JOURNAL OF
SYMPLECTIC GEOMETRY
Volume 17, Number 3, 793–856, 2019

Interface asymptotics of partial Bergman kernels on S^1-symmetric Kähler manifolds

Steve Zelditch and Peng Zhou

This article is concerned with asymptotics of equivariant Bergman kernels and partial Bergman kernels for polarized projective Kähler manifolds invariant under a Hamiltonian holomorphic S^1 action. Asymptotics of partial Bergman kernel are obtained in the allowed region $\mathcal{A}$ resp. forbidden region $\mathcal{F}$, generalizing results of Shiffman-Zelditch, Shiffman-Tate-Zelditch and Pokorny-Singer for toric Kähler manifolds. The main result gives scaling asymptotics of equivariant Bergman kernels and partial Bergman kernels in the transition region around the interface $\partial\mathcal{A}$, generalizing recent work of Ross-Singer on partial Bergman kernels, and refining the Ross-Singer transition asymptotics to apply to equivariant Bergman kernels.

Research partially supported by NSF grant DMS-1541126 and by the Stefan Bergman trust.

This article is concerned with the asymptotics of *partial Bergman kernels* for positive Hermitian holomorphic line bundles $(L,h)\to(M,\omega)$ over a Kähler manifold of complex dimension m carrying a Hamiltonian holomorphic S^1 action, where $S^1=\mathbf{T}=\mathbb{R}/2\pi\mathbb{Z}$,

$$\exp t(\xi_H/2\pi):\mathbf{T}\times M\to M,\quad \iota_{\xi_H}\omega=dH,\ \ \exp t(\xi_H/2\pi)(z)=:e^{2\pi it}z,$$

where $H:M\to P_0:=H(M)\subset\mathbb{R}$ is the Hamiltonian and ξ_H is its Hamilton vector field. The $\mathbf{T}$-action[1] preserves the data (L,h) and can be 'quantized' to give a unitary representation of $\mathbf{T}$

$$U_k(\theta)=e^{ik\theta\hat{H}_k}:\mathbf{T}\times H^0(M,L^k)\to H^0(M,L^k) \tag{1}$$

on the spaces $H^0(X,L^k)$ of holomorphic sections of the tensor powers L^k, equipped with the L^2 norm Hilb_{h^k} induced by the Hermitian metric h. The self-adjoint generator of $U_k(\theta)$ is denoted by

$$\hat{H}_k:=H+\frac{i}{2\pi k}\nabla_{\xi_H}:H^0(M,L^k)\to H^0(M,L^k), \tag{2}$$

where $\nabla_{\xi_H}s$ is the covariant derivative of a section s and Hs is the product of s with H [Ko, GS]. When ξ_H generates a holomorphic $\mathbf{T}$ action, $\hat{H}_k$

[1] We denote it by $\mathbf{T}$ rather than by S^1 because we use that notation for a different circle action on L^*. We also use the terms Bergman kernel and Szegö kernel interchangeably.

preserves holomorphic sections and coincides with the Toeplitz operator

$$\hat{H}_k s = \Pi_k \hat{H}_k \Pi_k s, \quad s \in H^0(M, L^k)$$

with principal symbol H (see §1.3). Here,

$$\Pi_k : L^2(M, L^k) \to H^0(M, L^k)$$

is the orthogonal projection (or Szegö -Bergman kernel).

We define the eigenspaces of $\hat{H}_k$ (= the weight spaces of the $\mathbf{T}$ action) by

$$\begin{aligned} V_k(j) &= \{s \in H^0(M, L^k) : U_k(\theta)s = e^{ij\theta} s\} \\ &= \left\{ s \in H^0(M, L^k) : \hat{H}_k s = \frac{j}{k}\, s \right\}; \end{aligned} \tag{3}$$

it is known that $V_k(j) \neq \{0\}$ if and only if $\frac{j}{k} \in P_0 = H(M)$, and their dimensions have been computed in articles on "quantization commutes with reduction" [GS]. In Lemma 2.1 we show that $H(M) = [0, a]$ for a positive integer a which is equal to the symplectic area of a generic $\mathbb{C}^*$ orbit. We define the associated weight space projections (termed *equivariant Bergman kernels*)

$$\Pi_{k,j}(z, w) : L^2(M, L^k) \to V_k(j). \tag{4}$$

These equivariant Bergman kernels are the smallest components of the full Bergman kernel (or Szegö projector)

$$\Pi_k(z, w) = \sum_{j : \frac{j}{k} \in P_0} \Pi_{k,j} : L^2(M, L^k) \to H^0(M, L^k) \tag{5}$$

to possess strong asymptotic expansions when $\frac{j}{k} \to E$ for some value E of H. The norm contraction of $\Pi_{k,j}(z, z)$ on the diagonal is denoted by $\Pi_{k,j}(z)$ and is called the equivariant density of states.[2] As proved in Theorems 1 and 2, the normalized equivariant density of states $k^{-m+\frac{1}{2}} \Pi_{k,j}(z)$ resembles a Gaussian bump concentrated on the energy level $H^{-1}(E)$ in the sense of being essentially equal to 1 on $H^{-1}(\frac{j}{k})$ and having "Gaussian decay" $e^{-k b_E(z)}$ away from $H^{-1}(\frac{j}{k})$ along gradient lines $\sigma \to e^{-\sigma/2} \cdot z$ of H, where b_E is defined by (9), and is like distance-squared to the hypersurface $H^{-1}(E)$.

[2] The norm contraction of any kernel $K(z, w)$ on the diagonal is denoted $K(z)$.

This is the analogue for S^1 actions of the result of [STZ] showing that joint eigensections z^α of the torus action of a toric Kähler manifold are Gaussian-like bumps centered on the tori $\mu^{-1}(\alpha)$ (the inverse image of $\alpha \in \mathbb{Z}^m$ under the moment map μ), a fact also used in [PS, RS].

The partial Bergman kernels of the title are projectors

$$\Pi_{k,P}(z,w) := \sum_{j:\frac{j}{k}\in P} \Pi_{k,j}(z,w). \tag{6}$$

onto subspaces

$$\mathcal{S}_{k,P} := \bigoplus_{j:\frac{j}{k}\in P} V_k(j) \subset H^0(M, L^k) \tag{7}$$

corresponding to proper sub-intervals $P \subset P_0 = H(M)$. They behave like sums of Gaussian bumps centered at the the inverse images $H^{-1}(\frac{j}{k})$ of the "lattice points" $\frac{j}{k} \in P$.

The main problem is to relate the asymptotic properties of $\Pi_{k,P}(z,w)$ to the geometry of the Hamiltonian flow of H and its complexification as a $\mathbb{C}^*$ action. The analogous problem for toric Kähler manifolds was studied in [ShZ], with P a sub-polytope of the Delzant moment polytope of (M,ω). As in the toric case, we prove in Theorem 3 the norm contraction $\Pi_{k,P}(z)$ of $\Pi_{k,P}(z,z)$ has standard asymptotics in the *allowed region* $\mathcal{A}_P$ and exponentially decaying asymptotics in the *forbidden region* $\mathcal{F}_P$ where

$$\mathcal{A}_P := \mathrm{int}\{z \in M : H(z) \in P\}, \quad \mathcal{F}_P := \mathrm{int}(M \backslash \mathcal{A}_P).$$

On the boundary, or "interface" $\partial\mathcal{A}_P$, Ross-Singer in [RS] showed that $k^{-m}\Pi_{k,P}(z)$ decreases from ~ 1 to ~ 0 in a tube of radius $k^{-\frac{1}{2}}$. In the special case where the minimum set of H is a complex hypersurface, Theorem 1.2 of [RS] asserts that if $\sqrt{k}(H(z)-E)$ is bounded, then

$$k^{-n}\Pi_{k,(-\infty,E]}(z) = \frac{1}{\sqrt{2\pi|\xi_H(z)|^2}} \int_{-\infty}^{\sqrt{k}(H(z)-E)} e^{-\frac{t^2}{2|\xi_H(z)|^2}} dt + O(k^{-\frac{1}{2}}). \tag{8}$$

Here, $|\xi_H|$ is the norm of ξ_H with respect to the Kähler metric ω. The integral on the right is an incomplete Gaussian integral closely related to the error function $\mathrm{erf}(x) = \frac{1}{\sqrt{\pi}}\int_{-x}^{x} e^{-t^2} dt$, which is odd and smoothly interpolates between -1 at $-\infty$ to 1 at $+\infty$. The right side above involves the slight modification $\mathrm{Erf}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2} dt$, which interpolates between

0 at $-\infty$ and 1 at $+\infty$. It often arises in interface problems involved in packing quantum states in a domain (see e.g. [J, W]).

One of the principal motivations for this article is to establish this transition law for all Hamiltonian holomorphic S^1 actions, with no conditions on the fixed point set or on the analyticity of the Kähler metric ω. We obtain the interface asymptotics from the Gaussian asymptotics of the equivariant kernels (4). The Gaussian asymptotics of the equivariant Bergman kernels in Theorems 1 and 2 are used in Theorem 4 to give Erf asymptotics for partial Bergman kernels (8), which are essentially integrals of equivariant Bergman kernels. In Theorem 3 we give exponentially decaying asymptotics of the partial Bergman kernels $\Pi_{k,P}$ in the forbidden region.

Asymptotics of equivariant Szegö kernels have also been studied by R. Paoletti in several settings, of which the closest to this article are contained in [P, P2]. Equivariant Szegö kernels were not explicitly defined or studied by Ross-Singer [RS]; as discussed in §0.7, they constructed kernels $G_{j,k}$ which play the role of $\Pi_{k,j}$.

0.1 Set-up

To state our results we introduce some notation. Let (L, h, M, ω) be a Kähler manifold with a positive line bundle (L, h) with C^∞ Hermitian metric h and with $\omega = i\partial\bar{\partial} \log h$ a C^∞ Kähler form. Let $H : M \to \mathbb{R}$ be a Hamiltonian function generating the holomorphic $\mathbf{T}$-action. We shift H by a constant such that the minimum of H is zero. In §2.1 and §2.2 the complex and real Morse theory of Hamiltonians generating holomorphic S^1 actions is reviewed. The Hamiltonian and gradient flows of H commute and generate a $\mathbb{C}^*$ action. We denote the $\mathbb{C}^*$ action by $e^w z = e^{\rho+i\theta} \cdot z$, and the $\mathbb{R}$-action of gradient flow (and $\mathbf{T}$-action of Hamiltonian flow, resp) of H by e^ρ (and $e^{i\theta}$, resp), and infinitesimal generators for $\mathbb{R}$ and $\mathbf{T}$ action by ∂_ρ and ∂_θ.

We assume that E is a regular value of H, i.e. H has no critical points on $H^{-1}(E)$. The subinterval P of $H(M)$, is taken as $P = [0, E)$. The allowed region and forbidden region are then

$$\mathcal{A}_E = \{z \in M \mid H(z) < E\}, \quad \text{and} \quad \mathcal{F}_E = \{z \in M \mid H(z) > E\}.$$

Results for general P can be obtained from this easily.

Let $M^E_{\max}$ be the set of points whose $\mathbb{C}^*$-orbit (hence $\mathbb{R}$-orbit) intersects with hypersurface $H^{-1}(E)$. Let $\mathcal{F}^E_{\max} = M^E_{\max} \cap \mathcal{F}_E$.

Definition 0.1. For $z \in M^E_{\max}$, we define $z_E \in H^{-1}(E)$, and real number $\tau_E(z)$ and $b_E(z)$ as follows:
(1) z_E is the intersection of the $\mathbb{R}$-orbit $\{e^\rho \cdot z\}$ with $H^{-1}(E)$. We define the projection

$$q_E : M^E_{\max} \to H^{-1}(E), \quad z \mapsto z_E.$$

(2) $\tau_E(z)$ is the 'flow-time' from z_E to z, $z = e^{\tau_E(z)} \cdot z_E$.
(3) $b_E(z)$ is an analog of distanced-squared to $H^{-1}(E)$, defined by

$$b_E(z) = 2 \int_0^{\tau_E(z)} \left(H(e^\sigma \cdot z_E) - H(z_E)\right) d\sigma. \tag{9}$$

For ease of notation, we sometimes write $b(z, E) = b_E(z)$, $\tau(z, E) = \tau_E(z)$.
(4) For E a regular value of H, we define the largest $(1/k)\mathbb{Z}$ lattice points in $P = [0, E)$, as

$$E_k := \max \left\{ \frac{1}{k}\mathbb{Z} \cap [0, E) \right\}. \tag{10}$$

For any point $z \in M$ that is not a fixed point of $\mathbf{T}$, we fix a local $\mathbb{C}^*$-invariant holomorphic section $e_L \in \Gamma(U, L)$ in an open neighborhood U of z and define the Kähler potential φ by $e^{-\varphi} = \|e_L\|^2_h$. For any subspace $\mathcal{S}_k \subset \Gamma(M, L^k)$, the Bergman density for $\mathcal{S}_k$ can be written as

$$\Pi_{\mathcal{S}_k}(z) = \sum_{j=1}^{\dim \mathcal{S}_k} \|s_j(z)\|^2_{h^k} = \sum_{j=1}^{\dim \mathcal{S}_k} |f_j(z)|^2 e^{-k\varphi(z)}$$

where $\{s_j : j = 1, \ldots, \dim \mathcal{S}_k\}$ is an orthonormal basis for $\mathcal{S}_k$ and $s_j(z) = f_j(z) \cdot e_L(z)^{\otimes k}$ for a local holomorphic function $f_j(z)$ on U.

0.2 Asymptotics of equivariant Bergman kernels

Our first result is the precise statement that $k^{-m+1/2}\Pi_{k,j}(z, z)$ is a kind of Gaussian bump along $H^{-1}(\frac{j}{k})$, i.e. in the tangential directions along $H^{-1}(\frac{j}{k})$ it is essentially constant while it has Gaussian decay at speed k in the normal directions (i.e. along the ∇H flow lines). Note that the 'lattice points' $\frac{j}{k}$ are Bohr-Sommerfeld type energy levels and $H^{-1}(\frac{j}{k})$ are the corresponding classical energy surfaces. As $k \to \infty$ they become denser and approximate any energy level E. We now give the Gaussian asymptotics of $k^{-m+1/2}\Pi_{k,j}(z, z)$ as $\frac{j}{k} \to E$.

Theorem 1. *Let $(L,h) \to (M,\omega)$ be a positive line bundle over a Kähler manifold, $\omega = i\partial\bar{\partial}\log h \in C^\infty$, and let $H : M \to \mathbb{R}$ generate a holomorphic $\mathbf{T}$-action. Let E be a regular value of H, and $z \in M^E_{\max}$. Then for any sequence $j_1, j_2, \ldots$ such that $|j_k/k - E| < C/k$ for some constant C, then the equivariant density of states has the following asymptotics.*

$$\Pi_{k,j_k}(z) = \begin{cases} k^{m-\frac{1}{2}}\sqrt{\frac{2}{\pi\partial_\rho^2\varphi(z_E)}}(1+O(k^{-1})), & z \in M^E_{\max} \cap H^{-1}(E), \\ k^{m-\frac{1}{2}}\sqrt{\frac{2}{\pi\partial_\rho^2\varphi(z_E)}}e^{-kb(z,j_k/k)}(1+O(k^{-1})), & z \in M^E_{\max} \backslash H^{-1}(E). \end{cases}$$

Here, z_E and $b(z,E)$ are defined in Definition 0.1.

Remark 1. The right side of the bottom asymptotics can be re-written in terms of $b(z,E)$ but the coefficient changes. For $z \notin H^{-1}(E)$, $|b(z,j_k/k) - b(z,E)| = \partial_E b(z,\cdot)(\frac{j_k}{k} - E) + O(\frac{1}{k^2})$. Hence

$$e^{-kb(z,j_k/k)} \simeq e^{-kb(z,E)}e^{-\partial_E b(z,E)[k(\frac{j_k}{k}-E)]}\left(1+O\left(\frac{1}{k}\right)\right).$$

Hence, the coefficient of the exponential decay $e^{-kb(z,E)}$ depends both on the geometric coefficient $\partial_E b(z,E)$ and on the degree of approximation of E by the nearest 'lattice point' $\frac{j_k}{k}$.

The next result concerns the scaling asymptotics of the equivariant Bergman kernels in a $\frac{C}{\sqrt{k}}$-neighborhood of $H^{-1}(E)$.

Theorem 2. *With (L,h,M,ω) , (H,E) and (k,j_k) as in Theorem 1. Let $z_E \in M_{\max} \cap H^{-1}(E)$ and $z_k = e^{\frac{\beta}{\sqrt{k}}} \cdot z_E$ be a sequence of points approaching z_E. Then,*

$$\Pi_{k,j_k}(e^{\frac{\beta}{\sqrt{k}}} \cdot z_E) = k^{m-\frac{1}{2}}\sqrt{\frac{2}{\pi\partial_\rho^2\varphi(z_E)}}e^{-\beta^2\partial_\rho^2\varphi(z_E)}(1+O(k^{-1/2})).$$

0.3 Asymptotics of partial Bergman kernels

In this section, we state analogues of Theorems 1 and 2 for partial Bergman kernels (6); the first is the analogues of Theorem 1.2 of [ShZ] for partial Bergman kernels of toric Kähler manifolds. Our aim is to obtain exponentially accurate asymptotics in the forbidden region.

