Section 2.4)

$$Q_{/H} : \mathbb{Z}^N / \mathbb{Z}^{[N]_H} \rightarrow \mathbb{Z}^k / (\mathbb{Z}^k \cap H)$$

where the chamber C descend to a chamber C/H , with corresponding quotient $X_{C/H}$, the relevant subspace H quotient to a point H/H , and $Z_{H/H} = \text{pt}$.

Theorem 1.2 (Lemma 2.18). *Let Q be any toric GIT problem, C a chamber, H a relevant subspace. Then the multiplicity is invariant under passing to the Coulomb problem $Q_{/H}$*

$$[X_C : Z_H] = [X_{C/H} : Z_{H/H}].$$

Theorem 1.3 (Theorem 3.4). *Let W be a codimension one cone in the GKZ fan of a CY problem. For any minimal face F of the polytope Π , let $n_{W,F}^A$ denote the intersection multiplicity (A-side multiplicity) defined by the tropical complex, and $n_{W,F}^B$ denote the semiorthogonal decomposition multiplicity (B-side multiplicity). Then*

$$n_{W,F}^A = n_{W,F}^B.$$

In addition, we obtain a recursive formula relating multiplicities to the ranks of K-theory, hence volumes of stacky fans (see Section 2.3.2).

Theorem 1.4 (Proposition 2.19). *Let C be a GKZ chamber, and Σ the corresponding stacky fan. For each minimal face $F \in \mathcal{F}$, let $n_F := [X_\Sigma : Z_{/F}]$ denote the SOD multiplicity of $\text{Coh}(Z_{/F})$ in $\text{Coh}(X_\Sigma)$. Then we have a system of linear equations labelled also by minimal faces \mathcal{F} ,*

$$\text{rank}([X_{\Sigma \cap F}]) := \sum_{F' \leq F} n_{F'} \text{rank}([Z_{F/F'}]). \quad (1.1)$$

where equation labelled by F only involves $n_{F'}$ with face $F' \leq F$, and the coefficient of 'leading term' n_F is $\text{rank}([Z_{F/F}]) = 1$.

1.2. **Related Work.** Homological mirror symmetry for toric variety has been extensively studied, using Floer theoretic technique [Abo06, Han19, HH20] and using the microlocal sheaf method [FLTZ11, FLTZ12, Kuw20, Zho19, GPS18], see a recent review using GIT quotient [She21].

In the context of mirror symmetry to toric GIT problem, it is well-understood on the B-side [HHP08, BFK19, HL15, Seg11] how to use window subcategory to do wall-crossing between adjacent chambers, and sometimes in nice cases (e.g. quasi-symmetric case) how to do wall-crossing among all chambers simultaneously [HLS20, ŠVdB19].

On the A-side mirror to toric GIT, if we use microlocal sheaf as the A-model, then thanks to the functoriality of coherent constructible correspondence [Bon06, FLTZ11, Kuw20, Zho19], we can translate all the B-side VGIT result to the A-side [Zho20, HZ20], window categories into window skeletons etc. If we use the more traditional Fukaya category A-model, then the question is more subtle and harder to solve [DKK16, BDF⁺15, Ker17]. The program of Ballard-Diemer-Favero-Katzarkov-Kerr is about matching the A-side and B-side SOD.

In general, the above result on B-model (and on microlocal A-model) only sees the codimension-1 wall crossing, and did not see the entire GKZ discriminant locus, the recent work of Kite-Segal [KS22] in some sense remedies the above deficiency.

The numerical version of the full conjecture $n_{W,F}^A = n_{W,F}^B$ has been partially proven by Kite-Segal when $\dim_{\mathbb{C}} T = 1, 2$. And Horja-Katzarkov [HK22] proved an 'integrated' numerical equality

$$\sum_F n_{W,F}^A \text{rank}(Z_{/F}) = \sum_F n_{W,F}^B \text{rank}(Z_{/F}),$$

which does most of the heavy-lifting for us.

To get the full categorical conjecture [APW17, KS22, HK22], one need to identify the A-model irreducible SOD component mirror to the B-model counterpart $Z_{/F}$, and find a way to book-keep the relations between various SOD components. We leave these to future work.

1.3. Outline. We will mainly work on the B-side. In the next section we introduce the necessary notation for toric GIT and review the notion of Coulomb and Higgs problems for a toric GIT, then we prove our main Lemma. Most of the content are review or slight generalization of [KS22]. Then, in the last section, we compare the A-side and B-side multiplicity for toric CY GIT wall-crossing. There are examples at the end of each section, which might help to counter the heavy notations.

2. THE B-SIDE

In this section, we first setup the general toric GIT problem, then introduce the GKZ fan to relate various phases of GIT quotients (M-side). These data can be equally well encoded in certain triangulations problem (N-side). Next, we introduce the notion of a Coulomb-Higgs GIT problem, which represent a sub-quotient of the original GIT problem. See [KS22, HK22, GKZ] for more details. Then we prove our main lemma, namely the SOD multiplicity is invariant under passing to the Coulomb problem associated to a minimal face. Given the lemma, we then deduce a recursive formula for SOD multiplicities.

2.1. Toric GIT setup. Our starting point is a torus $(\mathbb{C}^*)^k$ acting on $V = \mathbb{C}^N$ with weights $q_1, \dots, q_N \in \mathbb{Z}^k$. More invariantly, we have a rank k lattice $L \simeq \mathbb{Z}^k$, and $L_{\mathbb{C}^*} = L \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq (\mathbb{C}^*)^k$ acts on \mathbb{C}^N by factoring through $L_{\mathbb{C}^*} \rightarrow (\mathbb{C}^*)^N$. This induces a map on the cocharacter lattices and character lattices

$$Q^{\vee} : L \rightarrow \mathbb{Z}^N, \quad Q : (\mathbb{Z}^N)^{\vee} \rightarrow L^{\vee}.$$

We assume the coker(Q) is finite, or equivalently Q^{\vee} is injective.

We have a short exact sequence (SES), called 'N-sequence'

$$0 \rightarrow L \xrightarrow{Q^{\vee}} \mathbb{Z}^N \xrightarrow{A} \mathbf{N} \rightarrow 0$$

where \mathbf{N} might have torsion

$$0 \rightarrow \mathbf{N}_{tors} \rightarrow \mathbf{N} \rightarrow \mathbf{N}_{free} \rightarrow 0.$$

Let $\bar{A} : \mathbb{Z}^N \rightarrow \mathbf{N}_{free}$ denote the obvious composition. Apply the $\text{Hom}(-, \mathbb{Z})$ to the 'N-sequence', and use injective resolution $\mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z}$, then we have a long exact sequece, the 'M-sequence'

$$0 \rightarrow \mathbf{M}_0 \xrightarrow{A^{\vee}} (\mathbb{Z}^N)^{\vee} \xrightarrow{Q} L^{\vee} \rightarrow \mathbf{M}_1 \rightarrow 0,$$

where $\mathbf{M}_0 = \text{Hom}(\mathbf{N}, \mathbb{Z}) = \text{Hom}(\mathbf{N}_{free}, \mathbb{Z})$ is a lattice, and $\mathbf{M}_1 = \text{Ext}^1(\mathbf{N}, \mathbb{Z}) = \text{Hom}(\mathbf{N}_{tors}, \mathbb{Q}/\mathbb{Z})$ is the Pontryagin dual to the torsion subgroup \mathbf{N}_{tors} .

If G is an abelian group, we denote $(\dots)_G := (\dots) \otimes_{\mathbb{Z}} G$, where G can be $\mathbb{R}, \mathbb{Z}, \mathbb{T} = \mathbb{R}/\mathbb{Z}, \mathbb{C}, \mathbb{C}^*$. For example, if we apply $(\dots)_{\mathbb{R}}$ to $Q : (\mathbb{Z}^N)^{\vee} \rightarrow L^{\vee}$, then we will get $Q_{\mathbb{R}} : (\mathbb{R}^N)^{\vee} \rightarrow L_{\mathbb{R}}^{\vee}$.

Let the co-character lattice $X_*((\mathbb{C}^*)^N) \simeq \mathbb{Z}^N$ be equipped with the standard basis e_1, \dots, e_N , and let the character lattice $X^*((\mathbb{C}^*)^N) \simeq (\mathbb{Z}^N)^\vee$ be equipped with the dual basis $e_1^\vee, \dots, e_N^\vee$. Define

$$\tilde{a}_i = A(e_i) \in \mathbb{N}, \quad a_i = \bar{A}(e_i) \in \mathbb{N}_{free}, \quad q_i = Q(e_i^\vee) \in L^\vee.$$

2.2. M-side, chambers and walls. By the M-side, we mean the objects living on spaces in the M-sequence, for example about the map $Q_{\mathbb{R}} : (\mathbb{R}^N)^\vee \rightarrow L_{\mathbb{R}}^\vee$. Here for the simplicity of notation, we write $(\mathbb{R}^N)^\vee$ as \mathbb{R}^N .

