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# Gravitational fermion production in inflationary cosmology

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ABSTRACT

We revisit the gravitational production of massive Dirac fermions in inflationary cosmology with a focus on clarifying the analytic computation of the particle number density in both the large and the small mass regimes. For the case in which the masses of the gravitationally produced fermions are small compared to the Hubble expansion rate at the end of inflation, we obtain a universal result for the number density that is nearly independent of the details of the inflationary model. The result is identical to the case of conformally coupled scalars up to an overall multiplicative factor of order unity for reasons other than just counting the fermionic degrees of freedom.

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## 1. Introduction

Gravitational particle production (as reviewed e.g. in [1,2]) and string production (see e.g. [3–10]) are generic phenomena for quantum fields in a curved spacetime background and are analogs of particle creation in strong electric fields (see e.g. [11,12]). In the case of Friedmann–Robertson–Walker (FRW) cosmology without inflation, it was found [13–17] that the production of fermion and conformally coupled scalar fields near the radiation dominated (RD) universe singularity occurs when the particle masses $m$ are comparable to the Hubble expansion rate $H$, with a number density $n \sim m^3$ that dilutes as $a^{-3}$ due to expansion. The fractional relic density of these particles at the time of radiation–matter equality is $\Omega_X \sim (m_X/10^9\ \text{GeV})^{5/2}$ [18]. Hence, the requirement of $\Omega_X < 1$ puts an upper bound of $10^9$ GeV on the stable particle mass.[1]

In contrast, in inflationary cosmology the previously unbounded rapid growth of $H$ as one moves backward in time towards the RD singularity is replaced by a nearly constant $H_e$ during the quasi-de Sitter (dS) era. In such cases, the possibility of superheavy dark matter in a wide range of masses including $m > H_e$ was emphasized in [21,22]. In fact, natural superheavy dark matter candidates existed in the context of string phenomenology before the gravitational production mechanism was appreciated [23,24]. Furthermore, many extensions of the Standard Model also possess superheavy dark matter candidates (see e.g. [25–34]), which can have interesting astrophysical implications (see e.g. [29,35–39]). In such contexts, analytic relic density formulae have been computed in the heavy and the light mass regimes for conformally coupled scalars [40,41].

In this work, we turn our attention to the gravitational particle production of long-lived Dirac fermions in inflationary cosmology. Gravitational particle production of Dirac fermions has been studied numerically within the context of specific chaotic inflationary models [22]. Our purpose is to clarify the analytic computation and to derive a universal result for the light mass scenario that is nearly independent of the details of the inflationary model. Our result is identical up to an overall $O(1)$ multiplicative factor to that obtained for conformally coupled light scalar fields in [41], despite the fact that the Dirac structure naively imposes a different spectral (momentum scaling) property on the equations governing the particle production.[2] In comparison to the conformally coupled scalar case, no special non-renormalizable coupling to gravity nor possibility of tadpole instabilities concern the fermionic scenario in the light mass limit because the fermion kinetic operator is conformally invariant and fermions cannot obtain a non-vanishing vacuum expectation value.

We also derive the particle production spectrum for the heavy mass scenario and find it to be identical to the result of [40] (again up to an $O(1)$ multiplicative constant) despite a different momentum dependence of the starting point of the equations.

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[1] Physics quite similar to this is reported in [19,20].

[2] Although the aim of [41] is to consider a hybrid inflationary scenario, it also contains a universal result, Eq. (44), applicable to generic inflationary scenarios. There is also a misprint in [41] in stating that the situation is for minimal coupling rather than for conformal coupling.

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As expected, the heavy mass number density falls off exponentially. In contrast with the light mass limit, this case is sensitive to the details of the transition out of the inflationary era. To emphasize the simplicity and the novel analytic arguments of the light mass scenario, we relegate the heavy mass results to Appendices A–C.

It should be noted that the production of fermions in inflationary cosmology has been extensively considered during the recent past, but most analyses have focused on the non-gravitational interactions. For example, [42–49] focused on both numerical and analytic analyses of fermion production during preheating. [50] considered the production effects when the fermion mass passes through a zero during the quasi-dS phase. The effects of radiative corrections that modify the fermion dispersion relationship and its connection to particle production were considered in [51]. Gravitino production has also been considered by many authors (see e.g. [52–57]). The main thrust of this Letter differs in that it focuses on the minimal gravitational coupling and derives a simple bound analogous to Eq. (44) of [41]. Indeed, our results will aid in future investigations similar to [32] which would benefit from a more accurate simple analytic estimate of the dark matter abundance.

The outline of this work is as follows. In Section 2, we discuss the intuition behind the general formalism for the gravitational production of massive Dirac fermions in curved spacetime. In Section 3, we discuss the generic features of the spectrum and derive the main result, which is that for a given mode with comoving wave number $k$, the Bogoliubov coefficient magnitude $|\beta_k|^2 \sim O(1/2)$ if $H(\eta) > m$ when $k/a(\eta) \sim m$. We test this analytic result within a toy inflationary model in Section 4, and discuss the dependence on reheating and the implications for the relic density in Section 5. Finally, in Section 6 we summarize our results and present our conclusions. Appendix A contains a collection of useful results for fermionic Bogoliubov transformation computations. Appendix B contains a complementary argument (which relies more on the spinorial picture of the fermions) for the universality of the Bogoliubov coefficient in the light mass region. Appendix C contains the particle density spectrum for the heavy mass limit.

## 2. Fermion particle production: Background and intuition

To compute the particle production of Dirac fermions in curved spacetime, we follow the standard procedure as outlined for example in [1,2] to calculate the Bogoliubov coefficient $\beta_k$ between the in-vacuum corresponding to the inflationary adiabatic vacuum and the out-vacuum corresponding to the adiabatic vacuum defined at post-inflationary times. The details of this formalism and our conventions are presented in Appendix A, with the expression for $\beta_k$ given in Eq. (A.28).