Theorem 3. *Let (L,h,M,ω) and (H,E) be as in Theorem 1, with $h,\omega \in C^\infty$. Let $P = H(M)\cap(-\infty,E)$ and $z \in M^E_{\max}$. Then the partial Bergman density is given by the asymptotic formulas:*

$$\Pi_{k,P}(z) = \begin{cases} \Pi_k(z) + O(k^{-\infty}) & H(z) < E \\ k^{m-1/2}\sqrt{\frac{2}{\pi \partial_\rho^2 \varphi(z_E)}} \frac{e^{-kb(z,E_k)}}{1-e^{-|2\tau_E(z)|}}(1+O(k^{-1})) & H(z) > E \end{cases}$$

where $z_E, \tau_E(z), b_E(z), E_k$ are given in Definition 0.1, and the remainder estimates are uniform on compact subsets of $M^E_{\max}$.

Remark 2. As in Remark 1, the decaying exponent of $e^{-kb(z,E_k)}$ in the forbidden region (bottom equation in Theorem 3) can be replaced by $e^{-kb(z,E)}$ by changing E_k to E at the cost of introducing an $O(1)$ multiplicative factor,

$$e^{-kb(z,E_k)} = e^{-kb(z,E)} e^{\partial_E b(z,E) k(E-E_k)}(1+O(1/k)).$$

However, when doing computation involving $k^{-1}\log \Pi_{k,P}(z)$ or its derivatives, one may replace $b(z,E_k)$ by $b(z,E)$ with only an $O(1/k)$ error.

0.4 Interface asymptotics

Interface asymptotics concerns scaling asymptotics of the partial Bergman density function $\Pi_{k,P}(z)$ for z in a $\frac{1}{\sqrt{k}}$-tube around the interfaces $H^{-1}(\partial P)$. It suffices to consider intervals of the form $(-\infty,E]$ or $[E,\infty)$. We parametrize the tube around $H^{-1}(E)$ using points $z_k = e^{\beta/\sqrt{k}} z_E$ with $z_E \in H^{-1}(E)$.

Theorem 4. *Let (L,h,M,ω) and (H,E) be as in Theorem 1. In particular, $h,\omega\in C^\infty$. Let $z_E \in M_{\max}\cap H^{-1}(E)$ and let $z_k = e^{\frac{\beta}{\sqrt{k}}}\cdot z_E$ be a sequence of points approaching z_E along an $\mathbb{R}_+$ orbit, where $\beta\in\mathbb{R}$. Then,*

$$\begin{aligned} \text{(11)} \qquad \Pi_{k,(-\infty,E]}(z_k) &= k^m \operatorname{Erf}\left(\sqrt{4\pi k}\frac{E-H(z_k)}{|\nabla H|(z_E)}\right)(1+O(k^{-1/2})) \\ &= k^m \operatorname{Erf}\left(\frac{-\beta|\nabla H(z_E)|}{\sqrt{\pi}}\right)(1+O(k^{-1/2})). \end{aligned}$$

We present two approaches to prove Theorem 3 and 4 on partial Bergman kernel $\Pi_{k,P}(z)$. The first approach (see Section 6) is based on asymptotics of the equivariant Bergman kernels $\Pi_{k,j}(z)$ for $j/k \in P$ of Theorem 2. For fixed z, we use the localization Lemma 5.1 to identify the relevant cluster of weights j that contributes to leading order and calculate their contribution. The second approach (see Section 8) makes use of the *polytope character* $\chi_{k,P}(e^w) = \sum_{j \in kP} e^{jw}$ and the Euler-MacLaurin formula as in [ShZ, RS]. It is a more global approach. Convolution of $\chi_{k,P}(e^w)$ with the full Bergman kernel sifts out the relevant equivariant modes. A key point is that the $\chi_{k,P}$ is a 'semi-classical Fourier integral with complex phase,' as is the Bergman kernel, so that asymptotics can be obtained by use of the Boutet-de-Monvel-Sjöstrand parametrix (see Section 3.3) and the stationary phase method.

There are interesting variations on the $\frac{1}{\sqrt{k}}$-scaling, which do not seem to have been studied before, and which will be developed in [ZZ17]. In Theorem 4, one takes a fixed interval of eigenvalues and studies the behavior of pointwise Weyl sums for z in a $\frac{1}{\sqrt{k}}$-tube around the classical interface $H^{-1}(E)$. But one might also study spectral sums where the eigenvalues are also constrained to lie $\frac{1}{\sqrt{k}}$ close to E. This is done in Propositions 8.1 and 8.5 for smooth, resp. sharp interval, constraints. It is shown in the "Localization Lemma" 5.1 (2)–(3) that weights $\frac{j}{k} \leq E$ in the sum (11) which are 'far' from E in the sense that $|\frac{j}{k} - E| \geq \frac{\log k}{\sqrt{k}}$ do not contribute to the asymptotics and, moreover, that those satisfying $|\frac{j}{k} - E| \geq \frac{C}{\sqrt{k}}$ do not contribute to the leading order asymptotics for C sufficiently large. Hence a smooth model for the interface sums is to use weights $f(\sqrt{k}(\frac{j}{k} - E))$ with Schwartz test functions f. For $f = \mathbf{1}_{[-M,M]}$ we use the Euler-MacLaurin sums method (Proposition 8.5). As in Theorem 4, the sums in Propositions 8.1 and 8.5 exhibit Erf asymptotics. As explained further in [ZZ17], the $\sqrt{k}$ scaling of smooth Weyl sums gives rise to a kind of Central Limit Theorem for deterministic Weyl sums.

The asymptotics in Theorem 3 improve on Theorem 1.1 of [RS] by giving exponentially accurate asymptotics in the forbidden region and give the analogue of the mass density results of [ShZ]. The interface asymptotics of Theorem 4 extend the result of [RS] to general holomorphic S^1 actions. The method of proof is rather different and, in particular, Proposition 8.1 extends to general Hamiltonian flows ([ZZ17]).

0.5 Zero locus of a Random section

As in [ShZ], one may deduce the formula for the asymptotic distribution of zeros of Gaussian random sections of $\mathcal{S}_k = \mathcal{S}_{k,P}$, the subspace of $H^0(M, L^k)$ spanned by eigensection of H_k with eigenvalue in the subinterval $P = [0, E]$ (c.f (7)). The definition of random sections is precisely as in [ShZ]: Let $s = \sum_{j=1}^{\dim \mathcal{S}_k} a_{k,j} s_{k,j}$ where $a_{k,j}$ are i.i.d. complex $N(0,1)$ random variables and $\{s_{k,j}\}$ is an orthonormal basis of $\mathcal{S}_k$. Let Z_s be the zero set of s and let $[Z_s]$ be the current of integration over Z_s.

Recall the setup from §0.1, let E be a regular value of H, $M^E_{\max}$ the set of points whose $\mathbb{C}^*$ orbits intersect the hypersurface $H^{-1}(E)$. We define $q_E : M^E_{\max} \to H^{-1}(E)$ sending a point z along its $\mathbb{R}$-orbit to the hypersurface $H^{-1}(E)$, and define

$$\overline{q}_E : M^E_{\max} \to M^E_{\max}/\mathbb{C}^* = H^{-1}(E)/S^1$$

to be its image under the further quotient by the S^1-action.

Theorem 5. *For any $z \in \mathcal{A}_E$ or $z \in \mathcal{F}_E \cap M^E_{\max}$, we have the following weak-* convergence result*

$$\lim_{k\to\infty} \frac{1}{k}\mathbb{E}([Z_s])(z) = \begin{cases} \omega, & \text{for } z \in \mathcal{A}_E. \\ \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\left[\varphi(\overline{q}_E(z)) + 2E\tau_E(z)\right], & \text{for } z \in \mathcal{F}_E \cap M^E_{\max}. \end{cases}$$

The limiting current in $\mathcal{F}_E \cap M^E_{\max}$ is a smooth $(1,1)$-form of rank $(n-1)$.

Above, φ is a local Kähler potential for ω, i.e. $\omega = (i/2\pi)\partial\bar{\partial}\varphi$. It is S^1 invariant and descends to a potential on the reduced Kähler manifold $H^{-1}(E)/S^1$. Since the Gaussian ensemble is S^1 invariant, the limiting currents in both regions are S^1 invariant. In $\mathcal{F}_E$ the first term is invariant under the $\mathbb{C}^*$ action.

Remark 3. In the case where the fixed locus $Y = H^{-1}(E_{\max})$ is a divisor, and there is no other critical value of H in $(E, E_{\max})$, the limiting current of zero locus can be written as

$$\lim_{k\to\infty} \frac{1}{k}\mathbb{E}([Z_s])(z) = \frac{i}{2\pi}\partial\bar{\partial}\varphi_{Y,\epsilon},$$

where $\epsilon = E_{\max} - E$ and $\varphi_{Y,\epsilon}$ is the envelope psh function

$$\begin{aligned}\varphi_{Y,\epsilon}(z) = \varphi(z) + \sup\{\psi : \omega + (i/2\pi)\partial\bar{\partial}\psi \geq 0,& \\ \psi \leq 0,\ \nu(\psi)_z \geq \epsilon \text{ for all } z \in Y\}.&\end{aligned}$$

In [RS] Theorem 5.16, they give an explicit expression for $\varphi_{Y,\epsilon}$, which in our notation can be written as

$$\varphi_{Y,\epsilon}(z) = \begin{cases} \varphi(z) & z \in \mathcal{A} \\ \varphi(z) - b_E(z) = \varphi(z_E) + 2E \cdot \tau_E(z) & z \in \mathcal{F}. \end{cases}$$

See also sections 2.5 and 2.6 for the notation b_E, τ_E and the relation with symplectic potential $u(I,z)$[3].

0.6 Relations to Bernstein polynomials

The interface result of Theorem 4 reduces to a classical theorem on Bernstein polynomial approximations of characteristic functions of intervals in the special case where $M = \mathbb{CP}^1$ and $\varphi(z) = \log(1+|z|^2)$, resp. $M = \mathbb{C}$ and $\varphi(z) = |z|^2$. We briefly review these classical results and their relation to the present article.

We recall that Bernstein polynomials of one variable give canonical uniform polynomial approximations to continuous functions $f \in C([0,1])$:

$$B_N(f)(x) = \sum_{j=0}^{N} \binom{N}{j} f(\frac{j}{N}) x^j (1-x)^{N-j}, \tag{12}$$

and

$$\lim_{N\to\infty} B_N(f) \to f \text{ uniformly on } [0,1].$$

It is explained in [Ze, Fe] that (12) can be put in the form of the kernels in Theorem 3. More precisely, Bernstein polynomials in the sense of [Ze, Fe] are functions

$$B_k(f)(z) := \sum_{j=1}^{N_k} f\left(\frac{j}{k}\right) \frac{\Pi_{k,j}(z)}{\Pi_k(z)} = \frac{(\Pi_k f(\hat{H}_k)\Pi_k)(z,z)}{\Pi_k(z,z)}, \tag{13}$$

where for f is smooth there exists asymptotic expansion for $k \to \infty$. Here, $f(\hat{H}_k)$ on $H^0(M, L^k)$ is defined by the spectral theorem, so that $f(\hat{H}_k)\hat{s}_{k,j} = f\left(\frac{j}{k}\right)\hat{s}_{k,j}$ if $s_{k,j}$ is an eigensection of $\hat{H}_k$ with eigenvalue j/k.

Partial Bergman kernels are Bernstein polynomials (up to a normalization constant) in the case where f is a step function $\mathbf{1}_P$. Since $\mathbf{1}_P$ above is a characteristic function with a jump, $B_N(\mathbf{1}_P)(x)$ cannot approach $\mathbf{1}_P(x)$ at the jump. In fact, there is a kind of mean value formula at the jump involving incomplete Gaussian integrals $\mathrm{Erf}(x) = \int_{-\infty}^{x} e^{-x^2/2} dx/\sqrt{2\pi}$. In the classical setting of Bernstein polynomials on $[0,1]$, the jump formula is proved in [Ch, Lo, Lev]. The interface asymptotics of [RS] and of this article are generalizations of Theorem 1.5.2 of [Lo] in the one-variable setting.

---

[3]We thank the referee for pointing out this connection.

0.7 Remarks on the proof, on related work and open problems

The main idea of the present article is to use the spectral theory of the S^1 action and in particular the eigenspace projections (equivariant Bergman kernels) to obtain asymptotics. The spectral viewpoint generalizes in many respects to any Hamiltonian $H : M \to \mathbb{R}$ on any compact Kähler manifold, including cases where the Hamiltonian does not generate an S^1 action. In the general case, the gradient flow and Hamiltonian flow do not commute or define a $\mathbb{C}^*$ action, and the eigenspace projections do not have individual asymptotics. Consequently, much of the analysis of this article does not generalize. However it can be replaced by a more difficult analysis using Toeplitz operators [ZZ17]. At this time, the spectral approach has been the only feasible approach to asymptotics of partial Bergman kernels in forbidden regions or to interface asymptotics.

In this paper we have avoided critical points $\nabla H(z) = 0$ of H. The interface results would change at a critical point. Roughly speaking, one would have to use the quadratic approximation (the metaplectic representation) rather than the linear approximation (the Heisenberg representation). It seems to be an interesting problem to study the local interface behavior around a critical point.

In [RS] the role of the equivariant kernels is played by the terms $G_{n,k}(z, w)$ $\sigma^n(z) \otimes \overline{\sigma(w)}^n$ in Definition 5.21, where σ is defined in Section 5.2 as the section $\sigma \in H^0(Y, \mathcal{O}(Y))$ defining a hypersurface component of $\mathrm{Fix}(\mathbf{T})$. We do not assume $\mathrm{Fix}(\mathbf{T})$ contains a hypersurface component and do not make use of σ. We also do not make any constructions of $G_{n,k}$ or construct special parametrices adapted to the hypersurface Y, as is done in [RS].

The analysis in Ross-Singer [RS] was largely motivated by a more general unsolved problem of determining Bergman kernel asymptotics for subspaces of sections defined in terms of vanishing order along a divisor $Y \subset M$. The partial Bergman kernels are Schwartz kernels of the orthogonal projections

$$\Pi_k^{Y,t}(z, w) : L^2(X, L^k) \to H^0(X, \mathcal{O}(L^k) \otimes \mathcal{I}_Y^{tk}) \tag{14}$$

onto the subspace of $s \in H^0(M, L^k)$ which vanish to order tk on a complex hypersurface Y. The main question is to find the asymptotics of the density of states,

$$\rho_{h^k}^{Y,t}(z) := \Pi_k^{Y,t}(z, z)_{h_z^k \otimes h_z^k}, \tag{15}$$

defined by contracting the Szegö kernel along the diagonal with the metric. The asymptotics depend on whether z lies in the *allowed region* $\mathcal{A}_t$ far from the divisor Y or whether it lies in the *forbidden region* $\mathcal{F}_t$ near the divisor Y but it is more difficult to define these regions in the absence of a $\mathbf{T}$ action.

The general definition of allowed/forbidden regions (due to R. Berman [Ber] and developed several articles of Ross-Witt-Nystrom) is that the allowed region is the set $\mathcal{A}_t := D_{Y,t} = \{\phi_{e,Y,t} = \phi\}$ where a certain equilibrium potential $\phi_{e,Z,t}$ equals the original Kähler potential. As pointed out in [PS], in this generality, there is no information about the smoothness of $\partial D_{Y,t}$ nor about the 'transition behaviour' of (15) near $\partial D_{Y,t}$. In [Ber] it is suggested to employ singular Hermitian metrics with singularities and with *negative curvature* concentrated along the divisor Y. At the present time, this program has only partially been carried out in [Ber, RS, CM] and remains largely open.

The spectral viewpoint towards (14) has not been developed beyond the S^1 case of this article, and may not admit generalizations to non-symmetric cases. The interface might be quite irregular in general (as the boundary of an envelope). We briefly discuss how to re-cast vanishing order in terms of spectral theory. When the Hamiltonian flow is holomorphic and periodic and when one component of the fixed point set $M^{\mathbf{T}}$ of the $\mathbf{T}$ action is a divisor Y, then the allowed and forbidden regions are those defined above in terms of the Hamiltonian and the interface asymptotics are given by (8) in [RS]. The hypersurface Y is necessarily the minimum set of the classical Hamiltonian. The link between the spectral theory and the definition of partial Bergman kernels in terms of vanishing order is given in the following Proposition (closely related to Lemma 5.4 of [RS].)

Proposition 0.2. *Suppose that the minimum set of H is a complex hypersurface Y. Then $H^0(X, \mathcal{O}(L^k) \otimes \mathcal{I}_Y^{tk}) = \bigoplus_{j \geq tk} V_k(j)$ is the direct sum of eigenspaces of $\hat{H}_k$ for eigenvalue $\geq t$.*

Proof. Since Y is $\mathbf{T}$-invariant, both L^k and $\mathcal{I}_Y^{tk}$ are $\mathbf{T}$ invariant and so the action fo $\mathbf{T}$ on $H^0(X, \mathcal{O}(L^k) \otimes \mathcal{I}_Y^{tk})$ decomposes into weight spaces,

$$H^0(X, \mathcal{O}(L^k) \otimes \mathcal{I}_Y^{tk}) = \bigoplus_{j : \frac{j}{k} \in P_0} V_k(j) \otimes \mathcal{I}_Y^{tk}.$$

Thus it suffices to determine the summands which are non-zero. An element $s \in V_k(j)$ transforms by w^j under the action of $w \in \mathbb{C}^*$. We restrict it to $\mathbb{C}^*$ orbit which tends to a point $y \in Y$. In holomorphic coordinates (w, y) where $w = 0$ on Y, it is given by $c_y w^j$. Thus it vanishes to order $\geq tk$ if and only if $j \geq tk$. $\square$

Of course, it is only in special cases that the minimum set of H is a hypersurface. To take a simple model example, the hypersurface $\{z_1 = 0\}$ is a component of the fixed point set of the S^1 action on $\mathbb{C}^m$ defined by $e^{i\theta}(z_1, \ldots, z_m) = (e^{i\theta} z_1, z_2, \ldots, z_m)$, but for the 'isotropic Harmonic oscillator' $e^{i\theta}(z_1, \ldots, z_m) = (e^{i\theta} z_1, e^{i\theta} z_2, \ldots, e^{i\theta} z_m)$ generated by $H = \|Z\|^2/2$ only $\{0\}$ is in the fixed point set or minimum set of H.