We recall the definition of GKZ fan (rather GKZ stratification) $\Sigma_{GKZ}(Q)$ in $L_{\mathbb{R}}^\vee$ (aka GIT fan or secondary fan) for a toric GIT problem Q . We omit (Q) if there is no danger of confusion.

By the GKZ stratification $\Sigma_{GKZ}(Q)$, we mean the stratification of $L_{\mathbb{R}}^\vee$ generated by $Q_{\mathbb{R}}$ images of various faces of the positive quadrant $P_N = \mathbb{R}_{\geq 0}^N$. A top dimensional strata is called a chamber, and a codimension-1 strata is called a wall. The support of the GKZ fan is $Q_{\mathbb{R}}(\mathbb{R}_{\geq 0}^N)$. If the support is not the full $L_{\mathbb{R}}^\vee$, then its complement is still a GKZ strata, and we call it the empty chamber.

Here P_N is the image of the moment map of $(\mathbb{C}^*)^N$ acting on \mathbb{C}^N

$$\mu_N : \mathbb{C}^N \rightarrow \mathbb{R}^N, \quad (z_i)_i \mapsto (|z_i|^2)_i.$$

And the moment map of $L_{\mathbb{C}^*}$ acting on \mathbb{C}^N is $\mu = Q_{\mathbb{R}} \circ \mu_N$.

For any $c \in L_{\mathbb{R}}^\vee$, we define the GIT quotient stack

$$X_c = [(\mathbb{C}^N)_c^{ss} / L_{\mathbb{C}^*}], \quad (\mathbb{C}^N)_c^{ss} = \{z \in \mathbb{C}^N \mid \overline{L_{\mathbb{C}^*} \cdot z} \cap \mu^{-1}(c) \neq \emptyset\}.$$

More concretely, the positive quadrant P_N is stratified by faces τ , $P_N = \sqcup \tau$ which induces a stratification of \mathbb{C}^N into strata $(\mathbb{C}^N)_\tau = \mu_N^{-1}(\tau)$, and $(\mathbb{C}^N)_c^{ss} = \sqcup \{(\mathbb{C}^N)_\tau \mid Q(\bar{\tau}) \cap c \neq \emptyset\}$. From the latter description, it is clear that X_c is constant when c varies within a GKZ strata C , hence we also write X_c as X_C .

Definition 2.1. For any GKZ strata C , we call the GIT quotient X_C a 'phase' of the toric GIT problem.

Let $\det V = \sum_i q_i \in L^\vee$ denote the weight of $L_{\mathbb{C}^*}$ acting on $\det V$. The toric GIT is called Calabi-Yau (CY), if $\det V = 0$.

Let W be a wall separating two chambers C_+, C_- . We choose the \pm sign so that on the wall $\det V$ is pointing towards C_+ . Let $\lambda_W \in L$ be an integral primitive vector conormal to the wall. Let $d_W = |\langle \lambda_W, \det V \rangle|$. If $d_W = 0$, i.e. $\det V$ is parallel to W , we say there is a balanced wall-crossing and we have (non-canonical) derived equivalence $\text{Coh}(X_{C_+}) \simeq \text{Coh}(X_{C_-})$. If $d_W > 0$, then we have Semi-orthogonal decomposition

$$\text{Coh}(X_{C_+}) = \langle \text{Coh}(X_{C_+}), \text{Coh}(Z_W), \dots, \text{Coh}(Z_W) \rangle,$$

where Z_W is some toric variety defined in [...], and the factor $\text{Coh}(Z_W)$ repeats d_W times.

We will abuse notation and let X also denote $\text{Coh}(X)$.

Therefore, starting from any point $c \in L_{\mathbb{R}}^\vee$, we may form the ray in the direction of $-\det V$. This is called the 'straight-line' run in [DKK16]. If c is generic, then the run will only encounter walls, and ends in a chamber C that contains $-\det V$ in its closure. We call such a chamber C and the corresponding phase X_C minimal. It is possible that $X_C = \emptyset$ or that there are several minimal chambers, but upto derived equivalence, the minimal phase is unique. Let $X_{\min}(Q)$ denote the minimal phase (possibly empty) for toric GIT problem.

For a non-CY problem, we can associate a CY problem to by adding a weight $q_{N+1} = -\det V$. The new problem is

$$0 \rightarrow L \xrightarrow{\hat{Q}} \mathbb{Z}^{N+1} \xrightarrow{\hat{A}} \mathbb{N} \oplus \mathbb{Z} \rightarrow 0$$

We have $\hat{A}(e_i) = (A(e_i), 1)$ for $i \in [N]$, and $\hat{A}(e_{N+1}) = (0, 1)$.

Going the other way around, suppose we have a toric CY GIT problem, then pick any q_i (which by assumption are all nonzero), say q_{N+1} by relabelling, we can delete it and get a toric non-CY GIT problem.

Assuming Q is a non-CY problem, then the toric CY GKZ stratification $\Sigma_{GKZ}(\hat{Q})$ refines the non-CY GKZ stratification $\Sigma_{GKZ}(Q)$. The new walls in $\Sigma_{GKZ}(\hat{Q})$ are all parallel to $\det V$.

2.3. N-side, local triangulations and stacky fan. By the N-side, we mean the objects living on spaces in the N-sequence, for example about the map $A : \mathbb{Z}^N \rightarrow \mathbb{N}$. Recall $a_i = \bar{A}(e_i) \in \mathbb{N}_{free} \subset \mathbb{N}_{\mathbb{R}}$ and $\tilde{a}_i = A(e_i) \in \mathbb{N}$.

2.3.1. Localized Marked Polytope Subdivision. Let $\Delta_N = \text{conv}(0, e_1, \dots, e_N)$ be the standard N -simplex in \mathbb{R}^N . Let $\Pi = A_{\mathbb{R}}(\Delta_N)$ be the image of Δ_N in $\mathbb{N}_{\mathbb{R}}$, which is also the convex hull $\text{conv}(0, a_1, \dots, a_N)$.

We start with a pair of piecewise linear (PL) functions on \mathbb{R}^N and $(\mathbb{R}^N)^\vee$ related by Legendre transformation. Let

$$\varphi_N(x_1, \dots, x_N) := \min(0, x_1, \dots, x_N) : (\mathbb{R}^N)^\vee \rightarrow \mathbb{R}.$$

It is a concave function, with $\varphi_N^{-1}(0) = P_N = (\mathbb{R}_{\geq 0})^N$. By Legendre transformation, we have

$$\psi_N(y_1, \dots, y_N) = \max_{x \in \mathbb{R}^N} (\varphi(x) - (x, y)) : \mathbb{R}^N \rightarrow \mathbb{R}.$$

ψ_N is a convex function, and explicitly

$$\psi_N(x) = \begin{cases} 0 & x \in \Delta_N \\ +\infty & x \notin \Delta_N \end{cases}$$

Given any point $c \in L_{\mathbb{R}}^\vee$, choose its lift $b \in Q^{-1}(c)$. Then, we can identify $\mathbb{M}_{\mathbb{R}} \simeq Q^{-1}(c)$ by

$$f_b : \mathbb{M}_{\mathbb{R}} \rightarrow Q^{-1}(c), \quad \xi \mapsto b + A_{\mathbb{R}}^\vee(\xi) = (b_1 + (a_1, \xi), \dots, b_N + (a_N, \xi)).$$

The restriction of φ_N on $Q^{-1}(b)$ pullback to $\mathbb{M}_{\mathbb{R}}$ by f_b gives

$$\varphi_b(\xi) := \varphi_N(b + A_{\mathbb{R}}^\vee(\xi)) = \min(0, b_1 + a_1(\xi), b_2 + a_2(\xi), \dots, b_N + a_N(\xi)).$$

Its Legendre transformation is

$$\psi_b(\eta) = \max_{\xi \in \mathbb{M}_{\mathbb{R}}} (\varphi_b(\xi) - (\eta, \xi)).$$

For uniform treatment later, it is useful to introduce another pair $(a_{N+1}, b_{N+1}) = (0, 0) \in \mathbb{N}_{\mathbb{R}} \times \mathbb{R}$ for the origin.