However, to obtain a better intuitive picture of the particle production mechanism, here we present general physical arguments regarding the expected features of the spectrum. We begin by considering a Dirac fermion field $\Psi$ described by

$$\mathcal{L} = i\bar{\Psi}\gamma^{\mu}\nabla_{\mu}\Psi - m\bar{\Psi}\Psi \tag{1}$$

minimally coupled to gravity. As the action $S = \int d^4x\sqrt{g}\mathcal{L}$ is conformally invariant in the $\{m \to 0, \hbar \to 0\}$ limit (with $\delta g_{\mu\nu}(x) = -2\sigma(x)g_{\mu\nu}(x)$), physical quantities are necessarily independent of the FRW scale factor $a$ to leading order in $\hbar$. Hence, the leading $\hbar$ order Bogoliubov coefficient $\beta_k$ is zero in the $ma/k \to 0$ limit, since it is the metric that drives the particle production (i.e., it plays the role of the electric field in the analogy of particle creation by strong electric fields). This implies that particle production can only occur in significant quantities for non-relativistic modes.[3]

We next point out that the Dirac equation with a time-dependent mass term results in mixing between positive and negative frequency modes, similar to the case of the conformally coupled Klein–Gordon system with a time-dependent mass. To see this explicitly, consider the Dirac equation for the spinor mode functions $u_{A,B}$ that follows from Eq. (1):

$$i\partial_\eta \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} am & k \\ k & -am \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix}, \tag{2}$$

which is our Eq. (A.25) from Appendix A. Here $u_{A,B}$ span the complete solution space (they contain both approximate positive and negative frequency solutions in the adiabatic regime). Here we are working in conformal time, which is related to the comoving observer's proper time via $dt \equiv a(\eta)\,d\eta$. From Eq. (2), we see that the rotation matrix that diagonalizes the right-hand side is a function of the time-dependent quantity $am$. Hence, the Dirac equation *as a function of time* mixes approximate positive and negative frequency solutions leading to non-vanishing particle production.

To estimate the Bogoliubov coefficient, we can compute the effects of the time-dependent mixing matrix $\mathcal{U} \in O(2)$ as follows. We begin by inserting $1 = \mathcal{U}^T\mathcal{U}$ into Eq. (2) to obtain

$$i\mathcal{U}\partial_\eta\left[\mathcal{U}^T\mathcal{U}\begin{pmatrix} u_A \\ u_B \end{pmatrix}\right] = \mathcal{U}\begin{pmatrix} am & k \\ k & -am \end{pmatrix}\mathcal{U}^T\mathcal{U}\begin{pmatrix} u_A \\ u_B \end{pmatrix} \quad \Rightarrow \tag{3}$$

$$i\mathcal{U}\partial_\eta\mathcal{U}^T\begin{pmatrix} u'_A \\ u'_B \end{pmatrix} + i\partial_\eta\begin{pmatrix} u'_A \\ u'_B \end{pmatrix} = \begin{pmatrix} \sqrt{k^2+m^2a^2} & 0 \\ 0 & -\sqrt{k^2+m^2a^2} \end{pmatrix}\begin{pmatrix} u'_A \\ u'_B \end{pmatrix}, \tag{4}$$

in which the primed basis is defined to be

$$\begin{pmatrix} u'_A \\ u'_B \end{pmatrix} \equiv \mathcal{U}\begin{pmatrix} u_A \\ u_B \end{pmatrix}. \tag{5}$$

The Dirac equation is diagonal in the primed basis except for the appearance of the mixing term

$$\mathcal{U}\partial_\eta\mathcal{U}^T = \frac{a}{2}\begin{pmatrix} 0 & \frac{mHk_p}{k_p^2+m^2} \\ -\frac{mHk_p}{k_p^2+m^2} & 0 \end{pmatrix}, \tag{6}$$

with $k_p \equiv k/a$. From this result, we see that during inflation the mixing term approximately vanishes for a fixed comoving wave number $k$ as $a \to 0$, while after inflation it is the largest when $H$ is the largest. Using this result, it is straightforward to show that the Bogoliubov coefficients due to mixing take the following form:

$$\beta_k^{\mathrm{mix}} \sim \int dt\,\frac{mk_p}{k_p^2+m^2}He^{-2i\int dt\,\omega_k}, \tag{7}$$

in which $\omega_k = \sqrt{k_p^2+m^2}$. One may still ask whether there are any other sources of positive and negative frequency mixing since the diagonal terms of Eq. (4) are time dependent, just as conformally coupled scalar fields contain $\omega^2 = k^2 + m^2a^2$ in their mode equations. The answer is no *if* the fermionic particles are *defined* as modes that exactly satisfy the condition

$$i\partial_\eta\begin{pmatrix} u'_A \\ u'_B \end{pmatrix} = \begin{pmatrix} \sqrt{k^2+m^2a^2} & 0 \\ 0 & -\sqrt{k^2+m^2a^2} \end{pmatrix}\begin{pmatrix} u'_A \\ u'_B \end{pmatrix}. \tag{8}$$

[3] We neglect possible conformal symmetry breaking effects associated with preheating [44]. In that sense, there is a mild implicit model dependence here.

For example, the adiabatic vacuum positive frequency modes are defined to be

$$\begin{pmatrix} u'_A \\ u'_B \end{pmatrix} \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\int dt\sqrt{\frac{k^2}{a^2}+m^2}}. \tag{9}$$

Eq. (9) corresponds to a zeroth order adiabatic vacuum in which the adiabaticity parameter $\epsilon_A$ is defined as

$$\epsilon_A \equiv \frac{mHk_p}{(k_p^2+m^2)^{3/2}}, \tag{10}$$

in accordance with the usual conventions [1,17,21,58]. This parameter vanishes in the asymptotically far past (near when the in-vacuum is defined) and in the far future (near when the out-vacuum is defined). Eq. (9) coincides with

$$\begin{pmatrix} u_A \\ u_B \end{pmatrix} \to \begin{pmatrix} \sqrt{\frac{\omega+am}{2\omega}} \\ \sqrt{\frac{\omega-am}{2\omega}} \end{pmatrix} e^{-i\int^{\eta} d\eta'\,\omega} \tag{11}$$

in the basis of Eq. (2).

To summarize, the zeroth adiabatic order vacuum Bogoliubov coefficient is approximately given by Eq. (7). Compared to the conformally coupled bosonic case (see e.g. [40]), the long wavelength fermionic particle production is suppressed due to the appearance of $k_p$ in the numerator.

## 3. Light mass case and generic features of the spectrum

In this section, we present a universal result for the spectrum in the light mass scenario that is nearly independent of the details of the inflationary model. We will show that under a specific set of conditions, the Bogoliubov spectral amplitude (evaluated with observable particle state basis defined at time $t$) takes the approximate form

$$\left|\beta_k(t)\right|^2 \sim O(1/2). \tag{12}$$

An alternate argument emphasizing more of the spinorial nature of the fermions is presented in Appendix B.

For Eq. (12) to hold generically, the following conditions must simultaneously be satisfied. The fermions that are produced must be light (to be made precise below). After the end of inflation, the modes that are produced must become non-relativistic during the time when the expansion rate is the dominant mass scale. Finally, $t$ must be a time when particles with $k_p = k/a$ are non-relativistic.