Partial Bergman kernels corresponding to intervals of eigenvalues are closely related to Bergman kernels for the Kähler symplectic cut of M in $H^{-1}(P)$ in the sense of [BGL]. It would be interesting to compare the interface behavior of the partial Bergman kernel, and the Bergman kernel of the Kähler cut near the 'cut' but only for special cuts does the line bundle project to an ample bundle.

Acknowledgments

We would like to thank R. Paoletti, J. Ross, and M. Singer for their comments on earlier versions. We also thank the referee for carefully reading the paper and pointing out many mistakes and typos.

1. Hermitian line bundles, Kähler potentials and geometric quantization

In this section, we review some elementary facts about the Kähler geometry and geometric quantization, and also establish our notations and conventions.

1.1. Hermitian line bundles

Let $(L, h) \to (M, \omega = c_1(L))$ be an ample line bundle with a positive hermitian metric over a projective Kähler manifold. For any $z \in M$, and any open neighborhood U of z on which L is trivial, we may choose a local trivialization $e_L \in \Gamma(U, L)$, that is $e_L(z) \neq 0$ for all $z \in U$. Then we may define the corresponding local Kähler potential $\varphi : U \to \mathbb{R}$ as

$$h(e_L(z), e_L(z)) =: e^{-\varphi(z)} := h(z).$$

The Chern connection associated to h is

$$\nabla : C^\infty(L) \to \mathcal{A}^1(L)$$

where $C^\infty(L)$ is the sheaf of smooth sections valued in L and $\mathcal{A}^1(L)$ is the sheaf of smooth 1-forms valued in L, such that

$$\begin{aligned}&\nabla = \nabla^{(1,0)} + \nabla^{(0,1)}, \quad \nabla^{(0,1)} = \bar\partial, \quad \nabla^{(1,0)} = \partial + A^{(1,0)},\\ &A^{(1,0)} = \partial \log h = -\partial\varphi.\end{aligned}$$

The curvature associated with the Chern connection is

$$F_\nabla(z) = dA^{(1,0)} = d\partial \log h = \bar\partial\partial \log h = \partial\bar\partial\varphi.$$

We choose the Kähler form $\omega = Ric(L)$, more precisely

$$\omega = c_1(L) = \frac{i}{2\pi}F_\nabla = \frac{i}{2\pi}\partial\bar\partial\varphi = \frac{-1}{4\pi}dd^c\varphi.$$

where $d^c = i(\partial - \bar\partial)$, such that $d^c f = df \circ J$.

It often simplifies the analysis to lift sections of $L \to M$ and operators on sections to the unit circle bundle X_h of the Hermitian metric h, so that geometric pre-quantization of the S^1 action is pullback of scalar functions under a flow. In the next section we discuss the geometric aspects of the lift and in §3.1 we discuss the analytic aspects.

1.2. The disc bundle and the circle bundle of L^*

Let (L^*, h^*) be the dual bundle to L with the induced hermitian metric h^*, which we will also denote as h from now on. Let $e_L^* \in \Gamma(U, L^*)$ be the dual frame to e_L, then we can define the disc and circle bundle in L^*:

$$D_h = D(L^*) := \{(z,\lambda) \mid z \in M,\, \lambda \in L_z^*, \|\lambda\|_h \le 1\}, \quad X = X_h = \partial D_h.$$

The disc bundle D_h is strictly pseudoconvex in L^*, and hence X_h inherits the structure of a strictly pseudoconvex CR manifold. Let ψ be a smooth function defined in a neighborhood of ∂D inside D, such that $\psi > 0$ in D^o , $\psi = 0$ on ∂D and $d\psi \neq 0$ near ∂D. For example, one may take

$$\psi(x) = -2\log|\lambda| - \varphi(z), \tag{16}$$

where $x = (z, \lambda e_L^*(z))$. Associated to X_h is the contact form[4]

$$\alpha = \mathrm{Re}(i\partial\psi|_X) = \mathrm{Re}(-i\bar{\partial}\psi|_X) = d\theta + \mathrm{Re}(i\bar{\partial}\varphi(z)), \quad \pi^*\omega = \frac{1}{2\pi} d\alpha \tag{17}$$

where we used (z, θ) to denote $\big(z, e^{i\theta} e_L^*(z)/\|e_L^*(z)\|_h\big) \in X_h$, and we abused notation by omitting π^* in $\pi^*\bar{\partial}\varphi(z)$. The Reeb vector field R is uniquely defined by $\alpha(R) = 1, \iota_R d\alpha = 0$; here it is $R = \partial_\theta$, the fiberwise rotation. Since later we will use ∂_θ for the generator of the holomorphic circle action on M, we will always refer to the Reeb flow by R, and the group action by $r_\theta := \exp(\theta R)$.

A section s_k of L^k determines an equivariant function $\hat{s}_k$ on L^* by the rule

$$\hat{s}_k(\lambda) = \left(\lambda^{\otimes k}, s_k(z)\right), \quad \lambda \in L_z^*, \ z \in M,$$

where $\lambda^{\otimes k} = \lambda \otimes \cdots \otimes \lambda$.

1.3. Lifting the Hamiltonian flow to a contact flow on X_h.

Let H be a Hamiltonian function on (M, ω). Let ξ_H be the Hamiltonian vector field associated to H, that is,

$$dH(Y) = \omega(\xi_H, Y)$$

for all vector field Y on M. The sign convention for the Hamiltonian vector field and the corresponding Poisson bracket is

$$df(Y) = \omega(\xi_f, Y), \quad \{f, g\} = -\omega(\xi_f, \xi_g)$$

this choice ensures that $[\xi_f, \xi_g] = \xi_{\{f,g\}}$.

The purpose of this section is to lift ξ_H to a contact vector field $\hat{\xi}_H$ on X_h and to lift the Hamiltonian $\mathbf{T}$ action to a contact $\mathbf{T}$ action. Recall that the horizontal lift is defined by $\xi_H^h \in \ker \alpha$ and $\pi_* \xi_H^h = \xi_H$. We also denote the Reeb vector field generating the canonical S^1 action on $X_h \to M$ by R.

Lemma 1.1. *Define*

$$\hat{\xi}_H = \xi_H^h - 2\pi H R.$$

Then $\hat{\xi}_H$ is a contact vector field.

---

[4]If we used two different defining functions ψ_1 and ψ_2, the induced αs would differ as well. However, if $\psi_1 = f(\psi_2)$, then $\alpha_1 = f'(0)\alpha_2$ only differ by a constant factor.

Proof. Since $\pi_* \hat{\xi}_H = \pi_* \xi_H^h = \xi_H$, it suffices to check that $\hat{\xi}_H$ preserve the contact form α. By (17),

$$\begin{aligned}\mathcal{L}_{\hat{\xi}_H} \alpha &= \mathcal{L}_{\xi_H^h - 2\pi H R} \alpha = (\iota_{\xi_H^h - 2\pi H R} \circ d + d \circ \iota_{\xi_H^h - 2\pi H R})\alpha \\ &= \iota_{\xi_H^h} \pi^*(2\pi\omega) + d(-2\pi H \alpha(R)) = 0. \end{aligned}$$ □

In Lemma 2.6 we prove that the lifted flow is periodic of period 2π.

Lemma 1.2. *For any C^∞ section s of L^k, $\widehat{\hat{H}_k s} = \frac{i}{2\pi k} \hat{\xi}_H(\hat{s})$. If H defines a holomorphic S^1 action, then the spectrum of $\hat{H}_k$ is given by*

$$\mathrm{Sp}(\hat{H}_k) = \{\frac{j}{k} : j \in \mathbb{N}, \frac{j}{k} \in H(M)\}.$$

Proof. We write $\xi_H = \xi$. It is well-known that $\widehat{\nabla_\xi s} = d\hat{s}(\xi^h)$; we refer to [KN] for the proof. The equation follows from the fact s lifts to an equivariant function satisfying $R\hat{s} = ik\hat{s}$. □

Recall the definition (1) of $U_k(\theta) = e^{ik\theta \hat{H}_k}$, acting on $C^\infty(M, L^k)$, we have

Corollary 1.3. *For any smooth section $s \in C^\infty(M, L^k)$, and for any $\theta \in \mathbb{R}$,*

$$\widehat{U_k(\theta) s} = \exp\left(-\theta \frac{\hat{\xi}_H}{2\pi}\right) \hat{s} = \hat{s} \circ \exp\left(\theta \frac{\hat{\xi}_H}{2\pi}\right).$$

As mentioned above, if we have a holomorphic S^1 action, then the lifted flow is periodic of period 1 (Lemma 2.6). It follows that $U_k(\theta)$ is periodic of period 2π.

1.4. The $S^1 \times S^1$ action on X_h and its weights

The Reeb flow and the lifted $\mathbf{T}$ action together define an $S^1 \times S^1$ action on X_h. Its weights form the semi-lattice $\{(j,k) \in \mathbb{Z}_+ \times \mathbb{Z}_+, j \in kP_0\}$. This lifting and the approximation of energy levels by rays in $\mathbb{Z}_+ \times \mathbb{Z}_+$ is discussed in detail in [STZ] for toric varieties, and the same discussion applies almost verbatim to $\mathbf{T}$ actions.

The asymptotics of the equivariant Bergman kernels $\Pi_{k,j}$ involves pairs (j_n, k_n) of lattice points along a "ray" in the joint lattice. The simplest rays are the "rational rays" where $j/k \in \mathbb{Q} \cap P_0$. Somewhat more complicated are "irrational rays" where $\frac{j_n}{k_n} \to E \notin \mathbb{Q}$. In this case we consider lattice points with $|\frac{j}{k} - E| \leq \frac{C}{k}$.

2. Kähler manifolds carrying $\mathbb{C}^*$ actions

We begin by reviewing the geometry of $\mathbb{C}^*$ actions on Kähler manifolds and give examples where at least one component of the fixed point set is a hypersurface. We also consider the possible Hamiltonians H which generate such actions.

2.1. Bialynicki-Birula decomposition

Let (M, ω) be a compact Kähler manifold equipped with a holomorphic $\mathbb{C}^*$ action. The generator of the $\mathbb{C}^*$ action $\xi \in H^0(M, T^{1,0})$ is a holomorphic vector field. A holomorphic $\mathbf{T}$ action which preserves ω is necessarily an isometric $\mathbf{T}$ action for the Kähler metric. The closure of a non-trivial $\mathbb{C}^*$ orbit contains two fixed points and is a topological S^2 called a *gradient sphere*. A *free* gradient sphere is one whose generic point has trivial stabilizer.

By Frankel's theorem [F], if the action has a fixed point, then the real S^1 action is Hamiltonian. We denote the Hamiltonian by $H : M \to \mathbb{R}$ and its Hamilton vector field by $\xi = \xi_H$. Let $F_1, \ldots, F_r$ be the connected components of the fixed point set $M^{\mathbf{T}}$. Each F_j is a compact totally geodesic Kähler submanifold of (M, ω). Set

$$M_i^+ := \{x \in M : \lim_{t \to 0} tx \in F_i\}, \quad M_i^- := \{x \in M : \lim_{t \to \infty} tx \in F_i\}.$$

The so-called *Bialynicki-Birula decomposition* [BB, CS] states that the strata of the disjoint decomposition

$$M = \bigcup M_i^+ = \bigcup M_i^- \tag{18}$$

are locally closed analytic submanifolds. In Theorem II of [CS] it is proved that

$$T(M_j^{\pm})|_{F_j} = N(F_j)^{\pm} \oplus TF_j,$$

where $N(F_j)^+$ (resp. $N(F_j)^-$) is the weight space decomposition with positive (resp. negative) weights. Moreover, there is precisely one component M_{src}^+ of the plus-decomposition (called the *source*), resp. one component M_{sink}^- of the minus decomposition (called the *sink*) such that the associated stratum is Zariski open. We will denote the points in $M_{src}^+ \cap M_{sink}^-$ by $M_{\max}$ To paraphrase [BBS], the $\mathbb{C}^*$ action gives a 'flow' from the source to the sink, and the 'flowlines' are closures of 'generic' orbits and limits of such closures.

2.2. Morse theory and gradient flow

The same decomposition can be obtained from the real Morse theory of the Hamiltonian H. Kirwan proved that H^2 is a minimally degenerate Morse function. Since we are dealing with a real-valued moment map, we may simply use H and it is also a minimally degenerate (perfect) Bott-Morse function. The gradient flow of H with respect to the Kähler metric induces a Morse stratification of X, and in [Ki, Y] it is proved that this stratification is the same as the Bialynicki-Birula decomposition. That is, the Morse stratum

$$S_j^{\pm} = \left\{ x \in M : \lim_{t \to \pm\infty} \exp(t\nabla H) \cdot x \in F_j \right\}$$

is the same as $M_j^{\pm}$. We note that

$$M^{\mathbf{T}} = \{x : dH(x) = 0\}$$

so that F_j are the components of the critical point set. The sink corresponds to the minimum set of H. In [RS] it is assumed that one of the components of $M^{\mathbf{T}}$ is a hypersurface, and this hypersurface is necessarily the minimum set of H.

Above we defined the open dense set $M^E_{\max}$ of points whose forward or backward gradient trajectories intersect $H^{-1}(E)$. Its complement consists of the stable/unstable manifolds of critical points other than the minimum/maximum. These gradient trajectories can get hung up at the other critical points and not make it to $H^{-1}(E)$.

We will also identify the Lie algebra $\mathfrak{g}$ of $\mathbf{T}$ with $\mathbb{R}$, such that $-2\pi\partial_\theta \in \mathfrak{g} \mapsto 1 \in \mathbb{Z}$. Let H be the corresponding Hamiltonian for $\xi_H = -2\pi\partial_\theta$. H is determined only up to an additive constant. We fix the indeterminacy in H by defining $H = 0$ on its minimum set.

Remark 4. (Remark on periods) By definition, the vector field ∂_θ of the action $e^{i\theta} z$ has period 2π, so the above convention makes the period of ξ_H equal to 1.

2.3. The image $H(M)$

We normalized H so that the minimum of H is zero. The question then arises what is the maximum value of H or equivalently what is the interval $H(M)$. It must be a "lattice polytope", i.e. an interval which integer endpoints. Thus, the maximum of H must be an integer.

Lemma 2.1. *Let $z \in M_{\max}$ and let $\mathcal{O}_z \simeq \mathbb{CP}^1$ be the compactification of $\mathbb{C}^* z$. Let $\omega(\mathcal{O}_z) = \int_{\mathcal{O}_z} \omega$. Then*

- *$\omega(\mathcal{O}_z)$ is a positive integer and is constant in z for $z \in M_{\max}$.*
- *$\max H = \omega(\mathcal{O}_z)$;*

Proof. For each $z \in M_{\max}$ we obtain a polarized Kähler $\mathbb{CP}^1$ by $(O_z, L|_{O_z}, \omega|_{O_z})$ and it must be the case that $\omega|_{O_z} \in H^2(O_z, \mathbb{Z})$. This proves the first statement. We then restrict $H : O_z \to \mathbb{R}$. It generates the S^1 action restricted to O_z. Hence $\omega|_{O_z} = (2\pi)^{-1} dH \wedge d\theta$. If $\omega(O_z) = M$, then $M = \int_{O_z} (2\pi)^{-1} dH \wedge d\theta = \int_0^{H_{\max}} dH = H_{\max}$, or $H_{\max} = M$. □

2.4. The Hamiltonian and the T-invariant Kähler potentials

Following §0.2, for any $w = e^{\rho + i\theta} \in \mathbb{C}^*$, denote the $\mathbb{C}^*$ action on M by $z \mapsto e^{\rho+i\theta} \cdot z$. If we choose a local slice S of the $\mathbb{C}^*$ action (necessarily a symplectic manifold), then we may define *slice-orbit* coordinates (ρ, θ, y) by letting y be coordinates on the slice and identifying

$$e^{i\theta+\rho} y = z. \tag{19}$$

For instance, if we choose a slice S_E of the $\mathbf{T}$ action on $H^{-1}(E)$ then we may use $S_E \times \mathbb{C}^*$ to give local coordinates on a neighborhood of S_E. Also, $H^{-1}(E)$ is a slice of the gradient flow or $\mathbb{R}_+$ action on $M^E_{\max}$ and we use the coordinates $(\rho, z_E) \in \mathbb{R} \times H^{-1}(E)$ as well.

As in the introduction, for $z \in M^E_{\max}$, we define

$$\tau_E(z) \in \mathbb{R} := \text{ the unique time s.t. } z = e^{\tau_E(z)} \cdot z_E, \quad z_E \in H^{-1}(E). \tag{20}$$

As in §0.2, we denote the two global vector fields $\partial_\rho, \partial_\theta$ (not be confused with the Reeb flow R on the circle bundle), such that

(21)
$$\partial_\theta f(z) := \left.\frac{d}{d\theta}\right|_{\theta=0} f(e^{i\theta} \cdot z), \quad \partial_\rho f(z) := \left.\frac{d}{d\rho}\right|_{\rho=0} f(e^{\rho} \cdot z), \quad \forall f \in C^\infty(M)$$

Since the $\mathbb{C}^*$ action is holomorphic, we have $\partial_\theta = J\partial_\rho$.