The overgraph of ψ_b

$$\Gamma_{\geq \hat{\psi}_b} := \{(x, y) \in \Pi \times \mathbb{R} \mid y \geq \psi_b(x)\}$$

is the convex hull of the upward rays

$$\bigcup_{i=1}^{N+1} \tilde{a}_i \times \{y \geq b_i\} \subset \mathbb{N}_{\mathbb{R}} \times \mathbb{R}.$$

We record the subset $S \subset [N+1]$ where the tip of the ray touches the graph, i.e.

$$S = \{i \in [N+1] \mid \psi_b(a_i) = b_i\}.$$

For $x \in \Pi$, if $x = a_i$ for some $i \in S$, we abuse notation and say x is marked by a_i . It is possible x is marked by more than one a_i .

The maximal linearity domain of ψ_b defines a polytope subdivision of Π , and the vertices of the polytopes and possibly some interior points are marked by some a_i .

The polytope subdivision together with marking points S is called a coherent (multi-)marked polytope subdivision, and marked subdivision for short.

If we choose a different lift $b \in Q^{-1}(c)$, then ψ_b only change by a linear function and does not affect the coherent subdivision. For a generic choice of c , we have triangulations of Π with no interior marked points, and each vertex of the simplices is marked exactly once.

We say a marked subdivision uses a_i if a_i (rather i) appears in the marking. We say two marked subdivisions are **equivalent at a_i** if they have the same collection (possibly empty) marked polytopes that contains a_i in the marking.

Definition 2.2. *A marked subdivision localized at a_i is an equivalence class of marked subdivisions modulo equivalence relations at a_i .*

We refer to the index $N + 1$, and the point a_{N+1} , as the 'base point'.

For any $b \in L_{\mathbb{R}}^{\vee}$, we may use the above procedure to obtain a marked subdivision $T(b)$ of Π .

Proposition 2.3. (1) *For any $b \in L_{\mathbb{R}}^{\vee}$, the subdivision $T(b)$ uses the base point a_{N+1} if and only if b is not in the empty chamber of $\Sigma_{GKZ}(Q)$.*

(2) *For any $b_1, b_2 \in L_{\mathbb{R}}^{\vee}$, b_1 and b_2 are in the same GKZ strata of $\Sigma_{GKZ}(Q)$ if and only if $T(b_1)$ is equivalent to $T(b_2)$ at the base point a_{N+1} .*

Proof. $T(b)$ uses a_{N+1} if and only if $Q_{\mathbb{R}}^{-1}(b)$ intersect $\mathbb{R}_{\geq 0}^N$. b_1 and b_2 are in the same GKZ strata, if and only if as we move b straightly from b_1 to b_2 , the fiber $Q_{\mathbb{R}}^{-1}(b)$ intersects each face τ of $\mathbb{R}_{\geq 0}^N$ in the same way. Each non-empty intersection of $\tau \cap Q_{\mathbb{R}}^{-1}(b)$ corresponds to a polytope (possibly not of top dimension) in $T(b)$ marked by a_{N+1} . \square

Remark 2.4. *If the GIT problem Q is Calabi-Yau, then any marked subdivision of Π uses a_{N+1} and the equivalence relation is trivial. In this case, we have a bijection between GKZ chambers and marked triangulation of Π .*

If the GIT problem Q is not CY, then we may consider the associated CY problem \hat{Q} . Then chambers in $\Sigma_{GKZ}(\hat{Q})$ corresponds to marked triangulation of Π ; and chambers in $\Sigma_{GKZ}(Q)$ corresponds to marked triangulations of Π localized at a_{N+1} .

2.3.2. Stacky Fan and its Volume. Let C be a chamber of GKZ fan, and let $T(C)$ be the corresponding marked triangulation localized at the base point. This defines a stacky fan in \mathbb{N} as follows.

Recall the definition of simplicial stacky fan following [BCS05]. Let \mathbb{N} be a finitely generated abelian group, Σ be a rational simplicial polyhedran fan in $\mathbb{N}_{\mathbb{R}}$, and $\{v_i\}_{i \in \Sigma^1} \subset \mathbb{N}$ such that $(v_i)_{\mathbb{R}}$ generate the corresponding ray in Σ . The triple $\Sigma = (\mathbb{N}, \Sigma, \{v_i\})$ is called a stacky fan.

Roughly speaking, modulo torsion, we can think of a stacky fan Σ as a collection of simplices $\hat{\sigma} \subset \mathbb{N}_{\mathbb{R}}$ with shared vertices $(v_i)_{\mathbb{R}}$.

We write $|\Sigma|$ for the union of the cones σ , and we write $|\Sigma|$ for the union of the simplices $\hat{\sigma}$.

We normalize volume $\text{vol}_{\mathbb{R}}$ on $\mathbb{N}_{\mathbb{R}}$ such that a minimal simplex with vertices in \mathbb{N}_{free} has unit volume. Then we define the stacky volume by

$$\text{vol}(\Sigma) := |\mathbb{N}_{tors}| \cdot \text{vol}_{\mathbb{R}}(|\Sigma|).$$

By design of the volume, we have the following result

Proposition 2.5 ([HK22]). *The volume of the stacky fan equals the rank of the K_0 group of X_{Σ} ,*

$$\text{rank}(K_0(X_{\Sigma})) = \text{vol}(\Sigma).$$

2.3.3. *Minimal phases and Minimal Fans.* Given a toric GIT problem $A : \mathbb{Z}^N \rightarrow \mathbb{N}$ or $Q : (\mathbb{Z}^N)^\vee \rightarrow L^\vee$, on the M -side, we have some minimal chambers C , i.e. those containing $-\det(V)$. On the N -side, we have some minimal stacky fans corresponding to the minimal chambers. They have the same support $|\Sigma^{min}|$ which we describe now.

Let $S = A_{\mathbb{R}}(\mathbb{Z}_{\geq 0}^N)$ and $S_+ = A_{\mathbb{R}}(\mathbb{Z}_{\geq 0}^N \setminus \{0\})$ and define (slightly abusing notation)

$$|\Sigma^{min}| = \text{conv}(S) - \text{conv}(S_+).$$

We see $|\Sigma^{min}| = \emptyset$ if and only if $S = S_+$, or $0 \in \Pi = \text{conv}(a_i)$, or the support of the GKZ fan is not the whole $L_{\mathbb{R}}^\vee$.

Proposition 2.6 ([HK22]). *We have the rank-volume relation*

$$\text{rank}(K_0(X_{min})) = \text{vol}(|\Sigma^{min}|) = |\mathbb{N}_{tors}| \cdot \text{vol}_{\mathbb{R}}(\text{conv}(S) - \text{conv}(S_+)).$$

2.4. **Coulomb and Higgs GIT problems.** We follow [KS22, APW17] and introduce the a sub and quotient GIT problem, called Coulomb and Higgs problem respectively.

2.4.1. *From subsets of $[N]$.* For any subset $\Gamma \subset [N]$, we can define a Coulomb problem (as a GIT problem of a subspace acted on by a subgroup) and a Higgs problem (as a quotient space acted on by a quotient group).

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & L_\Gamma & \xrightarrow{Q_\Gamma^\vee} & \mathbb{Z}^\Gamma & \xrightarrow{A_\Gamma} & \mathbb{N}_\Gamma \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & L & \xrightarrow{Q^\vee} & \mathbb{Z}^N & \xrightarrow{A} & \mathbb{N} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & L/L_\Gamma & \xrightarrow{Q_{/\Gamma}^\vee} & \mathbb{Z}^N/\mathbb{Z}^\Gamma & \xrightarrow{A_{/\Gamma}} & \mathbb{N}/\mathbb{N}_\Gamma \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

Dualize the first two columns, and then apply the snake lemma, we get

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 & & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & (\mathbb{N}/\mathbb{N}_\Gamma)_0^\vee & \xrightarrow{A_{/\Gamma}} & (\mathbb{Z}^N/\mathbb{Z}^\Gamma)^\vee & \xrightarrow{Q_{/\Gamma}} & (L/L_\Gamma)^\vee & \longrightarrow & (\mathbb{N}/\mathbb{N}_\Gamma)_1^\vee \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M_0 & \xrightarrow{A^\vee} & (\mathbb{Z}^N)^\vee & \xrightarrow{Q} & L^\vee & \longrightarrow & M_1 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & (M_\Gamma)_0 & \xrightarrow{A_\Gamma^\vee} & (\mathbb{Z}^\Gamma)^\vee & \xrightarrow{Q_\Gamma} & L_\Gamma^\vee & \longrightarrow & (M_\Gamma)_1 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & \dots & & 0 & & 0 & & 0 \end{array}$$

Definition 2.7. Let $\Gamma \subset [N]$. We call the GIT problem associated to Q_Γ (or A_Γ) as the **Coulomb problem** for Γ , and the GIT problem associated to $Q_{/\Gamma}$ and $A_{/G_a}$ as the **Higgs problem** for Γ .