The evolution of the relevant physical scales is shown for clarity in Fig. 1. Here $t_e$ denotes the time of the end of inflation, $t_m$ is defined by $H(t_m) = m$, and $t_k$ stands for the time when $k_p(t) = m$. The two conditions under which Eq. (12) holds are $t_m > t_k > t_i$ and $t > t_k$, in which $t_i$ marks the beginning of inflation (not shown in the figure).

To show this more explicitly, we begin by noting that the modes that can be significantly produced by the FRW expansion satisfy $k_p \lesssim m$, since relativistic modes are approximately conformally invariant. Furthermore, during the time that $k_p \lesssim m$, Eq. (7) takes the form

$$\beta_k(t) \sim \int^{t} dt' \frac{k_p(t')}{m} H(t') e^{-2i\int^{t'} dt''\,\omega_k(t'')}. \tag{13}$$

Let us consider Eq. (13) for the time period with $H(t') > m$, such that $H^2 > \omega_k^2$. Here we take $k$ to be consistent with $k_p \lesssim m$; more precisely, $ma(t) > k > ma(t_i)$, where $t_i$ is the time when the initial vacuum is defined, which is typically at the beginning of inflation.[4] In this regime, the largest contribution to $\beta_k$ arises from the time $t_k$ when $k_p = k/a$ is at its largest while remaining non-relativistic: i.e., $k/a(t_k) = m$. When these conditions are satisfied, Eq. (13) results in

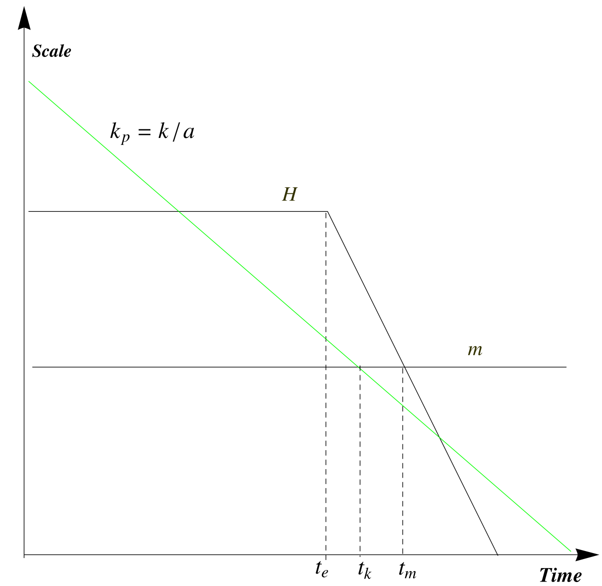


**Fig. 1.** The evolution of the physical scales $H(t)$, $k_p(t)$ and the corresponding time points. Modes with comoving wave number $k$ make the transition from relativistic to non-relativistic at time $t_k$. The Hubble rate drops below $m$ at $t_m$, and the end of inflation is at $t_e$.

$$\beta_k(t) \sim O\left(\frac{k/a(t_k)}{m}\right), \tag{14}$$

which we see is indeed of $O(1)$.

Our result indicates that the fermion creation saturates the Pauli exclusion principle, since $|\beta_k|^2$ represents the phase space density of the fermions created. The conditions leading to this result can be intuitively explained as follows. To have such a maximal production, we cannot excite $k_p \gg m$ modes because of conformal symmetry. Furthermore, we cannot excite $k_p \ll m$ modes because the violation of energy conservation is of order $F\Delta x \sim (Hk_p)k_p/(mH) \sim (k_p/m)k_p$, where $F$ is the force due to the expansion of the universe and $\Delta x$ is the distance over which the particle travels under this force. In addition, the force can act on the virtual particle only on a time scale shorter than the lifetime of the virtual state, which is of order $1/m$. This is equivalent to the condition that $H > m$ for this picture of particle production.

As Eq. (14) is independent of $H$, the result is insensitive to the details of the inflationary model. This insensitivity holds as long as the dominant contribution to $\beta_k(t)$ arises from the time period with $H(t')/m > 1$. However, $H(t')/m > 1$ clearly fails if $t' > t_m$. Thus, there is a mild inflationary model dependence, although it is largely insensitive. This is clear because the fermion mass can be made arbitrarily small compared to the expansion rate for any inflationary model. As we will see in Section 5, a stronger inflationary model dependence arises from the dilution factor $a(t_m)/a(t)$, which typically is a function of the reheating temperature.

Given that there is a general restriction that $|\beta_k|^2 < 1$ from quantization conditions, here $O(1)$ must mean a number less than

[4] The condition $k > ma(t_i)$ comes from the requirement of setting the adiabatic vacuum condition, which only applies for modes with subhorizon wavelengths.

unity.[5] To remind ourselves of this fact, we will refer to this $O(1) < 1$ number as $O(1/\sqrt{2})$. Putting all the conditions together with Eq. (14), we find

$$\left|\beta_k(t)\right|^2 \sim O(1/2) \quad \text{for } t_m > t_k > t_i \text{ and } t > t_k. \tag{15}$$

A more explicit restriction on $k$ that is consistent with the requirements of Eq. (15) can be written as follows:

$$ma(t_m) \gtrsim k > ma(t_i) \quad \text{and} \quad ma(t) \gtrsim k. \tag{16}$$

Eqs. (15) and (16) are the main results of this section.

For modes with $k > ma(t_m)$, $|\beta_k|^2$ is smaller since Eq. (13) is suppressed by an additional factor of $H/m$. The exact high $k$ behavior of $\beta_k$ is sensitive to the adiabatic order of the vacuum boundary condition as well as the details of the scale factor during the transition out of the quasi-dS era. However, what is generic is that the spectral contribution to the particle density no longer grows appreciably when $k > ma(t_m)$. Hence, we can define the critical momentum $k_* \equiv ma(t_m)$, which satisfies

$$k_*/a_e = (H_e/m)^{2/n_a} m, \tag{17}$$

where we have parameterized the energy density after the end of inflation as $\rho \propto a^{-n_a}$. Integrating over $d^3k/(2\pi a)^3$ to obtain the energy density of the fermions, for an order of magnitude estimate we can introduce a step function $\Theta(k_* - k)$ as follows:

$$\rho_\Psi(t) \sim 4 \times \frac{m}{4\pi^2}\frac{1}{a^3}\int dk\, k^2 \Theta(k_* - k), \quad t_{ma(t_i)} \ll t_m < t, \tag{18}$$

in which $t_{ma_i}$ is the time at which $k = ma(t_i)$. Assuming that the lower limit of Eq. (18) contributes negligibly to the integral, we obtain

$$\rho_\Psi(t) \sim 4 \times \frac{m^4}{12\pi^2}\left(\frac{a(t_m)}{a(t)}\right)^3, \tag{19}$$

which contains the mild inflationary scenario dependence discussed previously.