Our choice of coordinates is such that on $\mathcal{O}$,

$$\iota^*_{\mathcal{O}} \omega = \frac{i}{2\pi} \partial\bar\partial \iota^*_{\mathcal{O}} \varphi = \frac{1}{4\pi} \partial^2_\rho \, \varphi \;\; d\rho \wedge d\theta. \tag{22}$$

Recall that the gradient vector field ∇H is related to the Hamiltonian vector field ξ_H, for any $Y \in Vect(M)$,

$$dH(Y) = \omega(\xi_H, Y), \quad dH(Y) = g(Y, \nabla H), \quad g(X, Y) = \omega(X, JY)$$

hence $\nabla H = J\xi_H = -2\pi J\partial_\theta = 2\pi\partial_\rho$. Thus the limit point of downward gradient flow ∇H is the same as $\lim_{\rho\to-\infty} e^\rho \cdot z = z_\infty$.

The following lemma relates H with the local Kähler potential.

Lemma 2.2. *Fix any $z \in M_{max}$. Then*

$$H(z) = \frac{1}{2}\partial_\rho\varphi(z)$$

Proof. As in [GS] (5.5) we define a $\mathbf{T}$-invariant potential using a $\mathbf{T}$-invariant holomorphic section $s \in H^0_{\mathbf{T}}(M, L)$ in the sense of Lemma 1.3, i.e. so that $\hat{H}s = 0$. Then,

$$0 = \hat{H}s = \frac{i}{2\pi}\nabla_{\xi_H} s + Hs = \frac{i}{2\pi}\langle A, \xi_H\rangle s + Hs = \frac{i}{2\pi}\langle -\partial\varphi, -2\pi\partial_\theta\rangle s + Hs$$

where we used the Chern connection 1-form with respect to the basis frame s is given by $A = -\partial\varphi$, and our convention of $\xi_H = -2\pi\partial_\theta$ (see above). Hence

$$\begin{aligned}(23)\qquad H &= -i\langle\partial_\theta, \partial\varphi\rangle = \left\langle \partial_\theta, \frac{i}{2}(\bar\partial - \partial)\varphi \right\rangle \\ &= \left\langle \partial_\theta, \frac{-J}{2}(\partial + \bar\partial)\varphi \right\rangle = \left\langle J\partial_\theta, \frac{-1}{2}(d\varphi) \right\rangle = \frac{1}{2}\partial_\rho\varphi,\end{aligned}$$

This definition is unambiguous because any two $\mathbf{T}$-invariant holomorphic sections give the same Hamiltonian $\frac{1}{2}\partial_\rho\varphi(z)$. Indeed, let $s_1, s_2 \in H^0(M, L)$ be two $\mathbf{T}$-invariant (hence $\mathbb{C}^*$ invariant) holomorphic sections. Then $s_1 = fs_2$ for some $\mathbb{C}^*$-invariant meromorphic function f. Then $\partial_\rho f = \partial_\theta f = 0$, so $\varphi_1(z) = -\log\|s_1\|^2 = -\log|f|^2 + \varphi_2(z)$, and $\partial_\rho\varphi_1(z) = \partial_\rho(-\log|f|^2 + \varphi_2(z)) = \partial_\rho\varphi_2(z)$. □

2.5. The second derivative of φ and the action integral b_E

We now consider the relation of $\partial_\rho^2\varphi$ and $b_E(z)$ (9). Let $\mathcal{O}(z)$ denote the $\mathbb{C}^*$ orbit of z, and let $\mathcal{O}_\mathbb{R}(z)$ denote the gradient trajectory of z. If $\mathcal{O}_\mathbb{R}(z) \cap H^{-1}(E) \neq \emptyset$, then they intersect at the unique point z_E (20).

Lemma 2.3. *If the* $\mathbb{C}^*$ *orbit of* z *intersects* $H^{-1}(E)$*, let* $z = e^{\tau_E(z)} \cdot z_E$ *where* $\tau_E(z) \in \mathbb{R}$ *and* $z_E \in H^{-1}(E)$*, then*

$$\begin{aligned} b_E(z) &= \varphi(z) - \varphi(z_E) - \tau_E(z)\partial_\rho \varphi(z_E), \\ \partial_\rho b_E(z) &= \partial_\rho \varphi(z) - \partial_\rho \varphi(z_E), \\ \partial_\rho^2 b_E(z) &= \partial_\rho^2 \varphi(z) \end{aligned} \tag{24}$$

Hence, $b_E(e^\rho \cdot z_E)$ *is a strictly convex function in* ρ*, with minimum at* $\rho = 0$ *and* $b_E(z_E) = 0$*. In particular, for* $\rho > 0$ *(resp.* $\rho < 0$*),* $b_E(e^\rho \cdot z_E)$ *is strictly increasing (resp. decreasing) in* ρ*, or equivalently* $\tau_E(z)$*, along a* $\mathbb{C}^*$ *orbit for* $z \in \mathcal{F}_E$ *resp.* $z \in \mathcal{A}_E$*.*

Proof. By Lemma 2.2, we have $H(z) = \frac{1}{2}\partial_\rho \varphi(z)$, hence $2E = \partial_\rho \varphi(z_E)$. Hence from (9), we have

$$\begin{aligned} b_E(z) &= -2E\tau_E(z) + \int_0^{2\tau_E(z)} \left[H(e^{-\sigma/2} \cdot z) \right] \cdot d\sigma \\ &= -2E\tau_E(z) + 2\int_0^{\tau_E(z)} \left[H(e^{\sigma} \cdot z_E) \right] \cdot d\sigma \\ &= -\tau_E(z)\partial_\rho \varphi(z_E) + \int_0^{\tau_E(z)} \left[\partial_\rho \varphi(e^{\sigma} \cdot z) \right] \cdot d\sigma \\ &= \varphi(z) - \varphi(z_E) - \tau_E(z)\partial_\rho \varphi(z_E). \end{aligned}$$

The other two identities follow from $\partial_\rho \tau_E(z) = 1$ and $\partial_\rho \varphi(z_E) = 0$.

From Lemma 2.2 and the fact that $\nabla H = 2\pi \partial_\rho$ (see §2.4), we get

$$\pi \partial_\rho^2 \varphi = |\nabla H|^2 = |\xi_H|^2. \tag{25}$$

A closely related formula is that

$$\begin{aligned} H(e^\sigma z_0) - H(z_0) &= \int_0^\sigma \frac{d}{ds} H(e^s z_0) ds \\ &= \int_0^\sigma g\left(\nabla H(e^s(z_0)), \frac{d}{ds}(e^s z_0) \right) ds \\ &= \frac{1}{2\pi} \int_0^\sigma |\nabla H(e^s(z_0))|^2 ds. \end{aligned} \tag{26}$$

Hence for $z = e^{\tau_E(z)} \cdot z_E$ where $z_E \in H^{-1}(E)$, we have

$$b_E(z) = \int_0^{\tau_E(z)} \int_0^\sigma \pi^{-1} |\nabla H|^2 (e^s \cdot z_E) ds d\sigma$$

Monotonicity of b_E in $\tau_E(z)$ is evident from the formula when $\tau_E(z) > 0$ i.e. $z \in \mathcal{F}_E$, hence b_E is monotone increasing in ρ. $\square$

2.6. The Leafwise Symplectic Potential and b_E

In this section we relate $b_E(z)$ to leaf-wise symplectic potentials. To define the symplectic potentials we use slice-orbit coordinates (θ, ρ, y) as in (19) and pull back the Kähler potential (22) to $\mathbb{C}^*$, by $\varphi(\rho, \theta; y) = \varphi(e^{\rho + i\theta} \cdot y)$. Since the Kähler potential is $\mathbf{T}$ invariant, $\varphi(\rho, \theta; y)$ is θ independent and is convex in ρ, and will be denoted by $\varphi(\rho; y)$ relative to a slice $S_E \subset H^{-1}(E)$. Note that $\varphi(\rho; e^{i\theta} y) = \varphi(\rho; y)$ and so $\varphi(\rho, z_E)$ is defined for all $z_E \in H^{-1}(E)$.

The leafwise symplectic potential is defined to be the Legendre transformation of $\varphi(\rho; y)$,

$$u(I; y) = \sup_{\rho \in \mathbb{R}} (I\rho - \varphi(\rho; y))$$

Since $\varphi(\rho; y)$ is a smooth convex function in ρ, we have

$$u(I; y) = I\rho(I; y) - \varphi(\rho(I; y); y), \tag{27}$$

where $\rho(I; y)$ is s.t. $I = \partial_\rho \varphi(\rho(I; y); y)$. The Legendre transformation is an involution,

$$\varphi(\rho; y) = \sup_{I \in \mathbb{R}} (\rho I - u(I; y)) = \rho I(\rho; y) - u(I(\rho; y); y), \tag{28}$$

where $I(\rho; y)$ is s.t. $\rho = \partial_I u(I(\rho; y); y)$. Also their second derivatives are related by

$$\partial_I^2 u(I; y) = \partial_I \rho(E; y) = \frac{1}{\partial_\rho I(\rho; y)} = \frac{1}{\partial_\rho^2 \varphi(\rho; y)} \tag{29}$$

where $I = I(\rho; y)$ and $\rho = \rho(I; y)$. We use the notation I as in the "action-variable' dual to the angle variable θ; (27) implies that $I/2$ lies in the range of H (see Lemma (2.5)).

The following Lemma relates $b_E(z)$ with the symplectic potential, and can be easily verified using Lemma 2.3.

Lemma 2.4. *Let $z_E \in H^{-1}(E)$ and use gradient flow-coordinates $z = e^\rho \cdot z_E$. Then*

$$b_E(z) = -u(2H(z), z_E) - \varphi(z_E) + 2(H(z) - E)\tau_E(z), \tag{30}$$

Using the symplectic potential, one can easily derive the dependence of $b_E(z)$ in terms of E for fixed z.

Lemma 2.5. *Fix $z \in M$ and $E \in H(M)$, such that the $\mathbb{R}^*$ orbit of z intersects $H^{-1}(E)$ at z_E, and let $\tau(z,E) = \tau_E(z) \in \mathbb{R}$ be such that $e^{\tau_E(z)} \cdot z_E = z$. Then $b(z,E) = b_E(z)$ can be written as*

$$b(z,E) = \varphi(z) + u(2E; z) \tag{31}$$

and we have the following properties

$$\partial_E b(z,E) = -2\tau(z,E), \quad \partial_E^2 b(z,E) = \frac{4}{\partial_\rho^2 \varphi(z_E)}. \tag{32}$$

Hence $b(z,E)$ is a strictly convex function in E with minimum being 0 at $E = H(z)$.

Proof. First we claim that $\rho(2E;z) = -\tau(z,E)$. Indeed by the definition of $\rho(2E;z)$ in (27), we have

$$2E = \partial_\rho \varphi(\rho(2E;z);z) = \partial_\rho \varphi(e^{\rho(2E;z)} \cdot z) = 2H(e^{\rho(2E;z)} \cdot z)$$

and $\tau(z,E)$ by definition satisfies $E = H(z_E) = H(e^{-\tau(z,E)} \cdot z)$, hence $\rho(2E;z) = -\tau(z,E)$. Using (27) we have

$$u(2E;z) = 2E(-\tau(z,E)) - \varphi(e^{-\tau(z,E)} \cdot z) = -2\tau(z,E)E - \varphi(z_E)$$

Combined with (24), this proves (31). Next, using $\rho = \partial_I u(I(\rho;y);y)$ from (28), we have

$$\partial_E(b(z,E)) = 2\partial_I u(I;z)|_{I=2E} = 2\rho(2E;z) = -2\tau(z,E),$$

and using (29) we have

$$\partial_E^2(b(z,E)) = 4\partial_I^2 u(2E;z) = \frac{4}{\partial_\rho^2 \varphi(\rho(2E;z);z)} = \frac{4}{\partial_\rho^2 \varphi(z_E)}. \qquad \square$$

2.7. Periodicity of the lifted flow

We can now prove the periodicity statement in §1.3. Recall that the contact vector field is $\hat{\xi}_H = \xi_H^h - 2\pi R$.

Lemma 2.6. *The lifted flow* $\exp t\hat{\xi}_H$ *is* 1*-periodic, or equivalently,* $U_k(\theta)$ *is* 2π*-periodic.*

Proof. The equivalence follows from Corollary 1.3. By our choice of generator, the common period of all ξ_H-orbits equals 1, hence flow by ξ_H^h return to the same fiber. Since on the circle bundle, the horizontal vector field ξ_H^h and vertical Reeb vector field R commute, and $H(z)$ is constant along the ξ_H orbit, we may first flow by ξ_H^h for time 1, then by $-2\pi HR$ for time 1. Let θ_γ be defined such that $\exp(\xi_H^h)(z,\lambda) = (z, e^{i\theta_\gamma}\lambda)$. Then,

$$\theta_\gamma = i\int_\gamma A = i\int_0^{2\pi} \langle -\partial\varphi, \partial_\theta\rangle d\theta = 2\pi H$$

where we used identities from (23). Hence flowing by $-2\pi HR$ sends $(z,\lambda)\mapsto (z, e^{-i2\pi H}\lambda) = (z, e^{-i\theta_\gamma}\lambda)$, the two $e^{i\theta_\gamma}$ factor cancels, hence $\hat{\xi}_H$ has period 1. ☐

2.8. Examples

To illustrate the variety of S^1-Kähler manifolds, we first start with model linear cases and then proceed to other types of examples.

(0): Linear actions on $\mathbb{C}^m$ On the non-compact Kähler manifold $\mathbb{C}^m$ with Euclidean metric the linear S^1 actions have the form,

$$e^{i\theta}\cdot(Z_1,\dots,Z_m) = (e^{ib_1}Z_1,\dots,e^{ib_m}Z_m), \quad b_j\in\mathbb{Z},$$

with Hamiltonians $H = \sum_j b_j|Z_j|^2$. Extreme cases are where all $b_m = 0$ except $b_1 = 1$, in which case the fixed point set is a hypersurface $Z_1 = 0$, and the isotropic Harmonic oscillator with all $b_j = 1$ and Hamiltonian $|Z|^2$ with fixed point set $\{0\}$.

(i) Standard S^1 actions on $\mathbb{CP}^m$ They arise from subgroups $S^1 \subset SU(m+1)$ of the form

$$e^{i\theta}\cdot[Z_0,\dots,Z_m] = [e^{ib_0}Z_0,\dots,e^{ib_m}Z_m], \quad b_j\in\mathbb{Z}.$$

With no loss of generality it is assumed that $b_0 = 0$. When $b_j \neq b_k$ when $j\neq k$, the action has $m+1$ isolated fixed points, $P_j = [0,\dots,0,z_j,0,\dots,0]$. The weights at P_j are $\{b_j - b_i\}_{j\neq i}$. The Hamiltonian is

$$\mu_{\vec{b}}([Z_0 : \cdots : Z_m]) = \frac{b_1|Z_1|^2 + \cdots + b_m|Z_m|^2}{|Z|^2}.$$

(ii) Cubic hypersurface in $\mathbb{CP}^4$

This example is taken from [Ki]. Consider the cubic hypersurface $X \subset \mathbb{CP}^4$,

$$x^3 + y^3 + z^3 = u^2 v,$$

in $\mathbb{CP}^4 = \{[x, y, z, u, v]\}$ and let $\mathbb{C}^*$ act on X via the action on $\mathbb{CP}^4$,

$$t \cdot [x, y, z, u, v] = [t^{-1}x, t^{-1}y, t^{-1}z, t^{-3}u, t^3 v].$$

Then $X^{\mathbf{T}}$ has three fixed point components,

$$\begin{aligned} &F_1 = \{[0,0,0,1,0]\}, \quad F_2 = \{[x,y,z,0,0] : x^3 + y^3 + z^3 = 0\}, \\ &F_3 = \{[0,0,0,0,1]\}, \end{aligned}$$

of which two (F_1, F_3) are isolated fixed points and F_2 is a hypersurface in X, i.e. a curve. The point $P = [0,0,0,0,1]$ is singular.

The corresponding stable sets $S_j = \{x \in X : \lim_{t\to 0} t \cdot x \in F_j\}$ are

$$\begin{cases} S_1 = \{[x,y,z,u,v] \in X : u \neq 0\}, \\ S_2 = \{[x,y,z,u,v] \in X : u = 0, (x,y,z) \neq 0\}, \\ S_3 = \{[0,0,0,0,1]\}, \end{cases}$$

Here, S_1 is Zariski open in X, S_2 is of codimension one, and $S_3 = F_3$ is a point.

The Hamiltonian $H : X \to \mathbb{R}$ is the restriction of the Hamiltonian for the $\mathbf{T}$ action on $\mathbb{CP}^m$ above.

(iii) Ruled surfaces [HS]

Let M_g be a Riemann surface of genus g, equipped with a constant curvature metric. Let $L \to M$ be a holomorphic line bundle. L carries a natural $\mathbb{C}^*$ action. Projectivize each line $L_z \to \mathbb{P}L_z \simeq \mathbb{CP}^1$ to get $\mathbb{P}L$. It still carries a $\mathbb{C}^*$ action. Examples of S^1-invariant Kähler metrics are the constant scalar curvature metrics.

3. The Szegö kernel and the Boutet de Monvel-Sjöstrand parametrix

This section is preparation for Theorem 1 and the subsequent asymptotic results. The equivariant Bergman kernels $\Pi_{k,j}$ have two positive integer indices, indicating a lattice point in $\mathbb{Z}_+ \times \mathbb{Z}_+$[5]. The asymptotics in k for a fixed energy level E implicitly involve pairs (j_n, k_n) of lattice points along a "ray" in the joint lattice.