More generally, if we have $\Gamma_1 \subset \Gamma_2 \subset [N]$, we may form the GIT problem

$$L_{\Gamma_2}/L_{\Gamma_1} \xrightarrow{Q_{\Gamma_2/\Gamma_1}^\vee} \mathbb{Z}^{\Gamma_2}/\mathbb{Z}^{\Gamma_1} \xrightarrow{A_{\Gamma_2/\Gamma_1}} \mathbb{N}_{\Gamma_2}/\mathbb{N}_{\Gamma_1}$$

as the Coulomb-Higgs problem for the pair $\Gamma_1 \subset \Gamma_2$.

2.4.2. *From subspaces and faces.* Although the Coulomb-Higgs problems can be defined for general subsets $\Gamma_1 \subset \Gamma_2 \subset [N]$, they often arises from subspaces $H \subset L_{\mathbb{R}}^\vee$ and faces $F \subset \Pi$.

Definition 2.8. A subspace $H \subset L_{\mathbb{R}}^\vee$ is called **weight supported** if $H = \text{span}_{\mathbb{R}}\{q_i \in H\}$.

A subset $F \subset \Pi$ is called a **face** if there is a linear function $l : \mathbb{N}_{\mathbb{R}} \rightarrow \mathbb{R}$, such that $l|_{\Pi} \geq 0$ and $F = l^{-1}(0) \cap \Pi$.

Recall $[N]_H = \{i : q_i \in H\}$ and $[N]_F = \{i : a_i \in F\}$.

Definition 2.9. If $H_1 \subset H_2 \subset L_{\mathbb{R}}^\vee$ are a pair of weight supported subspaces, we define the Coulomb-Higgs problem Q_{H_2/H_1}

$$Q_{H_2/H_1} : \mathbb{Z}^{[N]_{H_2} - [N]_{H_1}} \rightarrow (L^\vee \cap H_2)/H_1.$$

If $H_1 = 0$, we get a Higgs problem Q_{H_2} ; if $H_2 = L_{\mathbb{R}}^\vee$, we get a Coulomb problem $Q_{/H_1}$.

If $F_1 \subset F_2 \subset \Pi$ is a pair of faces, we define the Coulomb-Higgs problem A_{F_2/F_1}

$$A_{F_2/F_1} : \mathbb{Z}^{[N]_{F_2} - [N]_{F_1}} \rightarrow \mathbb{N}_{F_2}/\mathbb{N}_{F_1}, \quad \mathbb{N}_F := \mathbb{N}_{\Gamma=[N]_F}.$$

If $F_1 = 0$, we get a Coulomb problem A_{F_2} ; if $F_2 = \Pi$, we get a Higgs problem $A_{/F_1}$.

2.4.3. *Minimal faces and relevant subspaces.* Here we again follow [KS22, APW17] and introduce certain important Coulomb Higgs problem, that describes the SOD irreducible components.

Definition 2.10. Let S be a subset of $[N]$. Recall that $a_i = A_{\mathbb{R}}(e_i)$, $q_i = Q_{\mathbb{R}}(e_i^\vee)$.

- (1) We say S is **$A_{\mathbb{R}}$ -redundant**, if there exists $c_i \neq 0$ for each $i \in S$, such that $\sum_{i \in S} c_i a_i = 0$.
- (2) We say S is **$A_{\mathbb{R}}$ -saturated**, if there is a linear function $l : \mathbb{N}_{\mathbb{R}} \rightarrow \mathbb{R}$, such that $S = \{i : l(a_i) = 0\}$.
- (3) We say S is **$A_{\mathbb{R}}$ -extremally-saturated**, if there is a linear function $l : \mathbb{N}_{\mathbb{R}} \rightarrow \mathbb{R}$, such that $S = \{i : l(a_i) = 0\}$, and $l(a_i) \geq 0$ for all $i \in [N]$.
- (4) We say S is **$Q_{\mathbb{R}}$ -redundant**, if there exists $c_i \neq 0$ for each $i \in S$, such that $\sum_{i \in S} c_i q_i = 0$.
- (5) We say S is **$Q_{\mathbb{R}}$ -saturated**, if there is a linear function $l : L_{\mathbb{R}}^\vee \rightarrow \mathbb{R}$, such that $S = \{i : l(q_i) = 0\}$.
- (6) We say S is **$Q_{\mathbb{R}}$ -positively-redundant** if there exists $c_i > 0$ for $i \in S$, such that $\sum_{i \in S} c_i q_i = 0$.

Proposition 2.11. Let S be

- (1) S is $A_{\mathbb{R}}$ -redundant if and only if S^c is $Q_{\mathbb{R}}$ -saturated.
- (2) S is $Q_{\mathbb{R}}$ -redundant if and only if S^c is $A_{\mathbb{R}}$ -saturated.
- (3) S is $Q_{\mathbb{R}}$ -positively redundant if and only if S^c is $A_{\mathbb{R}}$ -extremally-saturated.

Proof. We only prove the first statement, as the second follows verbatim by changing .

For (1), by definition, S is $A_{\mathbb{R}}$ -redundant, if there exists $c_i \neq 0$ for each $i \in S$, such that $\sum_{i \in S} c_i a_i = 0$. By setting $c_i = 0$ for $i \notin S$, we get an element $\vec{c} = (c_i) \in \mathbb{R}^N$, such that $A(\vec{c}) = 0$, i.e, $\vec{c} \in L$. Hence \vec{c} defines a linear function $l : L_{\mathbb{R}}^\vee \rightarrow \mathbb{R}$, such that $l(q_i) = 0$ if and only if $c_i = 0$, i.e $i \in S^c$. Thus S^c is $Q_{\mathbb{R}}$ -saturated. The argument can also be reversed, hence we get the equivalence.

For (2), we only need to change the above argument by swapping $A_{\mathbb{R}}$ with $Q_{\mathbb{R}}$, and a_i with q_i . And (3) is the same as (2) where the two positivity conditions matches. \square

Recall $\Pi = A_{\mathbb{R}}(\mathbb{R}_+^N) \subset \mathbb{N}_{\mathbb{R}}$. A subset $F \subset \Pi$ is called a face if there is a linear function $l : \mathbb{N}_{\mathbb{R}} \rightarrow \mathbb{R}$, such that $l(\Pi) \geq 0$ and $F = l^{-1}(0)$. For example, Π is always a face of itself. And if Π contains a linear subspace H , then any face $F \supset H$.

Definition 2.12. Let F is a face of Π and define $[N]_F = \{i : a_i \in F\}$. By construction $[N]_F$ is $A_{\mathbb{R}}$ extremally saturated. We say F is a **minimal face** of Π if $[N]_F$ is $A_{\mathbb{R}}$ redundant.

Let $H \subset L_{\mathbb{R}}^{\vee}$ be a weight supported subspace, and define $[N]_H = \{i : q_i \in H\}$. We say H is a **relevant subspace** of $L_{\mathbb{R}}^{\vee}$ if $[N]_H$ is $Q_{\mathbb{R}}$ -positively-redundant and $Q_{\mathbb{R}}$ saturated.

Remark 2.13. Our definitions generalize the definitions of minimal faces and relevant subspaces in [KS22] in the toric CY setting.

We obtain the following slight generalization of Proposition 4.15 in [KS22].

Proposition 2.14. There is a bijection between minimal faces of Π and relevant subspaces of $L_{\mathbb{R}}^{\vee}$, such that if a minimal face F corresponds to a relevant subspace H then $[N]_F = [N]_H^c$.

Proof. Let F be a minimal face, then $[N]_F$ is $A_{\mathbb{R}}$ extremally-saturated and redundant, hence $[N]_F^c$ is $Q_{\mathbb{R}}$ -positively-redundant and saturated. Let $H = \text{span}\{q_i : i \in [N]_F^c\}$, we then get a relevant subspace. Going backward is similar. \square

If F is a minimal face (of Π), we will abuse notation and also call $[N]_F$ a minimal face. Similarly, if H is a relevant subspace, we will also call $[N]_H$ is a relevant subspace.

2.4.4. *GIT quotients.* Here we define some GIT quotients associated with the Higgs and Coulomb problems.

Definition 2.15. If H is a relevant subspace, and F is the corresponding minimal face, with $\Gamma = [N]_F$, then we use $Z_H = Z_{/F} = Z_{/\Gamma}$ to denote the minimal phase in the Higgs problem $Q_{/H} A_F$.

In general, for any GIT problem Q_* or A_* , where $* = F, H, F_1/F_2 \dots$, we use Z_* to denote the minimal phase in that problem.

If W is a wall in $\Sigma_{GKZ}(Q)$, let $H = \text{span}_{\mathbb{R}}(W)$, then we define Z_W as the phase W of the Higgs problem Q_H .

The following result and proof is essentially also due to [KS22].