## 4. Example of fermion production in a toy inflationary model

To test the analytic estimation of Section 3, we now numerically compute the particle production in a toy inflationary model with instantaneous reheating occurs (i.e., in which the quasi-dS phase connects instantaneously to the RD phase). As is well known, such non-analytic models have unphysical large momentum behavior [1], which for our purposes can be dealt with simply by cutting off the integration of the spectrum. We find there is an upper bound on the fermion mass if $m < H_e$ during inflation, similar to the case of fermion production in pure RD cosmology [18]. We will turn to the more realistic case in which the inflationary era exits to a transient pressureless era during reheating in Section 5.

Let us consider a background spacetime which is initially dS with a Hubble constant $H_e$ that is followed by RD spacetime. Although the junction between the dS and RD eras is instantaneous, the scale factor $a(t)$ and the Hubble rate $H(t)$ are continuous across the junction. In particular, if we set the junction time at the conformal time $\eta = 0$ and we set the scale factor at the junction time to be $a_e$, the scale factor and Hubble rates can be written as

[5] The Bogoliubov coefficients satisfy $|\alpha_{\vec{k},s}|^2 + |\beta_{-\vec{k},s}|^2 = 1$, while Eq. (14) effectively neglects this constraint.

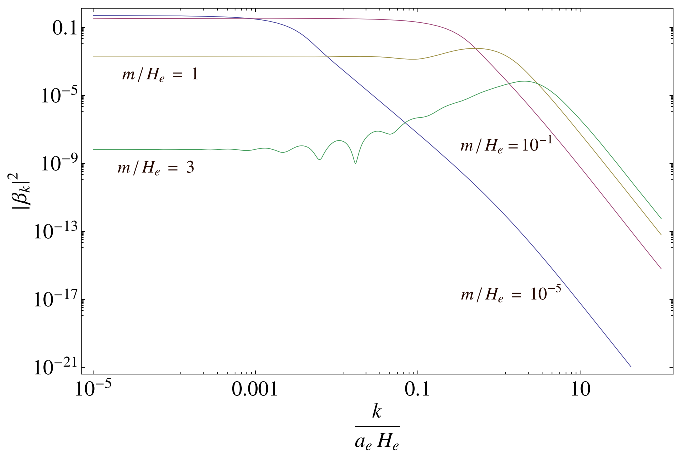


**Fig. 2.** The Bogoliubov coefficient amplitude $|\beta_k|^2$ as a function of $k/(a_e H_e)$ for various ratios of the fermion mass to the Hubble expansion rate during the dS era.

$$a(\eta) = \begin{cases} \left(\left(\frac{1}{a_e H_e} - \eta\right)H_e\right)^{-1} & \eta \leqslant 0 \quad (\text{dS}), \\ a_e^2 H_e\left(\eta + \frac{1}{a_e H_e}\right) & \eta > 0 \quad (\text{RD}), \end{cases}$$
$$H(\eta) = \begin{cases} H_e & \eta \leqslant 0 \quad (\text{dS}), \\ H_e\left(\frac{a_e}{a(\eta)}\right)^2 & \eta > 0 \quad (\text{RD}), \end{cases} \tag{20}$$

indicating that the leading discontinuity in $a$ occurs at second order in the conformal time derivative.

To compute $\beta_k$ using Eq. (A.28), it is necessary to fix the boundary conditions for the in-modes and the out-modes. For the in-modes, we require that in the infinite past, when a certain given mode's wavelength is within the horizon radius, its mode function must agree with the flat space positive frequency mode function. In other words, as $a(\eta) \to 0$,

$$\begin{pmatrix} u_A \\ u_B \end{pmatrix}^{\text{in}}_{k,\eta} \to \begin{pmatrix} \sqrt{\frac{\omega + a(\eta)m}{2\omega}} \\ \sqrt{\frac{\omega - a(\eta)m}{2\omega}} \end{pmatrix} e^{-i\int^\eta \omega(\eta')\,d\eta'}. \tag{21}$$

The in-modes' analytic expressions during the dS era thus take the form

$$\begin{pmatrix} u_A \\ u_B \end{pmatrix}^{\text{in}}_{k,\eta} = \begin{pmatrix} \sqrt{\frac{\pi}{4}\left(\frac{k}{aH_e}\right)} e^{i\frac{\pi}{2}\left(1 - i\frac{m}{H_e}\right)} H^{(1)}_{\frac{1}{2} - i\frac{m}{H_e}}\left(\frac{k}{aH_e}\right) \\ \sqrt{\frac{\pi}{4}\left(\frac{k}{aH_e}\right)} e^{i\frac{\pi}{2}\left(1 + i\frac{m}{H}\right)} H^{(1)}_{\frac{1}{2} + i\frac{m}{H_e}}\left(\frac{k}{aH_e}\right) \end{pmatrix} \tag{22}$$

where $H^{(1)}_\nu$ are Hankel functions of the first kind. Similarly, for the out-modes, as $k/a > H(\eta)$ in the RD era, we require the mode functions to agree with the flat space positive frequency mode functions, i.e., as $a(\eta) \to +\infty$,

$$\begin{pmatrix} u_A \\ u_B \end{pmatrix}^{\text{out}}_{k,\eta} \to \begin{pmatrix} \sqrt{\frac{\omega + a(\eta)m}{2\omega}} \\ \sqrt{\frac{\omega - a(\eta)m}{2\omega}} \end{pmatrix} e^{-i\int^\eta \omega(\eta')\,d\eta'}. \tag{23}$$

The out-mode analytic expressions during the RD era are given by

$$\begin{pmatrix} u_A \\ u_B \end{pmatrix}^{\text{out}}_{k,\eta} = \begin{pmatrix} e^{-\frac{\pi}{4}C} D_{-iC}\left(e^{i\pi/4}\sqrt{\frac{2m}{H(\eta)}}\right) \\ \sqrt{C} e^{-\frac{\pi}{4}C + i\frac{\pi}{4}} D_{-iC-1}\left(e^{i\pi/4}\sqrt{\frac{2m}{H(\eta)}}\right) \end{pmatrix}, \tag{24}$$

in which $C \equiv (k^2/a_e^2)/(2mH_e)$ characterizes the ratio of the momentum to the dynamical mass scale and the $D_\nu(x)$ are parabolic cylinder functions.