It is convenient to lift the sections of $H^0(M, L^k)$, resp. the equivariant kernels $\Pi_{k,j}$, as equivariant functions (resp. kernels) on the unit circle bundle $X_h \to M$ associated to the Hermitian line bundle (L^*, h^*), see §1.2. This circle bundle carries a canonical S^1 action. The Hamiltonian $\mathbf{T}$ action also lifts to X_h and thus the two commuting circle actions define an $S^1 \times S^1$ action, whose weights form the semi-lattice of $\{(j,k) \in \mathbb{Z}_+ \times \mathbb{Z}_+\}$. This lifting and the approximation of energy levels by rays in $\mathbb{Z}_+ \times \mathbb{Z}_+$ is discussed in detail in [STZ] for toric varieties, and the same discussion applies almost verbatim to $\mathbf{T}$ actions. We therefore summarize the key points and refer to [STZ] for further details.

3.1. The Szegö kernel and the Bergman kernel

We now discuss the analytic aspects of the lift to the circle bundle X_h and the disc bundle D_h in §1.2 and §1.3. We define the Hardy space $\mathcal{H}^2(X_h) \subset \mathcal{L}^2(X_h)$ of square-integrable CR functions on X_h, i.e., functions that are annihilated by the Cauchy-Riemann operator $\bar{\partial}_b$ and are $\mathcal{L}^2$ with respect to the inner product

$$\langle F_1, F_2 \rangle = \int_X F_1 \overline{F_2} dV_X\,, \quad F_1, F_2 \in \mathcal{L}^2(X)\,. \tag{33}$$

Equivalently, $\mathcal{H}^2(X)$ is the space of boundary values of holomorphic functions on D that are in $\mathcal{L}^2(X)$. Here, X_h is equipped with the contact volume form

$$dV_X = \frac{1}{m!}\frac{\alpha}{2\pi} \wedge \left(\frac{d\alpha}{2\pi}\right)^m = \frac{\alpha}{2\pi} \wedge dV_M, \quad \text{where} \quad dV_M = \frac{\omega^m}{m!}. \tag{34}$$

The S^1 action on X commutes with $\bar{\partial}_b$; hence $\mathcal{H}^2(X) = \bigoplus_{k=0}^\infty \mathcal{H}^2_k(X)$ where $\mathcal{H}^2_k(X) = \{F \in \mathcal{H}^2(X) : F(r_\theta x) = e^{ik\theta} F(x)\}$. As mentioned in §1.3, a

[5]The $\mathbf{T}$ action in general has $\mathbb{Z}$ weights, since we have chosen H such that $H(M) \geq 0$, the corresponding weights are in $\mathbb{Z}_+$.

section s_k of L^k determines an equivariant function $\hat{s}_l$ on L^* by the rule

$$\hat{s}_k(\lambda) = \left(\lambda^{\otimes k}, s_k(z)\right), \quad \lambda \in L_z^*, \ z \in M,$$

where $\lambda^{\otimes k} = \lambda \otimes \cdots \otimes \lambda$. We henceforth restrict $\hat{s}$ to X and then the equivariance property takes the form $\hat{s}_k(r_\theta x) = e^{ik\theta}\hat{s}_k(x)$. The map $s \mapsto \hat{s}$ is a unitary equivalence between $H^0(M, L^k)$ and $\mathcal{H}_k^2(X)$. (This follows from (34)–(33) and the fact that $\alpha = d\theta$ along the fibers of $\pi : X \to M$.)

We define the (lifted) Szegö kernel $\hat{\Pi}(x, y)$ to be the (Schwarz) kernel of the orthogonal projection $\hat{\Pi}_k : \mathcal{L}^2(X) \to \mathcal{H}^2(X)$. It is given by

$$\hat{\Pi}F(x) = \int_X \hat{\Pi}(x, y)F(y)dV_X(y), \quad F \in \mathcal{L}^2(X). \tag{35}$$

The Fourier components $\hat{\Pi}_k : \mathcal{L}^2(X) \to \mathcal{H}_k^2(X)$ of the Szegö projector can be extracted from $\hat{\Pi}(x, y)$ by

$$\hat{\Pi}_k(x, y) = \int_0^{2\pi} e^{-ik\theta}\hat{\Pi}(r_\theta x, y)\frac{d\theta}{2\pi}. \tag{36}$$

The Szegö (or Bergman)[6] kernel $\Pi_k(z, w)$ for the orthogonal projection $\Pi_k : \mathcal{L}^2(M, L^k) \to H^0(M, L^k)$ can be obtained via the isometry of $H^0(M, L^k) \cong \mathcal{H}_k^2(X)$.

In a local coordinate patch U with a holomorphic frame $e_L \in \Gamma(U, L)$, we introduce two scalar kernels $K_k(z, w)$ and $B_k(z, w)$, with respect to the holomorphic frame and unitary frame:

$$\Pi_k(z, w) =: K_k(z, w) \cdot e_L^k(z) \otimes \overline{e_L^k(w)} =: B_k(z, w) \cdot \frac{e_L^k(z)}{\|e_L^k(z)\|_h} \otimes \overline{\frac{e_L^k(w)}{\|e_L^k(w)\|_h}}.$$

The Bergman density function $\Pi_k(z)$ is the contraction of $\Pi_k(z, w)$ with the hermitian metric on the diagonal,

$$\Pi_k(z) := B_k(z, z)(:= \Pi_k(z, z)),$$

where in the second equality we record a standard abuse of notation in which the diagonal of the Szegö kernel is identified with its contraction.

[6]In the setting of line bundles, we use the terms interchangeably.

3.2. Equivariant Szegö kernels

Let e_L is a local $\mathbf{T}$-invariant holomorphic frame and we define equivariant Bergman kernel and densities,

$$\begin{cases} \Pi_{k,j}(z,w) = K_{k,j}(z,w) \cdot e_L^k(z) \otimes \overline{e_L^k(w)} \\ \qquad\qquad\quad = B_{k,j}(z,w) \cdot \dfrac{e_L^k(z)}{\|e_L^k(z)\|_h} \otimes \overline{\dfrac{e_L^k(w)}{\|e_L^k(w)\|_h}}, \\ \Pi_{k,j}(z) = B_{k,j}(z,z). \end{cases} \tag{37}$$

Equivariant Bergman kernels are closely related to Bergman kernels for the reductions of the level sets $H^{-1}(\frac{j}{k})$. For instance, the space of invariant sections

$$V_k(0) := H^0_{\mathbf{T}}(M, L^k) = \{s \in H^0(M, L^k) : e^{i\theta} s = s\}. \tag{38}$$

is isomorphic in a canonical way to the space of holomorphic sections of the reduced line bundle $L_{\mathbf{T}}$ on the reduced space $M_E := H^{-1}(E)/S^1$, i.e. $V_k(0) \simeq H^0(M_E, L^k_{\mathbf{T}}\}$, so that $\dim H^0_{\mathbf{T}}(M, L^k) = \mathrm{Vol}(H^{-1}(E)/S^1)\, k^{m-1}$.

3.3. The Boutet de Monvel-Sjöstrand parametrix

Near the diagonal in $X_h \times X_h$, there exists a parametrix due to Boutet de Monvel-Sjöstrand [BSj] for the Szegö kernel of the form,

$$\hat{\Pi}(x,y) = \int_{\mathbb{R}^+} e^{\sigma\psi(x,y)} \chi(z_x, z_y) s(x,y,\sigma) d\sigma + \hat{R}(x,y). \tag{39}$$

Here, $\chi(z_x, z_y)$ is a smooth cutoff supported in a neighborhood of the diagonal of $M \times M$. $\psi(x,y)$ is defined as (up to $2\pi\mathbb{Z}i$ ambiguity)

$$\psi(x,y) = +\log \lambda_x + \log \overline{\lambda_y} + \varphi(z_x, z_y)$$

where $x = \lambda_x \cdot e_L^*(z_x) \in X_h$ for $z_x \in M, \lambda_x \in \mathbb{C}^*$, similarly for y, with respect to a local trivialization $e_L \in \Gamma(U, L)$. And $\varphi(z,w)$ is the almost analytic extension of $\varphi(z)$ (we abuse notation), that is

$$\varphi(z,z) = \varphi(z), \bar{\partial}_z \varphi(z,w) = \partial_w \varphi(z,w) = 0$$
to infinite order on $\Delta_M \subset M \times M$.

On the co-circle bundle, we have $2\mathrm{Re}\log\lambda_x = -\varphi(z_x)$ and $2\mathrm{Re}\log\lambda_y = -\varphi(z_y)$, hence if we write $\theta_x = \arg\lambda_x, \theta_y = \arg\lambda_y$, we have

$$\psi(x,y) = i\theta_x - i\theta_y + \varphi(z_x, z_y) - \varphi(z_x)/2 - \varphi(z_y)/2.$$

The real part of ψ proportional to the Calabi-Diastasis,

$$\mathrm{Re}\psi(x,y) = -\frac{1}{2}D(z_x, z_y),$$

where

$$D(z,w) := -\varphi(w,z) - \varphi(z,w) + \varphi(z) + \varphi(w), \tag{40}$$

is defined near the diagonal of $M \times M$, and is positive and only vanishes when $z = w$. The amplitude s is a classical symbol,

$$s \sim \sum_{n=0}^{\infty} \sigma^{m-n} s_n(x,y). \tag{41}$$

Finally, the remainder term $\hat{R}(x,y)$ is C^∞.

From the parametrix for $\hat{\Pi}$ one can derive semi-classical parametrices for the Fourier components and thus for the semi-classical Szegö kernels on $H^0(M, L^k)$. If we substitute the first term of (39) into (36), one obtains the oscillatory integral,

$$\hat{\Pi}_k(x,y) \sim \int_{\mathbb{R}^+}\int_0^{2\pi} e^{\sigma\psi(x,r_\theta y)} e^{ik\theta} \chi(z_x, z_y) s(x, r_\theta y, \sigma) d\theta d\sigma, \tag{42}$$

Changing variables $\sigma \to k\sigma$ and eliminating the $d\theta d\sigma$ integral by the stationary phase method gives, at least formally, the off-diagonal expansion for the full Szegö kernel on M,

$$K_k(z,w) = e^{k\varphi(z,w)} k^m \ (1 + O(k^{-1})). \tag{43}$$

A direct construction of the parametrix is given in [BBSj] (where (43) is stated as (2.2)).

We only use the full Bergman kernel and the the parametrices (43)–(42) in Sections 4.1 and 7.1, in the proof of the Localization Lemma, and in Section 8.

4. Equivariant Bergman kernels: Proofs of Theorem 1 and Theorem 2

In this section, we prove that the equivariant Bergman kernel $\Pi_{k,j}(z,z)$ forms a Gaussian bump around the hypersurface $H^{-1}(j/k)$, with decay width $\sim 1/\sqrt{k}$.

Lemma 4.1. *For all $\alpha, \beta \in \mathbb{C}$, we have*

$$K_{k,j}(e^\alpha \cdot z, e^\beta \cdot w) = e^{j(\alpha+\bar{\beta})} K_{k,j}(z,w)$$

and

$$B_{k,j}(e^\alpha \cdot z, e^\beta \cdot w) = e^{j(\alpha+\bar{\beta})-k(\varphi(e^\alpha \cdot z)-\varphi(z))/2-k(\varphi(e^\beta \cdot w)-\varphi(w))/2} B_{k,j}(z,w).$$

In particular, if we set $\beta = -\bar{\alpha}$, we have

$$K_{k,j}(z,w) = K_{k,j}(e^\alpha \cdot z, e^{-\bar{\alpha}} \cdot w).$$

Proof. This is immediate from the definition of K and B. □

4.1. Proof of Theorem 1

Theorem 1 follows from the following two propositions. The first one establishes the decay property of $\Pi_{k,j}(z)$ away from the real hypersurface $H^{-1}(j/k)$. Recall the definition of $b_E(z)$ (9).

Proposition 4.2. *Fix (k,j) and $z \in H^{-1}(E)$ where $E = j/k$. Then, for any $\alpha \in \mathbb{R}$, we have*

$$\Pi_{k,j}(e^\alpha \cdot z) = e^{-kb_E(e^\alpha z)} \Pi_{k,j}(z).$$

Proof. Using Lemma 4.1 with $z = w$, $\alpha = \beta \in \mathbb{R}$, we have

$$B_{k,j}(e^\alpha \cdot z, e^\alpha \cdot z) = e^{2j\alpha - k(\varphi(e^\alpha \cdot z)-\varphi(z))} B_{k,j}(z,z).$$

Now, write $j = kE = kH(z) = k\partial_\rho \varphi/2$, we have

$$\begin{aligned}\Pi_{k,j}(e^\alpha \cdot z) = B_{k,j}(e^\alpha \cdot z, e^\alpha \cdot z) &= e^{-k(\varphi(e^\alpha \cdot z)-\varphi(z)-\alpha\partial_\rho\varphi(z))} B_{k,j}(z,z) \\ &= e^{-kb_E(e^\alpha z)} B_{k,j}(z,z) = e^{-kb_E(e^\alpha z)} \Pi_{k,j}(z).\end{aligned}$$

□

These are exact identities and do not involve parametrices. Next, we express $K_{k,j}$ as a Fourier coefficient of K_k with respect to the Hamiltonian S^1 action and give a parametrix formula,

Lemma 4.3. *For any j,*

$$\begin{aligned} K_{k,j}(z,z) &= \int_0^{2\pi} K_k(e^{i\theta}\cdot z, z)e^{-ij\theta}\frac{d\theta}{2\pi} \\ &= k^m \int_0^{2\pi} e^{k\varphi(e^{i\theta}z,z)}e^{-ij\theta}\chi(e^{i\theta}z,z)(1+O(1/k))\frac{d\theta}{2\pi} \end{aligned}$$

where $\chi(z,w)$ is a cut-off function supported in a neighborhood U of the diagonal of $M\times M$.

Proof. The first line is evident and the second uses (43). □

The next proposition studies the kernel $\Pi_{k,j_k}(z)$ when $z\in H^{-1}(E)$ and $j_k/k\to E$.

Proposition 4.4. *Fix a regular value E of $H: M\to\mathbb{R}$, and a sequence $\{j_k\}$ such that $|\frac{j_k}{k}-E|<C/k$ for some positive constant C. Then for any $z\in H^{-1}(E)$ with trivial stabilizer in the $\mathbf{T}$-action, we have*

$$\Pi_{k,j_k}(z) = k^{m-1/2}\sqrt{\frac{2}{\pi\partial_\rho^2\varphi(z)}}(1+O(1/k)).$$

Proof. Let $E_k = j_k/k$, and $z_k\in H^{-1}(E_k)$ $\rho_k\in\mathbb{R}$, such that $z=e^{\rho_k}z_k$. We have $|\rho_k| = O(1/k)$. Indeed,

$$C/k > |E_k - E| = \frac{1}{2}|\partial_\rho\varphi(z_k)-\partial_\rho\varphi(z)| = \frac{1}{2}\left|\int_0^{\rho_k}\partial_\rho^2\varphi(e^sz)ds\right| > C'|\rho_k|$$

where we used the fact φ is psh and $\mathbf{T}$-invariant, to get $\partial_\rho^2\varphi$ strictly positive, hence $|\rho_k| = O(1/k)$. Then using Proposition 4.2, we get

$$\begin{aligned} \Pi_{k,j_k}(z) &= \Pi_{k,j_k}(z_k)e^{-kb_{E_k}(e^{\rho_k}z_k)} \\ &= \Pi_{k,j_k}(z_k)e^{-kO(\rho_k^2)} = \Pi_{k,j_k}(z_k)(1+O(1/k)). \end{aligned} \tag{44}$$

Next, we evaluate $\Pi_{k,j_k}(z_k)$ using the parametrix of Lemma 4.3 and the stationary phase method.

Setting $j = j_k, z = z_k$ in Lemma 4.3, we get

$$K_{k,j_k}(z_k, z_k) = k^m \int_{-\pi}^{\pi} e^{k(\varphi(e^{i\theta}z_k,z_k)-iE_k\theta)}\chi(e^{i\theta}z_k, z_k)(1+O(1/k))\frac{d\theta}{2\pi}. \tag{45}$$

This is not quite a standard stationary phase integral because the phase

$$\Psi_k(i\theta) := \varphi(e^{i\theta}z_k, z_k) - iE_k\theta, \tag{46}$$

depends on k. However, all aspects of the stationary phase expansion (see [Hö]) extend with no essential change to (46).