Proposition 2.16. Let F be a face of Π , and A_F, Q_F be the Coulomb problem associated to F . Then we have map of lattices

$$\pi_F : L^{\vee} \rightarrow L_F^{\vee}.$$

The map $(\pi_F)_{\mathbb{R}}$ is compatible with the GKZ stratifications $\Sigma_{GKZ}(Q)$ and $\Sigma_{GKZ}(Q_F)$, i.e. image of a strata is a strata.

Proof. Let $\tilde{\Sigma}_{GKZ}(Q)$ be the pullback stratification of \mathbb{R}^N , and $\tilde{\Sigma}_{GKZ}(Q_F)$ be that of $\mathbb{R}^{[N]_F}$. Suffice to show that under the quotient map $\tilde{\pi}_F : \mathbb{R}^N \rightarrow \mathbb{R}^{[N]_F}$, the image of a strata is a strata.

Let $b \in \mathbb{R}^N$, and $b_F = \pi_F(b) \in \mathbb{R}^{[N]_F}$. Then b defines a PL convex function ψ_b on Π , and b_F defines a PL convex function ψ_{b_F} on F . It is easy to check $\psi_{b_F} = \psi_b|_F$. Hence when b varies within a strata, ie, induced localized marked subdivision remains invariant, the boundary F subdivision also remains invariant. Thus π_F sends a strata inside a strata.

On the other hand, consider the lift (a section of π_F)

$$\iota_F : \mathbb{R}^{[N]_F} \rightarrow \mathbb{R}^N, \quad x \mapsto (x - b_F) + b.$$

We claim that ι_F also sends a strata into a strata. Indeed, as we vary b_F such that the localized marked subdivision of F remains invariant, then we can extend the variation of b_F to that of b by keeping other b_i s not in the face F fixed. The resulting localized subdivision of Π remains fixed. \square

Given the above result, if C is a chamber corresponding to the and F is a face, then we have the a well-defined chamber C_F in the Coulomb problem Q_F . If C corresponds to a stacky fan Σ , then we use Σ_F for C_F . Concretely, Σ_F is the stacky fan $(N_\Gamma, \Sigma \cap (N_\Gamma)_\mathbb{R}, S \cap \Gamma)$. We restrict the simplices $\hat{\sigma}$ to the face F and coarsen $N \cap \mathbb{R}F$ to sub-lattice N_Γ .

2.5. Main Lemma. First we recall some results from [KS22].

Proposition 2.17 ([KS22]). *The irreducible SOD components in a toric GIT problem Q are labelled by the set of relevant subspaces, or equivalently, the set of minimal faces. Let X_C be any phase and $H \subset L_\mathbb{R}^\vee$ any relevant subspaces, then the multiplicity $n_{C,H}^B = [X_C : Z_H]$, as the number of times that $\text{Coh}(Z_H)$ appears in any complete SOD of $\text{Coh}(X_C)$, is well-defined.*

Our main lemma equate the multiplicity $[X_C : Z_H]$ to its counterpart $[X_{C/H} : Z_{H/H}]$ in the Coulomb problem $Q_{/H}$.

Recall we have a quotient map $\pi_H : L_\mathbb{R}^\vee \rightarrow L_\mathbb{R}^\vee/H$. As Kite-Segal observed, the quotient π_H is compatible with fans $\Sigma_{GKZ}(Q)$ and $\Sigma_{GKZ}(Q_{/H})$. Indeed, a strata in $\Sigma_{GKZ}(Q)$ represent a localized marked subdivision, and passing to $\Sigma_{GKZ}(Q_{/H})$ means we intersect the subdivision with the face F corresponding to H , hence is still a localized marked subdivision.

If C is a chamber in $\Sigma_{GKZ}(Q)$, let C/H denote the corresponding chamber in $\Sigma_{GKZ}(Q_{/H})$. Different chambers C can result in the same quotient chamber C/H .

Let $X_{C/H}$ denote the phase of C/H in Coulomb problem $Q_{/H}$, and let $Z_{H/H}$ denote the minimal phase in the Coulomb-Higgs problem $Q_{H/H}$, which actually is a point.

Lemma 2.18. *Let C be a non-empty GKZ chamber and H a relevant subspace, then*

$$[X_C : Z_H] = [X_{C/H} : Z_{H/H}].$$

Proof. We induct on $r = \text{codim}_\mathbb{R} H$. If $r = 0$, $H = L_\mathbb{R}^\vee$. Since by assumption H is relevant, hence $Z_H = X_{\min} \neq \emptyset$. We have $[X_C : Z_H] = [X_C : X_{\min}] = 1$. And we also have $X_{C/H} = pt$, hence $[X_{C/H} : Z_{H/H}] = 1$.

Suppose the statement hold when $r < n$, and consider the case $r = n$. For any chamber C , consider a sequence of ‘‘monotone decreasing run’’ $C = C_0 \rightsquigarrow C_1 \rightsquigarrow \dots \rightsquigarrow C_m$, where C_m is a minimal phase, and let W_i denote the wall separating C_{i-1} and C_i . For any wall W , write $W \parallel H$ if $\text{span}_\mathbb{R}(W) \supset H$.

For any wall W with $W \not\parallel H$, we have $[Z_W : Z_H] = 0$. Thus

$$[X_C : Z_H] = \sum_W d_W [Z_W : Z_H] = \sum_{W \parallel H} d_W [Z_W : Z_H],$$

where summation of W runs over all the wall crossing in the run i.e. $W = W_i$ for some i .

For W with $W \parallel H$, $[Z_W : Z_H]$ is computed in the Higgs problem $\mathcal{Q}_{\text{span}_{\mathbb{R}} W}$. By our induction hypothesis, we have $[Z_W : Z_H] = [Z_{W/H} : Z_{H/H}]$. Hence

$$[X_C : Z_H] = \sum_{W \parallel H} d_W [Z_{W/H} : Z_{H/H}].$$

On the other hand, the sequence of ‘‘upstairs chambers’’ in $L_{\mathbb{R}}^{\vee}$ is sent to a sequence of ‘‘downstairs chambers’’ in $L_{\mathbb{R}}^{\vee}/H$ (possibly with repetitions). In particular, if $W \nparallel H$, then the two chambers separated by W are sent to the same chamber in $L_{\mathbb{R}}^{\vee}/H$. Hence the downstairs wall crossing corresponds to those upstairs wall crossing with $W \parallel H$. We get

$$[X_{C/H} : Z_{H/H}] = \sum_{W \parallel H} d_{W/H} [Z_{W/H} : Z_{H/H}].$$

Hence we only need to prove $d_W = d_{W/H}$, and this follows from $\pi_H^*(\lambda_{W/H}) = \lambda_W$ (up to sign ambiguity of λ_W). \square

Since a GKZ chamber C corresponds to a coherent stacky fan Σ , and a relevant subspace H corresponds to a minimal face F , we may also denote $X_{\Sigma} = X_C$ and $Z_{/F} = Z_H$, and denote $n_{\Sigma, /F}^B = n_{C, H}^B$. Let $\Gamma = [N]_F$ denote set of a_i in the minimal face. Then, the above statement is

$$[X_{\Sigma} : Z_{/F}] = [X_{\Sigma \cap F} : Z_{F/F}]$$

2.6. Recursive Formula for Multiplicity. Let \mathcal{F} be the set of minimal faces indexing the SOD components, and let $\mathbb{Z}\mathcal{F}$ be formal \mathbb{Z} -linear combination of \mathcal{F} . We denote the basis of $\mathbb{Z}\mathcal{F}$ by $[Z_{/F}]$. There is a rank map by taking the rank of the K_0 group of SOD components

$$\text{rank} : \mathbb{Z}\mathcal{F} \rightarrow \mathbb{Z}, \quad \text{rank}([Z_{/F}]) = \text{rank}(K_0(\text{Coh}(Z_{/F})))$$

Explicitly, we have $\text{rank}([Z_{/F}]) = \text{vol}(\Sigma^{\min}(A_{/F})) = i(\Gamma)u(\Gamma)$, where $i(\Gamma)$ and $u(\Gamma)$ are define in [GKZ] or [HK22]. (See also Section 2.3.3).

Proposition 2.19. *Let C be a GKZ chamber, and Σ the corresponding stacky fan. For each minimal face $F \in \mathcal{F}$, let $n_F := [X_{\Sigma} : Z_{/F}]$ denote the SOD multiplicity of $\text{Coh}(Z_{/F})$ in $\text{Coh}(X_{\Sigma})$. Then we have a system of linear equations labelled also by minimal faces \mathcal{F} ,*

$$\text{rank}([X_{\Sigma \cap F}]) := \sum_{F' \leq F} n_{F'} \text{rank}([Z_{F'/F}]). \quad (2.1)$$

where equation labelled by F only involves $n_{F'}$ with face $F' \leq F$, and the coefficient of ‘‘leading term’’ n_F is $\text{rank}([Z_{F/F}]) = 1$.