The numerical results for $|\beta_k|^2$ are shown as a function of $k/(a_e H_e)$ for various choices of the fermion masses in Fig. 2. From these results, we first note that it can be determined that for heavy masses $m > H_e$, e.g. $m/H_e = 1$ or 3, the infrared end of the spectrum behaves as $|\beta_k|^2 \sim (1 + \exp(2\pi m/H_e))^{-1}$. Further details of the heavy mass case are given in Appendix C. As the heavy mass situation is likely to be more sensitive to the abrupt transition approximation made in this section, we restrict our attention here to the light mass case in which $m < H_e$.

For the light mass case (e.g. $m/H_e = 10^{-5}$ in Fig. 2), we can see there are three ranges of $k$ that each have qualitatively different behavior. For $k/a_e > H_e$, the modes are still inside the horizon at the end of inflation, and the spectrum falls off as $|\beta_k|^2 \propto k^{-6}$. In contrast, for $\sqrt{mH_e} < k/a_e < H_e$, the modes are outside of the horizon at the end of inflation and remain relativistic at the time when $m = H(\eta)$ during RD. In this case, the spectrum falls off as $|\beta_k|^2 \propto k^{-4}$. Finally, for $k/a_e < \sqrt{mH_e}$, the modes are outside the horizon at the end of inflation and have become non-relativistic before $m = H(\eta)$ during RD. This results in a constant spectrum of $|\beta_k|^2 \approx \frac{1}{2}$, in agreement with the results of Section 3. Generically, if the scale factor $a(\eta)$ is sufficiently continuous [16,58], the spectrum will fall off in the ultraviolet region faster than $k^{-3}$, such that the total number density $n \sim \int d^3k\, |\beta_k|^2$ is finite. The majority of the contribution arises from the region in which $k/a_e < \sqrt{mH_e}$ where $|\beta_k|^2 \approx \frac{1}{2}$, as anticipated in Section 3. The number density for particle masses in the range of $m < 0.1H_e$ is numerically determined to be (recall that $\eta_m$ is defined by $H(\eta_m) = m$)

$$n(\eta) = 4 \times 0.005 m^3 \left(\frac{a(\eta_m)}{a(\eta)}\right)^3, \tag{25}$$

which again agrees with the analytic estimate of Eq. (19).

## 5. Inflationary reheating dependence

We now consider the more realistic situation in which there is a smooth transition region between the dS and RD phases. When inflation ends, there is typically a period of coherent oscillations ($a_e < a < a_{\rm rh}$) during which the equation of state is close to zero (see e.g. [59–61]). During that period, the expansion rate behaves as $H \propto a^{-3/2}$ and not $a^{-2}$ as during RD. This difference will lead to an effective dilution of the dark matter particles by the time RD is reached. More precisely, the fermion number density will be diluted as $1/a^3$ as long as the fermion plus anti-fermion number is approximately conserved. As we will see below, the integrated dilution is typically a function of the reheating temperature during inflation.

Accounting for the dilution, in this section we estimate the relic abundance of fermionic particles (fermions plus anti-fermions).[6] The dilution consideration breaks up naturally into two cases: $a_{\rm rh} > a(t_m)$ and $a_{\rm rh} < a(t_m)$. The former case corresponds to the situation in which the dominant particle production occurs during the reheating period, while the latter case corresponds to the complementary situation, which we will see below is unlikely to be physically important.

Let us begin with the case of $a_{\rm rh} > a_m$, which corresponds to

$$H_e \gg m > H_{\rm rh} \sim \frac{\sqrt{g_*}}{3}\frac{T_{\rm rh}^2}{M_p} = \left(\frac{T_{\rm rh}}{10^9\ {\rm GeV}}\right)^2 \left(\frac{g_*}{100}\right)^{1/2} {\rm GeV}, \tag{26}$$

where $H_{\rm rh}$ is the expansion rate at the time radiation domination is achieved. In this case, we have

$$\rho_\Psi(t_{\rm eq}) \sim 0.03 m^4 \left(\frac{H_{\rm rh}}{m}\right)^2 \left(\frac{a_{\rm rh}}{a_{\rm eq}}\right)^3, \tag{27}$$

in which we have used the fact that $H \propto a^{-3/2}$ during reheating. We thus find the relic abundance today of fermionic particles to be

$$\Omega_\psi h^2 \sim 3\left(\frac{m}{10^{11}\ {\rm GeV}}\right)^2 \left(\frac{T_{\rm rh}}{10^9\ {\rm GeV}}\right). \tag{28}$$

This matches Eq. (44) of [41] (up to a factor of order of a few, part of which is expected from counting fermionic degrees of freedom), which was computed in the context of conformally coupled scalar fields. The match is interesting because the analog of Eq. (7) for the conformally coupled scalar field case has a different $k/a$ dependence that converts into an effective $m$ dependence due to the conformal invariance of the fermionic kinetic term. Eq. (28) also agrees with the model-dependent numerical results of [22] up to a factor of 10. The related ratio of the fermion energy density to the radiation energy density at matter–radiation equality, $\rho_\Psi(t_{\rm eq})/\rho_R(t_{\rm eq})$, is the same as Eq. (28) up to a factor of 10.

For the case with $a_{\rm rh} < a_m$, we have

$$\rho_\Psi(t_{\rm eq}) \sim 0.03 m^4 \left(\frac{a_{\rm m}}{a_{\rm eq}}\right)^3, \tag{29}$$

which leads to

$$\frac{\rho_\Psi(t_{\rm eq})}{\rho_R(t_{\rm eq})} \sim \left(\frac{m}{10^8\ {\rm GeV}}\right)^{5/2} \left(\frac{g_*(t_m)}{100}\right)^{-1/4} \tag{30}$$

which up to an order of magnitude is $\Omega_\psi$. However, since this applies only for

$$m < \left(\frac{T_{\rm rh}}{10^9\ {\rm GeV}}\right)^2 \left(\frac{g_*}{100}\right)^{1/2} {\rm GeV}, \tag{31}$$

the relic abundance is negligible in this case. For example, an $m \sim 1$ GeV benchmark point will render $\Omega_\Psi \sim 10^{-20}$.

## 6. Conclusions

In this Letter, we revisited the gravitational production of massive Dirac fermions in inflationary cosmology. For the situation in which the fermions are light compared to the Hubble expansion rate at the end of inflation, we obtained the analytic result that the Bogoliubov coefficient amplitude $|\beta_k(t)|^2 \sim 1/2$ if $H > m$ when $k/a \sim m$, as summarized in Eqs. (15) and (16). We used this result to compute the relic density assuming that the gravitationally produced fermions are superheavy dark matter particles. In cases of phenomenological interest, the dark matter relic abundance depends on the reheating temperature, as given in Eq. (28). Up to a multiplicative overall factor of $O(1)$, this result is identical to that obtained for conformally coupled scalars in [41]. In the case that the fermions are heavy compared to the Hubble expansion rate at the end of inflation, the relic abundance is given by Eq. (C.7).