We claim that $\theta = 0$ is a Morse critical point of (46). To see this, we use $H = -i\langle\partial_\theta, \partial\varphi\rangle$ from (23)'s first equality. Thus, the first derivative of $\Psi_k(i\theta)$ at $\theta = 0$ is

$$-i\partial_\theta\Psi_k(i\theta)|_{\theta=0} = H(z_k) - E_k = 0.$$

To calculate the second derivative at $\theta = 0$ we rewrite

$$\varphi(e^{i\theta}z, z) = \varphi(e^{i\theta/2}z, e^{-i\theta/2}z),$$

using the $\mathbf{T}$ invariance of φ, then we extend (46) to a holomorphic function

$$\Psi_k(\tau) = \varphi(e^{\tau/2}\cdot z_k, e^{\bar{\tau}/2}\cdot z_k) - E_k\tau. \tag{47}$$

The Taylor expansion of $\Psi_k|_{\mathbb{R}}$, has the form,

$$\begin{aligned}\Psi_k(t) &= \varphi(e^{t/2}z_k, e^{t/2}z_k) - tE_k \\ &= \varphi(e^{t/2}z_k) - tE_k = \varphi(z_k) + \frac{t^2}{8}\partial_\rho^2\varphi(z_k) + O(t^3).\end{aligned}$$

Thus $\theta = 0$ is a non-degenerate isolated critical point of $\Psi(i\theta)$, with Hessian $\partial_\theta^2|_{\theta=0}\Psi(i\theta) = -\Psi''(0) = -\frac{1}{4}\partial_\rho^2\varphi(z)$. Hence, we may choose $\epsilon > 0$ small enough such that for $|\theta| < \epsilon$ there is no other critical point than $\theta = 0$. Let $\eta(\theta) \in C_c^\infty(\mathbb{R})$, such that $\eta(\theta) \equiv 1$ for $|\theta| < \epsilon/2$ and $\eta(\theta) \equiv 0$ for $|\theta| > \epsilon$. Since $e^{i\theta}\cdot z \neq z$ only when $\theta \neq 0$ by the assumption on the stabilizer of S^1-action on z, and since (as in (40))

$$\mathrm{Re}\varphi(z, w) - \frac{1}{2}\varphi(z) - \frac{1}{2}\varphi(w) = -\frac{1}{2}D(z, w) \leq 0$$

and only vanishes when $z = w$, we have the following upper bound on the real part of the phase function

$$\sup\{\mathrm{Re}\varphi(e^{i\theta}\cdot z, z) - \varphi(z) \mid (e^{i\theta}\cdot z, z) \in U,\ |\theta| \in [\epsilon/2, \pi]\} = -c < 0. \tag{48}$$

for some positive constant c, where we used $\varphi(e^{i\theta}z) = \varphi(z)$ by the $\mathbf{T}$-invariance of φ. It follows that

(49)

$$\begin{aligned}
&\Pi_{k,j_k}(z_k) = e^{-k\varphi(z_k)} K_{k,j_k}(z_k, z_k) \\
&= k^m \int_{-\pi}^{\pi} e^{k(\varphi(e^{i\theta}z_k, z_k) - \varphi(z_k) - iE_k\theta)} \eta(\theta)\chi(e^{i\theta}z_k, z_k)(1 + O(1/k))\frac{d\theta}{2\pi} \\
&\quad + k^m \int_{-\pi}^{\pi} e^{k(\varphi(e^{i\theta}z_k, z_k) - \varphi(z_k) - iE_k\theta)} (1 - \eta(\theta))\chi(e^{i\theta}z_k, z_k)(1 + O(1/k))\frac{d\theta}{2\pi} \\
&= k^{m-1/2} \sqrt{\frac{2}{\pi \partial_\rho^2 \varphi(z_k)}}(1 + O(1/k)) \\
&= k^{m-1/2} \sqrt{\frac{2}{\pi \partial_\rho^2 \varphi(z)}}(1 + O(1/k))
\end{aligned}$$

where we applied stationary phase method to the first term and bound the second term by $O(e^{-ck})$ using (48). Since $z = e^{\rho_k} z_k$ with $|\rho_k| = O(1/k)$, we replaced z_k by z in the last step without changing the remainder estimate. Combining (44) and (49), we finish the proof of the proposition. $\square$

Remark 5. If the stabilizer G_z of z is non-trivial then it is a cyclic group generated by $\zeta = e^{\frac{2\pi i}{\ell}}$ for some positive integer ℓ. By Lemma 4.1 and by the stabilizer condition, $K_{k,j_k}(e^{\frac{2\pi i}{\ell}} \cdot z, z) = e^{\frac{2\pi i j_k}{\ell}} K_{k,j_k}(z, z) = K_{k,j_k}(z, z)$. Hence, either $K_{k,j_k}(z, z) = 0$ or $e^{\frac{2\pi i j_k}{\ell}} = 1$, i.e. $\frac{j_k}{\ell} \in \mathbb{Z}$.

The stationary phase method applies as well, and each element $\zeta^n, n = 0, \ldots, \ell - 1$ is a critical point of the $d\theta$ integral. The phase has the critical value $e^{\frac{2\pi i j_k n}{\ell}}$ at ζ^n. The Hessian is independent of n, so the leading term of the stationary phase expansion is

$$k^{m-1/2} \sqrt{\frac{2}{\pi \partial_\rho^2 \varphi(z)}} \sum_{n=0}^{\ell-1} e^{\frac{2\pi i j_k n}{\ell}}.$$

If $\frac{j_k}{\ell} \in \mathbb{Z}$ then each term is 1 and the sum is ℓ. Otherwise, $K_{k,j_k}(z, z) = 0$.

The above two propositions finish the proof of Theorem 1.

Remark 6. If $\varphi(z)$ is real analytic, then $\Psi(\tau)$ is holomorphic when $\mathrm{Im}(\tau)$ is small enough. If φ is only smooth, then $\Psi(\tau)$ is an almost analytic extension of $\Psi|_{\mathbb{R}}$. Although the proof uses the parametrix, it only uses Ψ in the real domain and only uses the C^∞ remainder. Hence, it does not require real analyticity.

4.2. Proof of Theorem 2

Theorem 2 follows from the following proposition.

Proposition 4.5. *For any fixed k, j, $z \in H^{-1}(j/k)$ and $\alpha \in \mathbb{R}$, we have*

$$\Pi_{k,j}(e^{\alpha/\sqrt{k}} \cdot z) = e^{-\frac{\alpha^2}{2}\partial_\rho^2\varphi(z)}\Pi_{k,j}(z)(1 + O(k^{-1/2}))$$

Proof. This follows from Proposition 4.2, and Lemma 2.3. We Taylor expand $b_E(e^\alpha \cdot z)$ in α around $\alpha = 0$, to get

$$b_E(e^\alpha \cdot z) = \frac{\alpha^2}{2}\partial_\rho^2\varphi(z) + g_3(z, \alpha), \text{ where } g_3(z, \alpha) = O(|\alpha|^3). \tag{50}$$

Then we plug in the expansion to the exponent $e^{-kb_E(e^\alpha \cdot z)}$ to get

$$\begin{aligned}
\Pi_{k,j}(e^{\alpha/\sqrt{k}} \cdot z) &= e^{-kb_E(e^{\alpha/\sqrt{k}} \cdot z)}\Pi_{k,j}(z) \\
&= e^{-k(\frac{\alpha^2}{2k}\partial_\rho^2\varphi(z) + g_3(z, \frac{\alpha}{\sqrt{k}}))}\Pi_{k,j}(z) \\
&= e^{-\frac{\alpha^2}{2}\partial_\rho^2\varphi(z)}\Pi_{k,j}(z)\left(1 + O\left(kg_3\left(z, \frac{\alpha}{\sqrt{k}}\right)\right)\right) \\
&= e^{-\frac{\alpha^2}{2}\partial_\rho^2\varphi(z)}\Pi_{k,j}(z)(1 + O(k^{-1/2})).
\end{aligned}$$

□

5. Lemma for Localization of sums

In this section we consider the sums in the partial Bergman kernels (6). We prove several localization formulae for these sums. Roughly speaking a localization formula says that, for a given z, only terms in the sums with $|\frac{j}{k} - H(z)| < \frac{M}{\sqrt{k}}$ contribute to the leading order asymptotics.

Lemma 5.1. *As in Theorem 1, let (L, h, M, ω) be a Kähler manifold with a positive line bundle, and H generates a holomorphic S^1-action on (L, M), with E a regular value of H, and $z \in H^{-1}(E)$ with $\mathbb{C}^*$ acting freely on z. Fix any smooth cut-off function $\chi : \mathbb{R} \to [0, 1]$, such that $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$. Then we have:*

(1) For any $1/2 \gg \epsilon > 0$, we have

$$\sum_{j/k \in H(M)} \left(1 - \chi\left(\frac{|j/k - H(z)|}{k^{-1/2+\epsilon}}\right)\right) \Pi_{k,j}(z) = O(k^{-\infty}). \tag{51}$$

(2) For any $R > 0$, there exists C large enough such that

$$\sum_{j/k \in H(M)} \left(1 - \chi\left(\frac{|j/k - H(z)|}{Ck^{-1/2}\sqrt{\log k}}\right)\right) \Pi_{k,j}(z) = O(k^{-R}). \tag{52}$$

(3) For any $\epsilon > 0$, there exists C large enough such that for large enough k

$$\sum_{j/k \in H(M)} \left(1 - \chi\left(\frac{|j/k - H(z)|}{Ck^{-1/2}}\right)\right) \Pi_{k,j}(z) < \epsilon k^m. \tag{53}$$

The above statements are also true for $\chi(x) = 1_{[0,1]}(x)$.

Proof. First we prove (1) and (2). From Proposition 4.2, we have

$$\Pi_{k,j}(z) = e^{-kb(z,j/k)} \Pi_{k,j}(z_j) = O(k^{m-1/2} e^{-kb(z,j/k)}).$$

If $j/k > H(z)$ and $1 - \chi\left(\frac{|j/k - H(z)|}{k^{-1/2}S}\right)$ is nonzero, e.g. for $S = k^\epsilon$ or $C\sqrt{\log k}$, then by monotonicity of $b(z, E)$ in E for $E > H(z)$ (Lemma 2.5), we have

$$kb(z, j/k) > kb(z, H(z) + k^{-1/2}S) = \frac{1}{2}\partial_E^2 b(z, H(z)) S^2 + O(k^{-1/2}S^3).$$

Similar statement is true for $j/k < H(z)$. Hence

$$\begin{aligned} &\sum_{j/k > H(z)} \left(1 - \chi\left(\frac{|j/k - H(z)|}{k^{-1/2}S}\right)\right) e^{-kb(z,j/k)} \Pi_{k,j}(z_j) \\ &= O(e^{-\frac{1}{2}\partial_E^2 b(z,H(z))S^2 + (m+1/2)\log k}). \end{aligned}$$

If $S = k^\epsilon$, then $-\frac{1}{2}\partial_E^2 b(z, H(z))S^2 + (m + 1/2)\log k < -ck^{2\epsilon}$ as $k \to \infty$, proving (51). If $S = C\sqrt{\log k}$, then for any $R > 0$, we can choose C large enough such that

$$\begin{aligned}
&-\frac{1}{2}\partial_E^2 b(z,H(z))S^2 + (m+1/2)\log k \\
&= \left(m + 1/2 - C^2\left(\frac{1}{2}\partial_E^2 b(z,H(z))\right)\right)\log k \\
&< -R\log k,
\end{aligned}$$

proving (52).

To prove (3), it is not enough to have a uniform bound on the summand, one needs to prove that the summand decays fast. Consider the range of j, where

$$I_{k,H(z)} = \{j : k^{-1/2}C < |j/k - H(z)| < k^{\epsilon-1/2}\},$$

then

$$\begin{aligned}
b(z,j/k) &= b(z,H(z)) + \partial_E b(z,H(z))(j/k - H(z)) \\
&\quad + \frac{1}{2}\partial_E^2 b(z,H(z))(j/k-H(z))^2 + O(|j/k-H(z)|^3) \\
&= \frac{1}{2}\partial_E^2 b(z,H(z))(j/k-H(z))^2 + O(k^{3\epsilon-3/2})
\end{aligned}$$

hence sum over $j \in I_{k,H(z)}$ gives

$$\begin{aligned}
&\sum_{j\in I_{k,H(z)}} \Pi_{k,j}(z) \\
&= \sum_{j\in I_{k,H(z)}} \Pi_{k,j}(z_j)e^{-kb(z,j/k)} \\
&= \sum_{j\in I_{k,H(z)}} k^{m-1/2}\sqrt{\frac{2}{\pi\partial_\rho^2\varphi(z_j)}}e^{-\partial_E^2 b(z,H(z))(\sqrt{k}(j/k-H(z)))^2}(1+O(k^{3\epsilon-1/2})) \\
&< c_1 k^{m-1/2}\sqrt{k}\int_C^\infty e^{-\partial_E^2 b(z,H(z))x^2}dx \\
&= c_1 k^m \delta_C,
\end{aligned}$$

where δ_C is a constant depending on C and $\partial_E^2 b(z,H(z))$, such that as $C\to\infty$, $\delta_C\to 0$. For any given $\epsilon>0$, we may take C large enough, such that $c_1\delta_C<\epsilon$. Thus combining with part (1) of the proposition, we finished the proof of part (3). $\square$

6. Proofs of Theorem 3 and 4: Summing equivariant Bergman kernels

In this section we prove results about partial Bergman kernel asymptotics in the interior of the allowed/forbidden regions (Theorem 3) and near the interface(4), using localization Lemma 5.1 and asymptotics of equivariant Bergman kernels in Theorem 1 and Theorem 2. Let $P = [0, E) \subset H(M)$, and recall the partial Bergman density as $\Pi_{k,P}(z) = \sum_{j/k \in P} \Pi_{k,j}(z)$.

We fix a standard smooth cut-off function $\chi : \mathbb{R} \to [0, 1]$, such that $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$.

Proof of Theorem 3 . (Allowed Region). If z is in the allowed region, we may use the localization formula (51) for the sum to write

$$\Pi_{k,P}(z) = \sum_{j \in kP \cap \mathbb{Z}} \chi\left(\frac{|H(z) - j/k|}{k^{-1/2+\delta}}\right) \Pi_{k,j}(z) + O(k^{-\infty}).$$

However, this is the same as the full Bergman kernel, up to another $O(k^{-\infty})$ error term.

(Forbidden Region). If z is in the forbidden region, $H(z) > E$, then only terms with $|H(z) - j/k|$ small will contribute. Recall as in Definition 0.1, we define $j_k = \max\{\mathbb{Z} \cap k[0, E)\}$ and $E_k = j_k/k$. Let $z_j \in H^{-1}(j/k)$ and τ_j be such that, $z = e^{\tau_j} z_j$. Since $H(z) > E > H(z_j)$, we have $\tau_j > 0$. Then using Proposition 4.2, we have

$$\begin{aligned}\frac{\Pi_{k,P}(z)}{e^{-kb(z,E_k)}\Pi_{k,j_k}(z_{j_k})} &= \sum_{j \in kP \cap \mathbb{Z}} \frac{\Pi_{k,j}(z)}{e^{-kb(z,E_k)}\Pi_{k,j_k}(z_{j_k})} \\ &= \sum_{j \in kP \cap \mathbb{Z}} \frac{e^{-kb(z,j/k)}\Pi_{k,j}(z_j)}{e^{-kb(z,E_k)}\Pi_{k,j_k}(z_{j_k})}.\end{aligned}$$

For any $1 \gg \epsilon > 0$, we claim the following localization result

$$\begin{aligned}&\frac{\Pi_{k,P}(z)}{e^{-kb(z,E_k)}\Pi_{k,j_k}(z_{j_k})} \\ &= \sum_{j \in kP \cap \mathbb{Z}} \frac{e^{-kb(z,j/k)}\Pi_{k,j}(z_j)}{e^{-kb(z,E_k)}\Pi_{k,j_k}(z_{j_k})} \chi\left(\frac{|j/k - E_k|}{k^{-1+\epsilon}}\right) + O(k^{-\infty}).\end{aligned} \tag{54}$$

Proof of the claim: By Taylor expansion of $b(z,E)$ in E, there exists $\delta, C > 0$, such that $\forall |E - E_k| < \delta$, and

$$b(z,E) = b(z,E_k) + (E - E_k)\partial_E b(z,E_k) + R_b^{(2)}(z,E,E_k), \tag{55}$$

where $|R_b^{(2)}(z,E,E_k)| \leq C|E - E_k|^2$. Then if $(j - j_k) > k^\epsilon$, and k large enough such that $k^{-1+\epsilon} < \delta$, then

$$\begin{aligned} k[b(z,j/k) - b(z,E_k)] &> k[b(z,j + k^\epsilon/k) - b(z,E_k)] \\ &= \partial_E b(z,E_k)k^\epsilon + kR_b^{(2)}(z,E_k + k^{\epsilon-1},E_k). \end{aligned}$$

Since

$$\partial_E b(z,E_k) = -2\tau(z,E_k) > 0,$$

and

$$kR_b^{(2)}(z,E_k + k^{\epsilon-1},E_k) < Ck^{-1+2\epsilon} \ll k^\epsilon,$$

we have

$$\sum_{j \in kP \cap \mathbb{Z}} \frac{e^{-kb(z,j/k)}\Pi_{k,j}(z_j)}{e^{-kb(z,E_k)}\Pi_{k,j_k}(z_{j_k})}\left(1 - \chi\left(\frac{|j - j_k|}{k^\epsilon}\right)\right) = O(k^{-\infty}).$$

This finishes the proof of the localization claim (54).