Proof. We define class $[X_{\Sigma}] \in \mathbb{Z}\mathcal{F}$ as

$$[X_{\Sigma}] := \sum_F [X_{\Sigma} : Z_{/F}] [Z_{/F}].$$

For each minimal face F , we have

$$[X_{\Sigma \cap F}] := \sum_{F' \leq F} [X_{\Sigma \cap F} : Z_{F'/F'}] [Z_{F'/F'}]. \quad (2.2)$$

By our main Lemma (2.18), we have that

$$[X_{\Sigma \cap F} : Z_{F'/F'}] = [X_{\Sigma \cap F \cap F'} : Z_{F'/F'}] = [X_{\Sigma \cap F'} : Z_{F'/F'}],$$

is independent of F . Denote $n_{F'} = [X_{\Sigma}, Z_{/F'}]$ as the unknown variables. Taking rank on both sides in Eq (2.2), we get the desired proposition. \square

2.7. Examples. Here we illustrate how to get the SOD multiplicity of $n_{\Sigma, F}^B$ in two different ways. The first way is by wall crossing, and count how each wall Z_W decompose into various $Z_{/F}$. The second way is to use the recursive formula we had.

Example 2.20. Consider the following a_i points in $N = \mathbb{Z}^2$. Our initial stacky fan is $\Sigma = \text{conv}(\{0\} \cup A)$, shown in yellow in the first figure. We have

$$X_{\Sigma} = [\mathbb{C}/\mathbb{Z}_2] \times [\mathbb{C}/\mathbb{Z}_3].$$

Then, we show a sequence of shrinking Σ (shown in yellow), and show how the lost volume during circuit transition contribute to various SOD components. The total volume of Σ is 6. There are 3 minimal faces

$$F_1 = \Pi, \quad F_2 = \text{conv}(0, a), \quad F_3 = \text{conv}(0, b), \quad F_4 = 0$$

where $a = (0, 3), b = (2, 0)$. They corresponds to

$$Z_{/F_1} = pt, \quad Z_{/F_2} = \mathbb{C}, \quad Z_{/F_3} = \mathbb{C}, \quad Z_{/F_4} = \mathbb{C}^2$$

They all have $\text{rank}(Z_{/F_i}) = 1$. And we have the decomposition of total volume as

$$6 = 2(F_1) + 2(F_2) + 1(F_3) + 1(F_4).$$

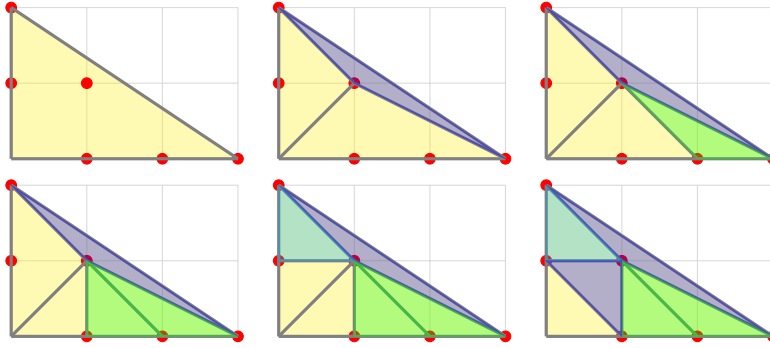


FIGURE 2. Multiplicities from Volume allocation.

Next, we compute the SOD multiplicities again using the recursive relation. Denote $n_i = [X_{\Sigma}, Z_{/F_i}]$, and denote $m_i = \text{rank}(Z_{/F_i})$. First, since F_4 is smallest face, we have $n_4 = 1$. Then, consider the equation for face F_3 . F_3 has a proper subspace F_4 , and we have $m_4 = 1, m_3 = 1$. We have

$$2 = \text{rank}(X_{\Sigma \cap F_3}) = n_3 + n_4 = n_3 + 1$$

hence $n_3 = 1$. Next, we compute n_2 using F_2 equation, we get

$$3 = n_2 + n_4 = n_2 + 1$$

hence $n_2 = 2$. Lastly, we compute n_1 , using F_1 , which gives

$$6 = n_1 + n_2 + n_3 + n_4 = n_1 + 2 + 1 + 1$$

Thus $n_1 = 2$. △

The next example shows that the irreducible SOD components really depends on the entire GIT problem, instead of just the initial phase X_{Σ} that we try to decompose.

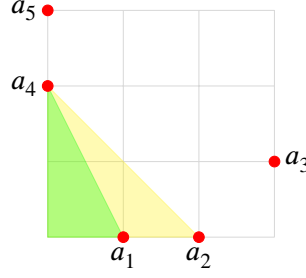


FIGURE 3. SOD components depend on the entire GIT problem

Example 2.21. Consider the following set of a_1, \dots, a_5 in $\mathbb{N} = \mathbb{Z}^2$. We have 4 minimal faces

$$F_1 = (0,0), \quad F_2 = \text{conv}(0, a_2), \quad F_3 = \text{conv}(0, a_5), \quad F_4 = \text{conv}(0, a_1, \dots, a_5),$$

with corresponding irreducible components $Z_i = Z_{/F_i}$

$$Z_1 = \mathbb{C} \times [\mathbb{C}/\mathbb{Z}_2], \quad Z_2 = \mathbb{C}, \quad Z_3 = \mathbb{C}, \quad Z_4 = \text{pt}.$$

The initial stacky fan $\Sigma = \text{conv}(0, a_2, a_4)$, with $X_\Sigma = [\mathbb{C}/\mathbb{Z}_2] \times [\mathbb{C}/\mathbb{Z}_2]$. And we have SOD and rank (volume) equation

$$X_\Sigma = Z_1 + Z_2 + Z_4, \quad 4 = 2 + 1 + 1.$$

Note that it is important that we have the point a_5 in the GIT problem. This makes $Z_2 = \mathbb{C}$.

If we change the problem by deleting a_5 . Then, a_i still generate $\mathbb{N} = \mathbb{Z}^2$, and we still have the above 4 minimal faces. However, we have different SOD components, in that $Z_2 = [\mathbb{C}/\mathbb{Z}_2]$. This results in a different SOD for X_Σ

$$X_\Sigma = Z_1 + Z_2, \quad 4 = 2 + 2.$$

△

3. MULTIPLICITY CONJECTURE FOR TORIC CY GIT

Consider a toric GIT problem $Q : (\mathbb{Z}^\vee)^N \rightarrow L^\vee$ that satisfies the Calabi-Yau condition, i.e. $\sum_{i=1}^N Q(e_i^\vee) = 0$.

In this case, GKZ defined a principal A -determinant E_A , an integer polynomial with N variables. E_A has many nice properties, its Newton polytope equals the secondary polytope, and its zero-loci descent to a hypersurface $\nabla_{GKZ} \subset L_{\mathbb{C}^*}^\vee$, the GKZ discriminant loci.

It is known that divisor ∇_{GKZ} has decomposition into irreducible components ∇_Γ ,

$$\nabla_{GKZ} = \sum_{\Gamma \in \mathcal{F}} m_\Gamma \nabla_\Gamma,$$

where \mathcal{F} denote the set of minimal faces, where $m_\Gamma = \text{rank}(K_0(Z_\Gamma))$ are certain multiplicities.

It is also known that the tropicalization of ∇_{GKZ} is the union of walls in Σ_{GKZ} . We will now decorate the walls by multiplicities in free abelian group generated by minimal faces.

Let $\mathbb{Z}\mathcal{F}$ denote the free abelian group generated by \mathcal{F} , with basis denoted by $[\Gamma]$. Then for each wall W , we define

$$[W] := \sum_{\Gamma} [Z_W : Z_\Gamma] \cdot [\Gamma] = \sum_{\Gamma} n_{W,\Gamma}^B \cdot [\Gamma].$$

Let \mathcal{W} denote the set of walls in $\Sigma_{GKZ}(\mathcal{Q})$. Let $\mathbb{Z}\mathcal{W}$ be the free abelian group generated by \mathcal{W} , with basis denoted as $\langle W \rangle$. Then any tropical complex supported on Σ_{GKZ} is equivalent to a non-negative linear combination of the walls. In particular, we have

$$\nabla_{\Gamma}^{trop} := \sum_{W \in \mathcal{W}} n_{W,\Gamma}^A \langle W \rangle$$

Definition 3.1. Let \mathcal{Q} be a toric CY GIT problem. A tropical complex is an element in $\mathbb{Z}\mathcal{W}$, and a decorated tropical complex is an element in $\mathbb{Z}(\mathcal{W} \times \mathcal{F})$. We define the A-model tropical complexes as

$$\nabla_{GKZ}^A = \sum_{\Gamma} \nabla_{\Gamma}^{trop} \cdot [\Gamma] = \sum_{\Gamma, W} n_{W,\Gamma}^A \langle W \rangle [\Gamma], \quad \nabla_{\Gamma}^A = \sum_W n_{W,\Gamma}^A \langle W \rangle$$

We define the B-model tropical complexes as

$$\nabla_{GKZ}^B = \sum_W \langle W \rangle [W] = \sum_{W,\Gamma} n_{W,\Gamma}^B \langle W \rangle [\Gamma], \quad \nabla_{\Gamma}^B = \sum_W n_{W,\Gamma}^B \langle W \rangle$$

We write $\nabla_{GKZ}^A(\mathcal{Q}), \nabla_{\Gamma}^A(\mathcal{Q}), \nabla_{GKZ}^B(\mathcal{Q}), \dots$ to emphasize the dependence on \mathcal{Q} .