It is also of interest to consider the isocurvature behavior of the gravitationally produced fermions in the case that they have suitable long-range non-gravitational interactions. Work along these lines is currently in progress [62].

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[6] This requires the fermion self-annihilation cross section rate to be smaller than the expansion rate throughout its history. Such weak interactions generically can be achieved for sufficiently large particle masses [21], which are allowed as long as the inflationary scale is sufficiently large.

## Appendix A. Formalism and conventions

Here we follow the strategy outlined in the classical review paper of DeWitt [2]. Consider the action of a four-component Dirac spinor in curved spacetime:

$$S=\int d^4x\sqrt{|g(x)|}\bar{\Psi}\left(i\gamma^a\nabla_{e_a}-m\right)\Psi \tag{A.1}$$

in which the gamma matrices $\gamma^a$ are chosen to be in the Dirac basis

$$\gamma^0=\begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix},\qquad \gamma^i=\begin{pmatrix} 0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix} \tag{A.2}$$

to simplify the derivation of the second order differential equation of the spinor mode functions. Extremizing the action with respect to $\delta\bar{\Psi}$ and $\delta\Psi$ yields the equations of motion:

$$\left(i\gamma^\alpha\nabla_{e_\alpha}-m\right)\Psi=0,\qquad \nabla_{e_a}\bar{\Psi}\left(-i\gamma^a\right)-\bar{\Psi}m=0. \tag{A.3}$$

The solution space can be endowed with a scalar product as

$$(\Psi_1,\Psi_2)_\Sigma=\int d\Sigma\, n_\mu e_a^\mu\bar{\Psi}_1\gamma^a\Psi_2 \tag{A.4}$$

in which $\Sigma$ is an arbitrary space-like hypersurface, $d\Sigma$ is the volume 3-form on this hypersurface computed with the induced metric, and $n_\mu$ is the future-pointing time-like unit vector normal to $\Sigma$. The current conservation condition

$$\nabla_{e_a}\left(\bar{\Psi}_1\gamma^a\Psi_2\right)=0 \tag{A.5}$$

implies the integral in the scalar product is independent of the choice of $\Sigma$. The conjugation map can also be defined in the solution space as $\Psi\mapsto -i\gamma^2\Psi^*$, which induces a pairing in the solution space.

Based on the scalar product and the conjugation map, one can construct an orthonormal basis for the solution space. It can be written as $\{U_i, V_i\equiv -i\gamma^2U_i^*\}$ ($i$ labels different solutions), with

$$(U_i,U_j)=\delta_{ij},\qquad (U_i,V_j)=0. \tag{A.6}$$

The Heisenberg picture field operator $\Psi(x)$ can then be expanded in this basis as follows:

$$\Psi(x)=\sum_i a_iU_i+b_i^\dagger V_i, \tag{A.7}$$

in which the canonical anticommutation relations imposed on equal-time surfaces and the orthonormality of the mode functions ensures that

$$\{a_i,a_j^\dagger\}=\delta_{ij},\qquad \{b_i,b_j^\dagger\}=\delta_{ij}. \tag{A.8}$$

The vacuum state is defined by $a_i|\text{vac}\rangle=b_i|\text{vac}\rangle=0$. The full Hilbert space can then be constructed as usual by applying the creation operators $a_i^\dagger$ and $b_i^\dagger$ to the vacuum state.

However, the choice of the orthonormal basis $\{U_i,V_i\}$ is not unique. Consider a different orthonormal basis $\{\tilde{U}_i,\tilde{V}_i\}$, which is related to the original basis as follows:

$$\tilde{U}_i=\sum_j\alpha_{ij}U_j+\beta_{ij}V_j,\qquad \tilde{V}_i=\sum_j\alpha_{ij}^*V_j+\beta_{ij}^*U_j. \tag{A.9}$$

The Bogoliubov coefficients $\alpha_{ij}$ and $\beta_{ij}$ can be extracted as

$$\beta_{ij}=(V_j,\tilde{U}_i),\qquad \alpha_{ij}=(U_j,\tilde{U}_i) \tag{A.10}$$

Note that the orthonormality relation on $\{U_i,V_i\}$ and $\{\tilde{U}_i,\tilde{V}_i\}$ implies the following relation:

$$\begin{pmatrix}\alpha & \beta\\ \beta^* & \alpha^*\end{pmatrix}^*\begin{pmatrix}\alpha & \beta\\ \beta^* & \alpha^*\end{pmatrix}^T=\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}. \tag{A.11}$$

Using $\Psi=\sum_i a_iU_i+b_i^\dagger V_i=\sum_i\tilde{a}_i\tilde{U}_i+\tilde{b}_i^\dagger\tilde{V}_i$, the following relation is obtained:

$$\begin{pmatrix}\tilde{a}\\ \tilde{b}^\dagger\end{pmatrix}=\begin{pmatrix}\alpha^* & \beta^*\\ \beta & \alpha\end{pmatrix}\begin{pmatrix} a\\ b^\dagger\end{pmatrix}. \tag{A.12}$$

Hence, the two mode functions result in inequivalent vacua. To see this more explicitly, consider the expectation value of the occupation number operator $\tilde{a}_i{}^\dagger\tilde{a}_i$ with respect to the vacuum defined using the $a_i, b_i$ operators:

$$\langle\text{vac}|\tilde{a}_i{}^\dagger\tilde{a}_i|\text{vac}\rangle=\sum_j|\beta_{ij}|^2. \tag{A.13}$$

The vacuum state corresponding to one definition thus is an excited state in the other definition.