Next, we claim that the sum in (54) can be approximated by an infinite geometric series with $O(1/k)$ error. Indeed, using Proposition 4.4, we have

$$\frac{\Pi_{k,j}(z_j)}{\Pi_{k,j_k}(z_{j_k})} = \sqrt{\frac{\partial_\rho^2\varphi(z_{j_k})}{\partial_\rho^2\varphi(z_j)}} + O(1/k) = 1 + R_1(z_j, z_{j_k}) + O(k^{-1}),$$

where $|R_1(z_j, z_{j_k})| < C|H(z_j) - E_k| = k^{-1} \cdot C|j - j_k|$. And from (32), we have

$$\frac{e^{-kb(z,j/k)}}{e^{-kb(z,E_k)}} = e^{-A(j_k - j)}(1 + R_2(j,j_k)), \quad A = -\partial_E b(z,E_k) = 2\tau(z,E_k),$$

where

$$R_2(j,j_k) = e^{kR_b^{(2)}(z,j/k,E_k)} - 1 < Ck \cdot |j/k - E_k|^2 = k^{-1} \cdot C|j - j_k|^2.$$

Hence we get

$$\begin{aligned}
&\sum_{j\in kP\cap\mathbb{Z}} \frac{e^{-kb(z,j/k)}\Pi_{k,j}(z_j)}{e^{-kb(z,E_k)}\Pi_{k,j_k}(z_{j_k})}\chi\left(\frac{|j/k-E_k|}{k^{-1+\epsilon}}\right)\\
&= \sum_{j\in kP\cap\mathbb{Z}} e^{-A(j_k-j)}(1+k^{-1}R(j-j_k,z_0))\chi\left(\frac{|j/k-E_k|}{k^{-1+\epsilon}}\right)\\
&= \left(\sum_{j\in kP\cap\mathbb{Z}} e^{-A(j_k-j)}\chi\left(\frac{|j/k-E_k|}{k^{-1+\epsilon}}\right)\right)(1+O(k^{-1}))\\
&= \sum_{j\in kP\cap\mathbb{Z}} e^{-A(j_k-j)}(1+O(k^{-1})) = (1-e^{-A})^{-1}(1+O(k^{-1})),
\end{aligned}$$

where $R(m,z_0)$ has at most polynomial growth in m, hence is integrable against the exponential decaying factor. Thus, we have proved

$$\Pi_{k,P}(z) = \Pi_{k,j_k}(z_{j_k})\frac{e^{-kb(z,E_k)}}{1-e^{-2\tau(z,E_k)}}(1+O(k^{-1})).$$

Using (49), we have

$$\Pi_{k,P}(z) = k^{m-1/2}\sqrt{\frac{2}{\pi\partial_\rho^2\varphi(z_k)}}\frac{e^{-kb(z,E_k)}}{1-e^{-2\tau(z,E_k)}}(1+O(k^{-1})).$$

Finally, one may replace $\partial_\rho^2\varphi(z_k)$ by $\partial_\rho^2\varphi(z_E)$ and $\tau(z,E_k)$ by $\tau(z,E)$ with an additional $(1+O(1/k))$ factor. This concludes the proof of Theorem 3. □

Proof of Theorem 4. We write z for the sequence $z_k = e^{\beta/\sqrt{k}}\cdot z_E$, for point $z_E \in H^{-1}(E)\cap M^E_{\max}$. By the Localization Lemma 5.1

$$\begin{aligned}
\Pi_{k,P}(z) &= \sum_{j\in kP\cap\mathbb{Z}} \Pi_{k,j}(z)\chi\left(\frac{|H(z)-j/k|}{k^{-1/2+\epsilon}}\right) + O(k^{-\infty})\\
&= \sum_{j\in kP\cap\mathbb{Z}} e^{-kb(z,j/k)}\Pi_{k,j}(z_j)\chi\left(\frac{|H(z)-j/k|}{k^{-1/2+\epsilon}}\right) + O(k^{-\infty})
\end{aligned}$$

Next we Taylor expand $b(z,E)$ around $E = H(z)$, $\exists \delta, C > 0$, such that for all $|H(z) - E| < \delta$, we have

$$\begin{aligned} b(z,E) &= \frac{|E-H(z)|^2}{2}\partial_E^2 b(z,E) + R_b^{(3)}(z,E) \\ &= \frac{|E-H(z)|^2}{2}\frac{4}{\partial_\rho^2\varphi(z)} + R_b^{(3)}(z,E) \end{aligned}$$

where we have used Lemma 2.5, and $|R_b^{(3)}(z,E)| < C|E-H(z)|^3$. Define

$$u_j = \sqrt{k}(j/k - H(z))$$

we have

$$kb(z,j/k) = A_2 u_j^2 + kR_b^{(3)}(z,j/k), \quad A_2 = \frac{2}{\partial_\rho^2\varphi(z)}$$

and

$$kR_b^{(3)}(z,j/k) < Ck^{-1/2}|u_j|^3 < Ck^{-1/2+3\epsilon}.$$

where we used $\chi(|u_j|/k^\epsilon) > 0$ only $|u_j| < 2k^\epsilon$. We have

$$\begin{aligned} \frac{\Pi_{k,P}(z)}{\Pi_{k,j_0}(z_{j_0})} &= \sum_{j\in kP\cap\mathbb{Z}} e^{-A_2u_j^2}e^{kR_b^{(3)}(z,j/k)}\cdot\frac{\Pi_{k,j}(z_j)}{\Pi_{k,j_0}(z_{j_0})}\cdot\chi(u_j/k^\epsilon) \\ &= \sum_{j\in kP\cap\mathbb{Z}} e^{-A_2u_j^2}(1+k^{-1/2}R(u_j))\chi(u_j/k^\epsilon) \\ &= \left(\sum_{j\in kP\cap\mathbb{Z}} e^{-A_2u_j^2}\chi(u_j/k^\epsilon)\right)(1+O(k^{-1/2})) \\ &= \left(\sum_{j\in kP\cap\mathbb{Z}} e^{-A_2u_j^2}\right)(1+O(k^{-1/2})) \end{aligned}$$

where $R(u_j)$ has at most polynomial growth in u_j, hence is integrable against the Gaussian decaying factor, and removing the cut-off function in the last step only will introduce an error of size $O(k^{-\infty})$. Finally, we replace the sum with the integral over u. Since $u_{j+1} - u_j = 1/\sqrt{k}$, and the integrand is smooth and has bounded derivative, the difference between the integral and the summation is again $O(k^{-1/2})$

$$\frac{\Pi_{k,P}(z)}{\Pi_{k,j_0}(z_{j_0})} = \int_{\sqrt{k}(-H(z))}^{\sqrt{k}(E-H(z))} \exp\left(-\frac{2u^2}{\partial_\rho^2\varphi(z)}\right)\sqrt{k}du(1+O(k^{-1/2})).$$

Using our assumption that $\sqrt{k}|E-H(z)|<C$, we may extend the lower limit of the integral to $-\infty$, with an $O(k^{-\infty})$ error. Using Theorem 1, we can estimate the denominator as

$$\Pi_{k,j_0}(z_{j_0}) = \Pi_{k,j_0}(z_E)(1+O(k^{-1/2})) = k^{m-\frac{1}{2}}\sqrt{\frac{2}{\pi\partial_\rho^2\varphi(z_E)}}(1+O(k^{-1/2})).$$

Then evaluate the incomplete Gaussian integral, we get

$$\Pi_{k,P}(z) \quad = \quad k^m \operatorname{Erf}\left(\sqrt{\frac{4k}{\partial_\rho^2\varphi(z_E)}}(E-H(z))\right)(1+O(k^{-1/2})).$$

Using (25), $\partial_\rho^2\varphi = |\nabla H|^2/\pi$, we finish the proof. □

7. Proof of Theorem 3: Euler-MacLaurin approach

In this section we prove Theorem 3 by the technique of [ShZ] of using 'polytope characters' to sift out the weights in the given interval. In dimension one we refer to these polytope characters as interval characters. They are very simple in dimension one and can be directly integrated.

We recall our normalization has $H(M)=[0,E_{\max}]$. Given a proper subinterval $P=[0,E)\subset H(M)$ the *interval characters* χ_{kP} defined on $(\mathbb{C}^*)$ by

$$\chi_{kP}(e^w) = \sum_{j=0}^{kE_k} e^{jw} = \sum_{j\in\mathbb{Z}\cap[0,kE)} e^{jw} \quad w\in\mathbb{C}. \tag{56}$$

where $E_k=\max\{\frac{1}{k}\mathbb{Z}\cap[0,E)\}$ as in (10). The next Lemma expresses the partial Bergman kernel in terms of the interval character:

Lemma 7.1. *For any proper subinterval $P=[0,E)\subset H(M)$, and $a,b\in\mathbb{R}$ such that $a+b=1$, we have the following contour integral expression*

$$K_{k,P}(z_1,z_2) = \int_{|e^w|=1} K_k(e^{-aw}\cdot z_1, e^{-b\bar{w}}\cdot z_2)\chi_{kP}(e^w)\,\frac{dw}{2\pi i}.$$

Proof. From the equivariant property of $K_{k,j}(z,w)$ in Lemma 4.1, we have

$$\begin{aligned} K_k(e^{-aw}\cdot z_1, e^{-b\bar{w}}\cdot z_2) &= \sum_j K_{k,j}(e^{-aw}\cdot z_1, e^{-b\bar{w}}\cdot z_2) \\ &= \sum_j e^{-jw} K_{k,j}(z_1,z_2). \end{aligned}$$

The contour integral then extracts the correct weights j from K_k. □

The particular case we use is $a = b = \frac{1}{2}$, where we have the following identity,

$$K_{k,P}(e^{\zeta}z, z) = K_{k,P}(e^{\zeta/2}z, e^{\bar{\zeta}/2}z), \quad (\zeta \in \mathbb{C}^*). \tag{57}$$

The main result is that interval characters are given by oscillatory integrals over P.

Proposition 7.2. *Let $P = [0,E) \subset H(M)$ be a proper subinterval of $H(M)$. Then, the interval characters* (56) *are oscillatory integrals*

$$\begin{aligned} &\chi_{kP}(e^{w}) = L(w)k\int_{[0,E_k]} e^{kwx}dx + \frac{1}{2}(1+e^{kE_k w}), \\ &\textit{for all} \quad w \in \mathbb{C}\backslash\{\pm 2\pi i, \pm 4\pi i, \dots\}, \end{aligned}$$

where

$$L(w) = \frac{w/2}{\tanh w/2}.$$

Proof. We recall the Euler-MacLaurin formula for lattice interval sum, for any $[a,b] \subset \mathbb{R}$, $a, b \in \mathbb{Z}$, we have

$$\sum_{n\in[a,b]} e^{nw} = L(w)\int_a^b e^{xw}dx + \frac{e^{aw}+e^{bw}}{2}.$$

Then plug in $a = 0, b = kE_k$ gives the desired result. The claim holds for all $w \in \mathbb{R}$ and by analytic continuation we get the desired results. □

Remark 7. See [KSW] and [ShZ] for the generalization to character sums over simple polytopes.

7.1. Proof of Theorem 3

Proof of Theorem 3. In this section we use the Euler-MacLaurin formula and the Boutet de Monvel-Sjöstrand parametrix discussed in section 3.3. Although we allow E to be any real number, Proposition 7.2 replaces $[0, E]$ by $[0, E_k]$ and we get integrals over the latter interval.

Combining Lemma 7.1 and Proposition 7.2, we obtain the following representation. For any $\tau \in \mathbb{R}$, we have

$$\begin{aligned}(58)\quad \Pi_{k,P}(z) &= e^{-k\varphi(z)} \int_{\tau-\pi i}^{\tau+\pi i} K_k(e^{-w/2}z, e^{-\bar{w}/2}z)\chi_{kP}(e^w)\frac{dw}{2\pi i} \\ &= \underbrace{e^{-k\varphi(z)} k \int_{\tau-\pi i}^{\tau+\pi i} \int_{[0,E_k]} K_k(e^{-w/2}z, e^{-\bar{w}/2}z)L(w)e^{kxw}\frac{dxdw}{2\pi i}}_{I_1} \\ &\quad + \underbrace{\frac{1}{2}(K_{k,0}(z) + K_{k,kE_k}(z))}_{I_2}\end{aligned}$$

where in the first step, we used Lemma 7.1 with $a = 1/2, b = 1/2$ (or the identity (57)), and shifted the integration contour from the unit circle to $|e^w| = e^{-\tau}$; this is possible because the integrand is holomorphic in w even for C^∞ metrics. We will see it is only necessary to shift the contour when z lies in the forbidden region. We use I_1, I_2 to denote the integral term and the boundary term.

Using the parametrix (43) for K_k, we obtain

$$I_1 = e^{-k\varphi(z)}k^{m+1} \int_{\tau-\pi i}^{\tau+\pi i} \int_{[0,E_k]} e^{k\varphi(e^{-w/2}z, e^{-\bar{w}/2}z)+kxw} L(w)A_k\frac{dxdw}{2\pi i}$$

where $A_k = (1 + O(k^{-1}))$ is a semi-classical symbol. The phase function is

$$\Psi(w, x) := \varphi(e^{-w/2}z, e^{-\bar{w}/2}z) + xw.$$

The asymptotics can be obtained in two ways. One is to apply stationary phase for oscillatory integrals with complex phase functions on the surface-with-boundary $S^1 \times [0, E_k]$ (see Appendix A for stationary phase method on half-space). The details are quite different in the allowed and forbidden regions, but the overall argument is to locate critical points (w, x) satisfying (59)

$$0 = \partial_x\Psi = w, \quad 0 = \partial_w|_{e^{-w}z=e^{-\bar{w}}z}\Psi = x - \frac{1}{2}\partial_\rho\varphi(e^{-w/2}z) = x - H(e^{-w/2}z)$$

and having maximal real part on the contour of integration. By (40) they occur near the diagonal $e^{-w}z = e^{-\bar{w}}z$, hence $\mathrm{Im} w = 0$. That is, at an interior critical point,

$$w = 0, \quad x = H(z).$$

The second way is to remove the dx integral using,

$$\int_{[0,E]} e^{kxw} dx = \frac{e^{kEw} - 1}{kEw}. \tag{60}$$

This formula is not useful when $w = 0$, which is a critical point for the integral in the allowed region. But it is useful when $w \neq 0$, which is true of critical points for the integral when z lies in the forbidden region. We now give the details.

Allowed Region. We assume that z is in the allowed region and that it is not a critical point of H. We set $\tau = 0$ in (58). The asymptotics of I_1 are thus determined by interior critical points (w, x).

The critical point equations (59) force $x = H(z)$ and $w = 0$. The Hessian matrix for $\Psi(\phi, x)$ $(w = i\phi)$ at $(\phi, x) = (0, H(z))$ is

$$\partial_{xx}\Psi = 0, \quad \partial_{xw}\Psi = 1, \quad \partial_{ww}\Psi = -\frac{1}{4}\partial_\rho^2\varphi(z).$$

The Hessian determinant in (x, ϕ) equals 1 . Note that $\Psi(0, H(z)) = \varphi(z)$ at the critical point, so the phase factor cancels the pre-factor $e^{-k\phi}$ in I_1. Hence the interior stationary phase formula ([Hö]) gives

$$I_1 = k^m L(0)(1 + O(k^{-1})) = k^m(1 + O(k^{-1})).$$

where the integration of $dwdx$ gives a factor of k^{-1}.

To complete the proof we show that $I_2 = O(k^{-\infty})$. Indeed, there exists constant $c > 0$, such that $|H(z) - 0| > c, |H(z) - E_k| > c$. By Theorem 1, $I_2 = O(k^{-\infty})$. Since the computation would be the same, if we had replaced $[0, E_k]$ by $[0, E_{\max}] = H(M)$, we get $K_{k,P}(z) = K_k(z)$ for z in the allowed region.

Forbidden Region. In the forbidden region, we have $H(z) > E$ and $z \in M^E_{\max}$. The phase of the integral I_1 has no critical points on the unit circle $|e^w| = 1$ and we must deform the integral to pick up the dominant critical point. The relevant value of τ is $\tau = 2\tau_E(z)$, where as above $z = e^{\tau_E(z)} \cdot z_E$ for $z_E \in H^{-1}(E)$, $\tau_E(z) > 0$. Then $w \neq 0$ on the contour and we can use (60) to remove the dx integral. Then the dominant critical point is on the boundary where $x = E_k$. The real part of the phase is smaller at 0 than E, and is therefore negligible. The critical equations are $\partial_w \Psi = x - H(e^{-\tau_{E_k}(z)} z) = x - E_k = 0$ is satisfied on the right boundary of $[0, E_k]$.

I_1 then can be explicitly written as

$$I_1 = e^{-k\varphi(z)} k^{m+1} \int_{-\pi}^{\pi} \int_{[0,E_k]} e^{k\varphi(e^{-i\phi/2} z_E, e^{i\phi/2} z_E) + 2kx\tau_{E_k}(z) + ikx\phi} \times L(2\tau_{E_k}(z) + i\phi)(1 + O(k^{-1})) \frac{dx d\phi}{2\pi}.$$

Alternatively, by (60), the dx integral equals $\frac{e^{2kE\tau_E(z) + ikE_k\phi} - 1}{2kE_k\tau_E(z) + ikE_k\phi}$. Since $e^{2kE_k\tau_E(z)}$ is exponentially larger than 1, we may absorb the second term of the numerator into the remainder estimate. The integral of $d\phi$ contributes

$$\frac{1}{\sqrt{k \cdot \frac{1}{4} \partial_\rho^2 \varphi(z_E)}}.$$

Recalling the definition of b_E (Definition 0.1), we obtain

$$\begin{aligned} I_1 &= k^{m-1/2} e^{-k(\varphi(z) - 2E_k\tau_{E_k}(z) - \varphi(z_E))} \frac{1}{\sqrt{2\pi}} \frac{1}{2\tau_{E_k}(z)} \frac{2}{\sqrt{\partial_\rho^2 \varphi(z_E)}} \\ &\quad \times \frac{\tau_{E_k}(z)}{\tanh(\tau_{E_k}(z))} (1 + O(k^{-1})) \\ &= k^{m-1/2} e^{-kb_{E_k}(z)} \frac{\sqrt{2}}{\sqrt{\pi \partial_\rho^2 \varphi(z_E)}} \left(\frac{1}{1 - e^{-2\tau_{E_k}(z)}} - \frac{1}{2} \right) (1 + O(k^{-1})). \end{aligned} \tag{61}$$

On the other hand, we can estimate the boundary terms

$$\begin{aligned} I_2 &= \frac{1}{2} K_{k,kE_k}(z) + O(k^{-\infty}) \\ &= k^{m-1/2} e^{-kb_{E_k}(z)} \frac{1}{\sqrt{2\pi \partial_\rho^2 \varphi(z_E)}} (1 + O(k^{-1})). \end{aligned} \tag{62}$$

Combining I_1, I_2, we see the $-1/2$ term in the parenthesis in (61) cancels out, and we get the result in the forbidden region. □

8. Interface asymptotics: Proof of Theorem 4

In Proposition 8.1, we first prove a smoothed version of Theorem 4 in which the characteristic function $\mathbf{1}_{[-\infty,E]}$ is replaced by a Schwartz test function

$f \in \mathcal{S}(\mathbb{R})$[7]. A density argument using the localization results of Section 5 then extends the asymptotic result from $f \in \mathcal{S}(\mathbb{R})$ to the characteristic function of any interval in $\mathbb{R}$ (Theorem 8.2), and in particular proves the leading order asymptotics stated in Theorem 4. In Section 8.3 we use the Euler-MacLaurin formula to obtain the stated remainder estimate for the intervals $[-\infty, E]$ (Proposition 8.5). The three main results of this section (Proposition 8.1, Theorem 8.2 and Proposition 8.5) are more or less independent: each uses a different techniques and sheds a different light on the Erf-asymptotics of Theorem 4.