Our goal is to show that $\nabla^A(\mathcal{Q}) = \nabla^B(\mathcal{Q})$, or more concretely for each Γ , $\nabla_{\Gamma}^A(\mathcal{Q}) = \nabla_{\Gamma}^B(\mathcal{Q})$. We first reduce the task to showing just the top dimensional minimal face.

Let $\mathcal{Q}_{\Gamma} : (\mathbb{Z}^{\vee})^{\Gamma} \rightarrow L_{\Gamma}^{\vee}$ denote the Coulomb problem for minimal face Γ . Let $\pi_{\Gamma} : L_{\mathbb{R}}^{\vee} \rightarrow (L_{\Gamma}^{\vee})_{\mathbb{R}}$. Define $\pi_{\Gamma}^* : \mathbb{Z}(\mathcal{W}_{\Gamma} \times \mathcal{F}_{\Gamma}) \rightarrow \mathbb{Z}(\mathcal{W} \times \mathcal{F})$. For a wall $W_{\Gamma} \in \mathcal{W}_{\Gamma}$, we define $\pi_{\Gamma}^*(\langle W_{\Gamma} \rangle) = \langle \pi^{-1}W_{\Gamma} \rangle = \sum_{W \in \mathcal{W}, \pi_{\Gamma}(W) = W_{\Gamma}} \langle W \rangle$. For any $\Gamma' \in \mathcal{F}_{\Gamma}$, since $\mathcal{F}_{\Gamma} \subset \mathcal{F}$, we have $\pi_{\Gamma}^*([\Gamma']) = [\Gamma']$.

Proposition 3.2. $\nabla_{\Gamma}^A(\mathcal{Q}) = \pi_{\Gamma}^*(\nabla_{\Gamma}^A(\mathcal{Q}_{\Gamma}))$ and $\nabla_{\Gamma}^B(\mathcal{Q}) = \pi_{\Gamma}^*(\nabla_{\Gamma}^B(\mathcal{Q}_{\Gamma}))$.

Proof. On the A-side, this follows from relation from the complex hypersurfaces $\nabla_{\Gamma}(\mathcal{Q}) = \pi_{\Gamma}^{-1}(\nabla_{\Gamma}(\mathcal{Q}_{\Gamma}))$.

On the B-side, we know from Lemma 2.18 that $[Z_W : Z_{\Gamma}] = [Z_{W_{\Gamma}} : Z_{\Gamma/\Gamma}]$. Thus

$$\nabla_{\Gamma}^B(\mathcal{Q}) = \sum_W [Z_W : Z_{\Gamma}] \langle W \rangle = \sum_{W:W \parallel \Gamma} [Z_{W_{\Gamma}} : Z_{\Gamma/\Gamma}] \langle W \rangle = \sum_{W_{\Gamma} \in \mathcal{W}_{\Gamma}} [Z_{W_{\Gamma}} : Z_{\Gamma/\Gamma}] \pi_{\Gamma}^*(\langle W_{\Gamma} \rangle) = \pi_{\Gamma}^*(\nabla_{\Gamma}^B(\mathcal{Q}_{\Gamma})).$$

□

Recall we have a rank map $\text{rank} : \mathbb{Z}\mathcal{F} \rightarrow \mathbb{Z}$, where

$$\text{rank}([\Gamma]) := \text{rank}(K_0(Z_{\Gamma})).$$

And we extends by linearity to $\text{rank} : \mathbb{Z}(\mathcal{W} \times \mathcal{F}) \rightarrow \mathbb{Z}(\mathcal{W})$.

Horja-Katzarkov proved a numerical version of the desired theorem

Proposition 3.3 ([HK22], Theorem 3.5). $\text{rank}(\nabla_{GKZ}^A) = \text{rank}(\nabla_{GKZ}^B)$

Proof. In loc.cit, we have $\text{rank}([W]) = \sum_{\Gamma} n_{W,\Gamma}^A \text{rank}([\Gamma])$. Hence

$$\text{rank}(\nabla_{GKZ}^B) = \sum_W \langle W \rangle \text{rank}([W]) = \sum_{W,\Gamma} n_{W,\Gamma}^A \text{rank}([\Gamma]) = \text{rank}(\nabla_{GKZ}^A).$$

□

Now we are ready to prove our main theorem.

Theorem 3.4. For any toric CY GIT problem \mathcal{Q} , we have $\nabla_{GKZ}^A(\mathcal{Q}) = \nabla_{GKZ}^B(\mathcal{Q})$, i.e. $n_{W,\Gamma}^A = n_{W,\Gamma}^B$ for any wall W and minimal faces Γ .

Proof. We induct on the rank of L . In the case $\text{rank } L = 0$, there is no wall, hence nothing to prove. Assume the case for $\text{rank } L < n$ is proven, and we have $\text{rank } L = n$.

Let Γ_0 be largest minimal face, concretely $\Gamma_0 = \{i : q_i \neq 0\}$. For any minimal face $\Gamma \neq \Gamma_0$, we have $\text{rank } L_\Gamma < n$, hence by induction $\nabla_\Gamma^A(Q_\Gamma) = \nabla_\Gamma^B(Q_\Gamma)$. By Proposition 3.2, we have $\nabla_\Gamma^A(Q) = \nabla_\Gamma^B(Q)$ for all minimal faces $\Gamma \neq \Gamma_0$. Thus, we have

$$\nabla_{GKZ}^A(Q) - \nabla_{GKZ}^B(Q) = \sum_{\Gamma} (\nabla_\Gamma^A(Q) - \nabla_\Gamma^B(Q))[\Gamma] = (\nabla_{\Gamma_0}^A(Q) - \nabla_{\Gamma_0}^B(Q))[\Gamma_0]$$

Now, take rank on both sides, using $\text{rank}([\Gamma_0]) = \text{rank}(K_0 \text{Coh}(Z_{\Gamma_0})) = \text{rank}(K_0 \text{Coh}(\mathbb{A}^{\Gamma_0^c})) = 1 \neq 0$, and Horja-Katzarkov result Proposition 3.3, we get

$$0 = \nabla_{\Gamma_0}^A(Q) - \nabla_{\Gamma_0}^B(Q).$$

Thus $\nabla_{GKZ}^A(Q) = \nabla_{GKZ}^B(Q)$. \square

3.1. Examples.

Example 3.5. Consider the example in [KS22, Example 3.7] with an additional weight $-\det V = (-1, -2)$ (marked in blue) to get a toric CY GIT problem. There are three relevant subspaces:

$$H_0 = L_{\mathbb{R}}^{\vee}, \quad H_1 = \text{span}_{\mathbb{R}}(q_3, q_4, q_6), \quad H_2 = \{0\}$$

We compute the tropical complexes in this CY problem.

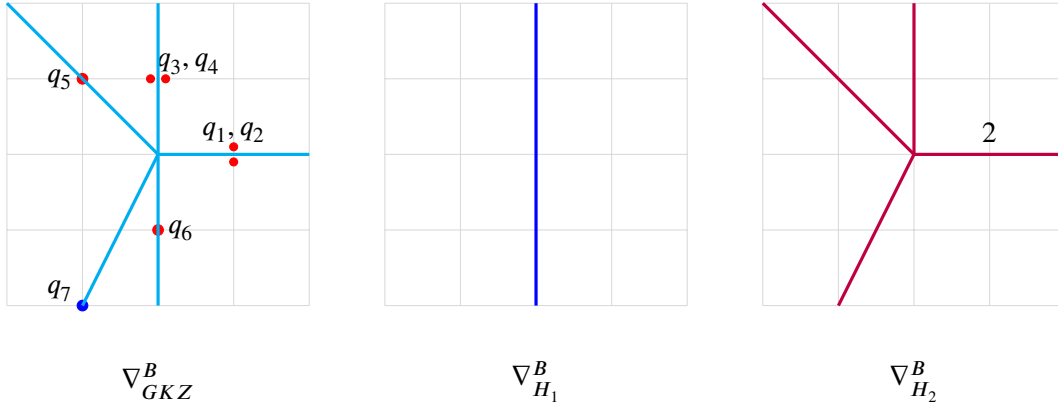


FIGURE 4. Decomposition of $\nabla_{GKZ}^B = \sum_H \nabla_H^B$, where H labels the relevant subspaces.