We turn now to FRW spacetime, in which the metric is conformally flat:

$$ds^2=g_{\mu\nu}\,dx^\mu\,dx^\nu=a(x_0)^2\eta_{\mu\nu}\,dx^\mu\,dx^\nu. \tag{A.14}$$

Since the action of Eq. (A.1) is covariant under Weyl transformations:

$$g_{\mu\nu}=\Omega^2(x)\tilde{g}_{\mu\nu},\qquad \Psi=\Omega(x)^{-3/2}\tilde{\psi},\qquad e_a{}^\mu=\Omega(x)^{-1}\tilde{e}_a{}^\mu, \tag{A.15}$$

a Weyl transformation with $\Omega(x)=a(x_0)$ can be used to rewrite the action as follows:

$$S=\int d^4x\,\bar{\psi}\left(i\gamma^\mu\partial_\mu-a(\eta)m\right)\psi, \tag{A.16}$$

where $\eta$ is the conformal time and $\psi$ is the rescaled spinor field. The equation of motion now takes the form

$$\left(i\gamma^\mu\partial_\mu-a(\eta)m\right)\psi=0. \tag{A.17}$$

The solution space is spanned by the orthonormal basis $\{U_{\vec{k},r},V_{\vec{k},r}\}$, which can be written as follows:

$$U_{\vec{k},r}(\eta,\vec{x})=\frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}}\begin{pmatrix} u_{A,k,\eta}h_{\hat{k},r}\\ ru_{B,k,\eta}h_{\hat{k},r}\end{pmatrix} \tag{A.18}$$

$$\equiv\frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}}\begin{pmatrix} u_{A,k,\eta}\\ ru_{B,k,\eta}\end{pmatrix}\otimes h_{\hat{k},r}, \tag{A.19}$$

in which $\hat{k}$ is the unit vector in the $\vec{k}$ direction ($\hat{k}=\hat{e}_z$ if $\vec{k}=0$), and $h_{\hat{k},r}$ is a 2-component complex column vector (called the helicity 2-spinor) that satisfies

$$\hat{k}\cdot\vec{\sigma}h_{\hat{k},r}=rh_{\hat{k},r},\qquad r=\pm1. \tag{A.20}$$

More concretely, if $\hat{k}=(\theta,\phi)$ in spherical coordinates, then the normalization factor can be chosen such that

$$h_{\hat{k},+1}\equiv\begin{pmatrix}\cos\frac{\theta}{2}e^{-i\phi}\\ \sin\frac{\theta}{2}\end{pmatrix},\qquad h_{\hat{k},-1}\equiv\begin{pmatrix}\sin\frac{\theta}{2}e^{-i\phi}\\ -\cos\frac{\theta}{2}\end{pmatrix}. \tag{A.21}$$

One can easily check that due to this phase convention

$$-i\sigma^2(h_{\hat{k},r})^*=-re^{-ir\phi}h_{\hat{k},-r},\qquad h_{-\hat{k},r}=-h_{\hat{k},-r}. \tag{A.22}$$

Using the above relations, one obtains

$$V_{\vec{k},r}(\eta,\vec{x})=\frac{e^{-i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}}\begin{pmatrix} -u_{B,k,\eta}^*\\ ru_{A,k,\eta}^*\end{pmatrix}\otimes h_{-\hat{k},r}\cdot\left(e^{-ir\phi}\right). \tag{A.23}$$

The normalization of the mode functions implies

$$h^{\dagger}_{\hat{k},r}h_{\hat{k},s}=\delta_{rs},\qquad |u_{A,k,\eta}|^2+|u_{B,k,\eta}|^2=1. \tag{A.24}$$

With this ansatz, Eq. (A.17) simplifies as follows:

$$i\partial_\eta\begin{pmatrix}u_A\\u_B\end{pmatrix}=\begin{pmatrix}am & k\\ k & -am\end{pmatrix}\begin{pmatrix}u_A\\u_B\end{pmatrix}. \tag{A.25}$$

Let $\tilde{U}_{\vec{k},s}$ be another basis in the form of Eq. (A.19). Due to the orthogonality of $h_{\hat{k},r}$ and $e^{i\vec{k}\cdot x}$, $\tilde{U}_{\vec{k},s}$ can only be a linear combination of $U_{\vec{k},s}$, $V_{-\vec{k},s}$:

$$\tilde{U}_{\vec{k},s}=\alpha_{(\vec{k},s)(\vec{k},s)}U_{\vec{k},s}+\beta_{(\vec{k},s)(-\vec{k},s)}V_{-\vec{k},s}. \tag{A.26}$$

The Bogoliubov coefficients are extracted using the scalar product of the mode functions evaluated at time $\eta$ as follows:

$$\alpha_{(\vec{k},s)(\vec{k},s)}=u^*_{A,k,\eta}\tilde{u}_{A,k,\eta}+u^*_{B,k,\eta}\tilde{u}_{B,k,\eta}, \tag{A.27}$$

$$\beta_{(\vec{k},s)(-\vec{k},s)}=e^{-is\phi(\hat{k})}(u_{A,k,\eta}\tilde{u}_{B,k,\eta}-u_{B,k,\eta}\tilde{u}_{A,k,\eta}). \tag{A.28}$$

Since we will only consider $|\beta_k|^2$ in this work, we can drop the $e^{-is\phi(\hat{k})}$ factor in the $\beta_k$ definition without loss of generality. Here one of the bases (corresponding to the Heisenberg state of the universe) is specified by asymptotic conditions such as the Bunch–Davies boundary condition as the in-vacuum (see e.g. Eq. (21).) Similarly, the other basis is the observable operator basis as specified by asymptotic conditions at late times, which is referred to as the out-vacuum.

## Appendix B. Demonstration that $|\beta_k|^2 \sim \frac{1}{2}$ for small $k$

We begin with the determination of $\beta_k$ from Eq. (A.28) evaluated at very late times when the out-modes can be directly replaced by their asymptotic values. In the limit in which $am/k\to\infty$, we see that we then only need to find the asymptotic values of the in-modes:

$$\begin{aligned}|\beta_k|&=\left|u^{\text{out}}_{A,k,\eta}u^{\text{in}}_{B,k,\eta}-u^{\text{out}}_{B,k,\eta}u^{\text{in}}_{A,k,\eta}\right|\\&=\left|\sqrt{\frac{\omega+am}{2\omega}}u^{\text{in}}_{B,k,\eta}-\sqrt{\frac{\omega-am}{2\omega}}u^{\text{in}}_{A,k,\eta}\right|\\&=\lim_{\eta\to\infty}\left|u^{\text{in}}_{B,k,\eta}\right|.\end{aligned} \tag{B.1}$$

Let us consider the evolution equations as given in Eq. (A.25) with boundary conditions as given in Eq. (21). For concreteness, we choose a time $\eta_i$ that is early enough such that $u_A(\eta_i)\approx u_B(\eta_i)\approx\frac{1}{\sqrt{2}}$. The system can be formally solved to obtain

$$\begin{pmatrix}u_A\\u_B\end{pmatrix}_f=T\exp\left\{-i\int d\Phi\,\sigma(\theta)\right\}\begin{pmatrix}u_A\\u_B\end{pmatrix}_i \tag{B.2}$$

in which $\omega\cos\theta=k$, $\omega\sin\theta=am$, $\omega\,d\eta=d\Phi$, and $\sigma(\theta)=\sigma_1\cos\theta+\sigma_3\sin\theta$ $(0\leqslant\theta\leqslant\pi/2)$. The time evolution is thus expressed as a series of infinitesimal $SU(2)$ rotations that act successively on the complex vector $u\equiv(u_A\ u_B)$.

For fixed $\theta$, the evolution corresponds to precession about the axis defined by $\sigma(\theta)$. However, throughout the evolution of the universe, $\sigma(\theta)$ evolves from its initial direction along $\sigma_1$ $(am\ll k)$ to its final direction along $\sigma_3$ $(am\gg k)$. If the switching of the axis is much faster than the precession time scale, $u$ remains in the $xy$-plane and rotates around the new axis $\sigma_3$, while if the switching is much slower compared with the precession time scale, $u$ adheres closely to the rotation axis and thus ends up in the $\sigma_3$ direction. The time scale of the axis switching is given by the Hubble expansion rate, since the universe needs to expand several e-folds for $am$ to overtake $k$, while the time scale of the precession is given by the physical frequency $\omega/a$, which is on the order of $m$ during the transition. Hence, fast transitions occur when $m\ll H$, for which $|u_B|^2$ stabilizes at $\frac{1}{2}$ and $|\beta_k|^2=\frac{1}{2}$. After $H(\eta)$ drops below $m$, only slow transitions occur and $|\beta_k|^2$ is small.

## Appendix C. Heavy mass case ($m > H_e$)

As we expect the particle production spectrum $|\beta_k|^2$ to be exponentially suppressed by $m/H$, we can adopt a similar approach as the heavy mass scalar case [40] to look for a one-pole approximation to the time integral that determines $\beta_k$. We shall consider the time-dependent Bogoliubov coefficients between the in-modes and the zeroth adiabatic modes with boundary conditions such that

$$\begin{pmatrix}u_A\\u_B\end{pmatrix}^{(\eta_1)}_{k,\eta=\eta_1}=\begin{pmatrix}\sqrt{\frac{\omega+am}{2\omega}}\\\sqrt{\frac{\omega-am}{2\omega}}\end{pmatrix}. \tag{C.1}$$

In the above, the superscript $(\eta_1)$ indicates the time that the boundary conditions are imposed. The in-modes can be decomposed into the zeroth adiabatic mode basis as follows:

$$\begin{pmatrix}u_A\\u_B\end{pmatrix}^{\text{in}}_{k,\eta_1}=\alpha_k^{in-(\eta_1)}\begin{pmatrix}u_A\\u_B\end{pmatrix}^{(\eta_1)}_{k,\eta_1}+\beta_k^{in-(\eta_1)}\begin{pmatrix}-u_B^*\\u_A^*\end{pmatrix}^{(\eta_1)}_{k,\eta_1}. \tag{C.2}$$

For $\eta_1\to\infty$, the instantaneous-modes will coincide with the out-modes up to an overall phase, and hence

$$|\beta_k|=\lim_{\eta_1\to\infty}\left|\beta_k^{in-(\eta_1)}\right|. \tag{C.3}$$

Inserting this decomposition into Eq. (A.25) (and writing $\alpha_k^{in-(\eta_1)}$ as $\alpha_k(\eta_1)$, etc., for notational simplicity) results in

$$\begin{aligned}\dot{\alpha}_k(\eta_1)&=-\frac{mk}{2\omega^2}\dot{a}e^{2i\int^{\eta_1}d\eta\,\omega(\eta)}\beta_k(\eta_1),\\\dot{\beta}_k(\eta_1)&=\frac{mk}{2\omega^2}\dot{a}e^{-2i\int^{\eta_1}d\eta\,\omega(\eta)}\alpha_k(\eta_1),\end{aligned} \tag{C.4}$$

with the initial conditions $\alpha_k(\eta_i)=1$, $\beta_k(\eta_i)=0$ for the time $\eta_i$ early enough that the mode is inside the dS event horizon. Since we expect $|\beta_k|\ll1$ and $a_k\approx1$, we can replace $\alpha=1$ in Eq. (C.4) and formally write the solution as

$$\beta_k(\eta_f)=\int_{\eta_i}^{\eta_f}d\tau\,\frac{mk}{2\omega^2}\dot{a}(\tau)e^{-2i\int^{\tau}d\eta\,\omega(\eta)}. \tag{C.5}$$

The steepest descent method can be applied to evaluate this integral in a similar fashion as was done for the scalar case in [40]. Despite the different $k$ dependence in Eq. (C.5), the result is the same as Eq. (41) of [40]:

$$|\beta_k|^2\approx\exp\left\{-4\left[\frac{[k/a(r)]^2}{m\sqrt{H^2(r)+R(r)/6}}+\frac{m}{\sqrt{H^2(r)+R(r)/6}}\right]\right\}, \tag{C.6}$$

in which $r$ is the real part of the complexified conformal time $\tilde{\eta}$ at which $\omega(\tilde{\eta})=0$ and $R$ is the Ricci scalar. This is approximately due to the fact that the branch point occurs when $\omega=0$, such that the dominant contribution occurs when $|k/a|\sim m$. Eq. (C.6) leads to the particle number density (fermion plus anti-fermion) as

$$\rho_\psi(t) \approx \frac{1}{2\pi^{3/2}}\left(\frac{a(r)}{a(t)}\right)^3 m\left[\frac{m}{4}\sqrt{H^2(r)+R(r)/6}\right]^{3/2} \times \exp\left(\frac{-4m}{\sqrt{H^2(r)+R(r)/6}}\right). \tag{C.7}$$

To estimate the relic abundance from this equation, one can use the formula

$$\Omega_\psi h^2 \sim 100\left(\frac{T_{\rm rh}}{10^9\ \text{GeV}}\right)\left(\frac{H(t_e)}{10^{13}\ \text{GeV}}\right)^{-2}\frac{\rho_\psi(t_e)}{(10^{12}\ \text{GeV})^4}, \tag{C.8}$$

where one is only formally evaluating $\rho_\psi(t_e)$ at the end of inflation time $t_e$ even though the particle densities are well defined at times far later than time. Unlike the formulae presented in the body of the text, the exponential sensitivity and the approximations made in obtaining the saddle-point does not allow one to guarantee an order of magnitude numerical accuracy, especially for large $m/H(r)$ [40]. However, the spectral and mass cutoffs can be well estimated by Eqs. (C.6) and (C.7).

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