(H_0) Since the problem is CY, every chamber is a minimal chamber. The Coulomb problem is the trivial problem $\mathbb{Z}^0 \xrightarrow{0} \mathbb{Z}^0$, and $\nabla_{H_0}^B = \emptyset$.

(H_1) From the Higgs problem $\Gamma_1^c = \{1, 3, 4\}$ we know $Z_{H_1} = \mathbb{A}^1$. The Coulomb problem for Γ_1^c is the Atiyah flop $Q = (1, 1, -1, -1)$. We claim $\nabla_{H_1}^B$ has multiplicity 1 on the contributing walls, which is a result proven in [KS22] when $\text{rank } L = 1$. (Indeed, any irreducible hypersurface in \mathbb{C}^* is a point with multiplicity 1).

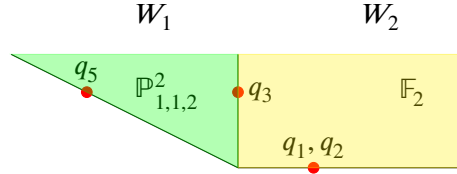
(H_2) In this case, the Coulomb problem is the entire GIT problem. The picture on the right is the tropical complex $\nabla_{H_2}^B$ where one of the walls has multiplicity 2. These multiplicities can be computed by looking at the Higgs problem. For the wall with multiplicity 2, $Z_W = \mathbb{P}^1$ and $Z_\Gamma = \text{pt}$, and $\text{Coh}(\mathbb{P}^1) = \langle \text{Coh}(\text{pt}), \text{Coh}(\text{pt}) \rangle$. For other walls, $Z_W = \text{pt}$ and $Z_\Gamma = \text{pt}$. \triangle

Example 3.6. *This example shows that the conjectural formula [HLS16, Remark 4.7] is false.*

Consider toric CY GIT problem $Q : \mathbb{Z}^5 \rightarrow \mathbb{Z}^3$, with weight matrix

$$Q = \begin{pmatrix} 1 & 1 & 0 & 0 & -n & n-2 \\ 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$

and let q_i be the column vectors. Assume $n \geq 2$. Consider the xy plane, which contains weights $q_1 = q_2 = (1, 0)$, $q_3 = (0, 1)$, $q_5 = (-n, 1)$ (omitting the 3rd coordinate). The following is a picture of the xy plane for $n = 2$.



There are two phases on this plane, corresponding to weighted projective space $Z_{W_1} = \mathbb{P}^2_{1,1,n}$ ($W_1 = \text{cone}(q_3, q_5)$) and Hirzebruch surface $Z_{W_2} = \mathbb{F}_n$ (for $W_2 = \text{cone}(q_1, q_3)$). We have SOD multiplicities (see also the example in the introduction of [BDF⁺15])

$$[\mathbb{P}^2_{1,1,n}] = (n+2)[pt], \quad [\mathbb{F}_n] = 4[pt].$$

This means a local curve transverse to wall W_1 will intersect ∇_{GKZ} at $n+2$ points, and a local curve transverse to wall W_2 will intersect ∇_{GKZ} at 4 points. On the other hand, the formula given by [HLS16] would always predict 4 (since the 4 weights q_i on the xy plane are all primitive vectors), no matter which wall the curve intersects.

One can also verify that ∇_{GKZ}^B here is a balanced tropical complex. Consider a local tropical curve C with tangent direction $(0, 0, 1)$ intersects ∇_{GKZ}^B near point q_3 . If C intersects at W_1 , then we get intersection multiplicity

$$C \cdot \nabla_{GKZ}^B = (C \cdot \langle W_1 \rangle) \cdot \text{rank}(Z_{W_1}) = 1 \cdot (n+2) = n+2,$$

where tropical intersection number $C \cdot W$ is defined as the unsigned volume $|\det(v_1 \wedge v_2 \wedge v_3)|$, where $v_i \in \mathbb{Z}^3$ and v_1 is primitive generator of C and v_2, v_3 are generators of the sublattice $\text{span}_{\mathbb{R}}(W) \cap \mathbb{Z}^3$. If we move the curve to the right, C intersects with W_2 and an additional wall $W_3 = \text{cone}(q_3, q_6)$, and we get

$$C \cdot \nabla_{GKZ}^B = (C \cdot W_2) \text{rank}(Z_{W_2}) + (C \cdot W_3) \text{rank}(Z_{W_3}) = 1 \cdot 4 + (n-2) \cdot 1 = n+2,$$

where $Z_{W_3} = pt$.

△

APPENDIX A. ADAPTATION OF GKZ'S SETUP

In the paper, we quoted results from [GKZ] whereas the setup do not quite match. In the appendix, we show that with minor modification, the generalized case reduces to the original GKZ setup, hence the GKZ conclusion still applies. In GKZ, one start with a finite set $A \subset \mathbb{Z}^{k-1}$, where as in our setup, we can have a map $A : \mathbb{Z}^N \rightarrow \mathbb{N}$, where \mathbb{N} can have torsion and $a_i = A(e_i)$ can coincide with each other. Here in the appendix we show that the difference in the setup is minor.

First, we generalize A to be a 'multi-set' while keeping \mathbb{N} a lattice. Suppose we have $A : \mathbb{Z}^N \rightarrow \mathbb{Z}^{k-1} \times \mathbb{Z}$, and let A' denote the image set $\{A(e_i)\}$, then we have a map $s : [N] \rightarrow A'$. We claim that the GKZ

discriminant loci for A is a pull-back of that for A' , where the fiber is a product of pair-of-pants. Indeed, consider the example,

$$A : \mathbb{Z}^3 \rightarrow \mathbb{Z}, \quad e_i \mapsto 1$$

then the corresponding $W = (c_1 + c_2 + c_3)z$, and the discriminants happens exactly at $c_1 + c_2 + c_3 = 0$, which after quotient by \mathbb{C}^* become the pair-of-pants in $L_{\mathbb{C}^*}^{\vee} = (\mathbb{C}^*)^2$.

Next, we generalize the case where N have torsion. Suppose $Q : (\mathbb{Z}^N)^{\vee} \rightarrow L^{\vee}$ has finite cokernel. Define $\hat{L}^{\vee} = \text{im}(Q) \subset L^{\vee}$, then we have

$$0 \rightarrow \hat{L}^{\vee} \rightarrow L^{\vee} \rightarrow M_1 \rightarrow 0. \quad (\text{A.1})$$

Dualize, we get

$$0 \rightarrow L \rightarrow \hat{L} \rightarrow N_{\text{tors}} \rightarrow 0 \quad (\text{A.2})$$

where we used that $N_{\text{tors}} = \text{Ext}^1(M_1, \mathbb{Z})$. In other words, \hat{L} is the saturation of L in \mathbb{Z}^N .

$$0 \rightarrow (M_0)_{\mathbb{C}^*} \rightarrow (\mathbb{C}^*)^N \rightarrow \hat{L}_{\mathbb{C}^*}^{\vee} \rightarrow 1. \quad (\text{A.3})$$

Apply $\text{Hom}(-, \mathbb{C}^*)$ to Eq (A.1),

$$0 \rightarrow M_1 \rightarrow \hat{L}_{\mathbb{C}^*}^{\vee} \rightarrow L_{\mathbb{C}^*}^{\vee} \rightarrow 1 \quad (\text{A.4})$$

Thus, although the fiber of $(\mathbb{C}^*)^N \rightarrow L_{\mathbb{C}^*}^{\vee}$ has $|M_1|$ disconnected component of $(M_0)_{\mathbb{C}^*}$, and the point $b \in L_{\mathbb{C}^*}^{\vee}$ is 'bad' if the function W restrict any component is singular (at finite place or at infinity), we can first work with $\hat{Q} : (\mathbb{Z}^N)^{\vee} \rightarrow \hat{L}^{\vee}$, and get $\nabla_{GKZ}(\hat{Q}) \subset \hat{L}_{\mathbb{C}^*}^{\vee}$, then pushforward it along $\hat{L}_{\mathbb{C}^*}^{\vee} \rightarrow L_{\mathbb{C}^*}^{\vee}$ to get $\nabla_{GKZ}(\hat{Q})$.

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