Recall the setup of Theorem 4, let E be a regular value of $H : M \to \mathbb{R}$, $z_E \in H^{-1}(E)$ and $z_k = e^{\beta/\sqrt{k}} \cdot z_E$ for some constant β. We define a sequence of measures,

$$d\mu_{k,z_E,\beta}(x) = \frac{1}{\Pi_k(z_k, z_k)} \sum_j \Pi_{k,j}(z_k)\delta_{\sqrt{k}(j/k-E)}(x), \quad (k = 1, 2, 3, \dots) \tag{63}$$

and the purported limiting measure,

$$d\mu_{\infty,z_E,\beta}(x) = e^{-\frac{1}{2}\left(\frac{2x}{\sqrt{\partial_\rho^2\varphi(z_E)}} - \beta\sqrt{\partial_\rho^2\varphi(z_E)}\right)^2} \frac{2dx}{\sqrt{2\pi\partial_\rho^2\varphi(z)}}. \tag{64}$$

In the following, we fix E, z_E, β and write μ_k and μ_∞ for $\mu_{k,z_E,\beta}$ and $\mu_{\infty,z_E,\beta}$, respectively.

For any bounded continuous function $f \in C_b(\mathbb{R})$, we define

$$I_{k,f}(z) := k^{-m} \sum_j f\left(\sqrt{k}\left(\frac{j}{k} - E\right)\right) \Pi_{k,j}(z). \tag{65}$$

Since $\Pi_k(z_k, z_k) = k^m(1 + O(k^{-1/2}))$, we have

$$\int f d\mu_k = I_{k,f}(z_k)(1 + O(k^{-\frac{1}{2}})).$$

8.1. Schwartz test functions

Although we state the main result for Schwartz test functions, it is easily seen that much less is required of the test functions for the asymptotics to be valid.

[7] $\mathcal{S}(\mathbb{R})$ denotes Schwartz space

Proposition 8.1. *With the same notation as in Theorem 4, and $f \in \mathcal{S}(\mathbb{R})$, we have*

$$I_{k,f}(e^{\beta/\sqrt{k}} \cdot z_E) = \int_{-\infty}^{\infty} f(x) e^{-\frac{1}{2}\left(\frac{2x}{\sqrt{\partial_\rho^2 \varphi(z_E)}} - \beta\sqrt{\partial_\rho^2\varphi(z_E)}\right)^2} \frac{2dx}{\sqrt{2\pi\partial_\rho^2\varphi(z_E)}} + O_f(k^{-1/2}).$$

where the constant in O_f depends on f.

Proof. By the Fourier inversion formula, we have

$$\begin{aligned} I_{k,f}(z) &= k^{-m}\sum_j \int_{\mathbb{R}} \hat{f}(t) e^{it\sqrt{k}(j/k-E)} K_{k,j}(z,z) e^{-k\varphi(z)} dt \\ &= k^{-m}\int_{\mathbb{R}} \hat{f}(t) e^{-iE\sqrt{k}t} K_k(e^{it/\sqrt{k}}z, z) e^{-k\varphi(z)} dt. \end{aligned}$$

where in the first step we used Fourier transformation and the definition $\Pi_{k,j}(z) = K_{k,j}(z,z)e^{-k\varphi(z)}$ (see (37)), and in the second step we used the equivariant property of Bergman kernel $K_{k,j}(z,z)$ (see Lemma 4.1). We note that $t \to \Pi_k(e^{it/\sqrt{k}}z, z)$ is $2\pi\sqrt{k}$-periodic (similarly for the parametrix and remainder terms), so the integrals converge when $\hat{f} \in \mathcal{S}(\mathbb{R})$. We periodize $g(t) = \hat{f}e^{-iEt\sqrt{k}}$ by means of the $\sqrt{k}$-periodization operator

$$\mathcal{P}_{\sqrt{k}} g(t) := \sum_{\ell\in\mathbb{Z}} g(t + 2\pi\sqrt{k}\ell), \quad g \in \mathcal{S}(\mathbb{R}),$$

which is periodic of period $2\pi\sqrt{k}$. In fact the right side converges as long as $|g(t)| \leq C(1+|t|)^{-1-\epsilon}$. We write

$$\mathcal{P}_{\sqrt{k}}(\hat{f}e^{-iEt\sqrt{k}}) = \sum_{\ell\in\mathbb{Z}} \hat{f}(t+2\pi\sqrt{k}\ell) e^{-iEt\sqrt{k}-2\pi ik\ell E} =: e^{-iEt\sqrt{k}}\hat{F}_k(t),$$

with $\hat{F}_k(t) = \sum_{\ell\in\mathbb{Z}} \hat{f}(t+2\pi\sqrt{k}\ell)e^{-2\pi i(k\ell E)}$. Then,

$$I_{k,f}(z) = k^{-m}\int_{-\pi\sqrt{k}}^{\pi\sqrt{k}} \hat{F}_k(t) e^{-iEt\sqrt{k}} K_k(e^{it/\sqrt{k}}z, z) e^{-k\varphi(z)} dt. \tag{66}$$

We then localize the last integral using a smooth cutoff $\chi(\frac{t}{(\log k)^2})$, where $\chi \in C_0^\infty(\mathbb{R})$ is supported in $(-1,1)$ and equals to 1 in $(-1/2, 1/2)$. When $\pi\sqrt{k} \geq |t| \geq (\log k)^2$, the off-diagonal Bergman kernel $K_k(e^{it/\sqrt{k}}z, z)$ is rapidly decaying at the rate $O(e^{-(\log k)^2})$. Here, we use the standard off-diagonal estimate, $|K_k(x,y)| \leq Ck^m e^{-\beta\sqrt{k}d(x,y)}$ for certain $\beta, C > 0$ (see Theorem B.1 of the Appendix). Hence,

$$I_{k,f}(z) = k^{-m} \int_{\mathbb{R}} \chi(\frac{t}{(\log k)^2})\ \hat{F}_k(t) e^{-iEt\sqrt{k}} K_k(e^{it/\sqrt{k}}z, z) e^{-k\varphi(z)} dt \\ + O_f(k^{-\infty}),$$

where the constant in $O_f(k^{-\infty})$ depends on $\|\hat{F}_k\|_{L^1(-\sqrt{k},\sqrt{k})} = \|\hat{f}\|_{L^1}$.

We then introduce the Boutet-de-Monvel-Sjöstrand parametrix for K_k,

$$I_{k,f}(z) = \int_{-\infty}^{\infty} \chi(\frac{t}{(\log k)^2})\ \hat{F}_k(t) e^{-iEt\sqrt{k}} e^{k\varphi(e^{it/\sqrt{k}}\cdot z, z) - k\varphi(z)} A_k(e^{it/\sqrt{k}}z, z) dt \\ + \int_{-\infty}^{\infty} \chi(\frac{t}{(\log k)^2})\ \hat{F}_k(t) e^{-iEt\sqrt{k}} R_k(e^{it/\sqrt{k}}z, z) dt + O_f(k^{-\infty}).$$

By the parametrix construction, $R_k \in k^{-\infty}C^\infty(M \times M)$, hence the second term is $O(k^{-\infty})$ and may be absorbed into the remainder estimate.

As in (47), the phase function of $I_{k,f}$ is

$$\Psi(it, z) = -it(\sqrt{k}E) + k\varphi(e^{it/2\sqrt{k}} \cdot z, e^{-it/2\sqrt{k}} \cdot z) - k\varphi(z). \tag{67}$$

Recall that $z_k = e^{\beta/\sqrt{k}} z_E$ with $H(z_E) = E$. Then as $k \to \infty$,

$$\begin{aligned} &\Psi(it, e^{\beta/\sqrt{k}} z_E) \\ &= -it(\sqrt{k}E) + k\left(\varphi(e^{(it/2+\beta)/\sqrt{k}} \cdot z_E, e^{(-it/2+\beta)/\sqrt{k}} \cdot z_E) - \varphi(e^{\beta/\sqrt{k}} \cdot z_E)\right) \\ &= -it(\frac{1}{2}\sqrt{k}\partial_\rho\varphi(z_E)) + k\left[\left(\frac{it/2+\beta}{\sqrt{k}}\right)\partial_\rho\varphi(z_E) + \frac{1}{2}\left(\frac{it/2+\beta}{\sqrt{k}}\right)^2 \partial_\rho^2\varphi(z_E)\right. \\ &\quad \left. - \left(\frac{\beta}{\sqrt{k}}\right)\partial_\rho\varphi(z_E) - \frac{1}{2}\left(\frac{\beta}{\sqrt{k}}\right)^2 \partial_\rho^2\varphi(z_E)\right] + g_3(it, z, \beta) \\ &= \frac{1}{2}((it/2+\beta)^2 - \beta^2)\partial_\rho^2\varphi(z_E) + g_4(it, z, \beta), \end{aligned}$$

where

$$g_3 = O(k^{-1/2}(|\beta|^3 + |t|^3)),\ \ g_4 = O(k^{-1/2}(|\beta|^3 + |t|^3)). \tag{68}$$

We substitute the Taylor expansion into the phase of the first term of $I_{k,f}(e^{\beta/\sqrt{k}} z_E)$, and also Taylor expand e^{g_4} to order 1. Let $e_1(x) = 1 - e^x$.

Since $|t| \leq (\log k)^2$ on the support of the integrand, $|g_4| \leq C(\frac{(\log k)^6}{\sqrt{k}})$ on $|t| \leq (\log k)^2$. Since $e^x = 1 + e_1(x)$ where $e_1(x) \leq 2x$ on $[0, C(\frac{\log k)^6}{\sqrt{k}})]$, $e^{g_4} = 1 + \tilde{g}_4$ where $\tilde{g}_4(k,t) \leq 2g_4 \leq C_0 k^{-\frac{1}{2}}(1+t^3)$ on $[0, (\log k)^2]$.

We get

$$\begin{aligned} &I_{k,f}(e^{\beta/\sqrt{k}} z_E) \\ &= \int_{\mathbb{R}} \chi(\frac{t}{(\log k)^2}) \hat{F}_k(t) e^{\frac{1}{2}((it/2+\beta)^2 - \beta^2)\partial_\rho^2 \varphi(z_E)} (1 + \tilde{g}_4)) dt + O_f(k^{-1/2}) \\ &= \int_{\mathbb{R}} \chi(\frac{t}{(\log k)^2}) \hat{F}_k(t) e^{\frac{1}{2}((it/2+\beta)^2 - \beta^2)\partial_\rho^2 \varphi(z_E)} dt + O_f(k^{-1/2}) \end{aligned}$$

where $\chi(\frac{t}{(\log k)^2})|\widetilde{g}_4| \leq C_0 k^{-1/2}(1+|t|^3)$ after integration against the Gaussian factor is of size $O(k^{-1/2})$.

Finally, we unravel the periodization $\hat{F}_k$ to evaluate the first term.

$$\begin{aligned} &\int_{\mathbb{R}} \chi(\frac{t}{(\log k)^2}) \hat{F}_k(t) e^{\frac{1}{2}((it/2+\beta)^2 - \beta^2)\partial_\rho^2 \varphi(z_E)} dt \\ &= \int_{\mathbb{R}} \chi(\frac{t}{(\log k)^2}) \hat{f}(t) e^{\frac{1}{2}((it/2+\beta)^2 - \beta^2)\partial_\rho^2 \varphi(z_E)} dt \\ &\quad + \sum_{\ell \in \mathbb{Z} \backslash 0} \int_{\mathbb{R}} \chi(\frac{t}{(\log k)^2}) \hat{f}(t + 2\pi\sqrt{k}\ell) e^{2\pi i k \ell E + \frac{1}{2}((it/2+\beta)^2 - \beta^2)\partial_\rho^2 \varphi(z_E)} dt \\ &= \int_{\mathbb{R}} \chi(\frac{t}{(\log k)^2}) \hat{f}(t) e^{\frac{1}{2}((it/2+\beta)^2 - \beta^2)\partial_\rho^2 \varphi(z_E)} dt + O_f(k^{-\infty}) \end{aligned}$$

where in bounding the terms with $\ell \neq 0$, we have used the fast decay property of the Schwarz function $\hat{f}(t)$, i.e., for any positive integer N, we have $|\hat{f}(t + 2\pi\sqrt{k}\ell)| < C_N(1 + |t + 2\pi\sqrt{k}\ell|)^{-N}$ for some C_N, hence the sum over ℓ is convergent by l^{-N} factor.

Finally, removing the cut-off $\chi(t/(\log k)^2)$ will introduce an error as $\int_{(\log k)^2}^{\infty} e^{-ax^2} dx = O(k^{-\infty})$. We have

$$\begin{aligned} I_{k,f}(e^{\beta/\sqrt{k}} z_E) &= \int_{\mathbb{R}} \hat{f}(t) e^{\frac{1}{2}((it/2+\beta)^2 - \beta^2)\partial_\rho^2 \varphi(z_E)} dt + O_f(k^{-1/2}) \\ &= \int_{-\infty}^{\infty} f(x) e^{-\frac{1}{2}\left(\frac{2x}{\sqrt{\partial_\rho^2 \varphi(z_E)}} - \beta\sqrt{\partial_\rho^2 \varphi(z_E)}\right)^2} \frac{2dx}{\sqrt{2\pi \partial_\rho^2 \varphi(z_E)}} \\ &\quad + O_f(k^{-1/2}) \end{aligned}$$

by the Plancherel theorem. This completes the proof of Proposition 8.1. □

8.2. Proof of Weak Convergence result.

We now use Proposition 8.1 to prove the following weak-convergence result:

Theorem 8.2. *The sequence of measures μ_k converges to μ_∞ weak* on $C_b(\mathbb{R})$. In particular, for any interval I, possibly unbounded,*

$$\mu_k(I) \to \mu_\infty(I).$$

This proves the leading order convergence statement of Theorem 4.

Proof. We first prove that $\int_{\mathbb{R}} f(x)d\mu_k(x) \to \int_{\mathbb{R}} f(x)d\mu_\infty(x)$ for $f \in C_c(\mathbb{R})$ and then $f \in C_b(\mathbb{R})$. We then use the results of Section 5 to show that $\{\mu_k\}_k$ is a tight family of probability measures. These steps are done in the following lemmas.

Lemma 8.3. *For any $f \in C_c(\mathbb{R})$*[8]*, we have*

$$\lim_{k\to\infty} \int f d\mu_k = \int f d\mu_\infty.$$

Proof. Let $\eta(x)$ be a smooth non-negative compactly supported function, such that $\int \eta(x)dx = 1$. Let $\eta_\epsilon(x) = \epsilon^{-1}\eta(x/\epsilon)$, and $f_\epsilon = \eta_\epsilon \star f$. Then $f_\epsilon \to f$ in the C^0-norm and $f_\epsilon \in C_c^\infty(\mathbb{R})$. Given $\delta > 0$, choose ϵ small enough such that $|f_\epsilon - f|_{C^0} < \delta$. Then,

$$\begin{aligned}\left|\int f(d\mu_k - d\mu_\infty)\right| &\leq \left|\int f_\epsilon(d\mu_k - d\mu_\infty)\right| + \int |f - f_\epsilon| d\mu_k + \int |f - f_\epsilon| d\mu_\infty \\ &\leq \left|\int f_\epsilon(d\mu_k - d\mu_\infty)\right| + 2\delta.\end{aligned}$$

By Proposition 8.1, $\left|\int f_\epsilon(d\mu_k - d\mu_\infty)\right| < \delta$ for k sufficiently large. Since δ is arbitrarily, Lemma 8.3 follows. □

Next,we prove that the sequence of measure μ_k is tight and extend the range of the test function from $C_c(\mathbb{R})$ to $C_b(\mathbb{R})$.

---

[8] $C_c(\mathbb{R})$ denotes continuous functions of compact support.

Lemma 8.4. *The sequence of measure $\{\mu_k\}$ is tight, and for any $f \in C_b(\mathbb{R})$,*

$$\lim_{k\to\infty} \int f d\mu_k = \int f d\mu_\infty.$$

Proof. To prove tightness, for any $\epsilon > 0$, we need to find $R > 0$ large enough, such that $\mu_k(\mathbb{R}\backslash[-R, R]) < \epsilon$ for all k. The existence of such R is an immediate consequence of Lemma 5.1 (3) on localization of sums.

We then prove weak convergence: Let ϵ, R be as above. Let $\chi(x)$ be a cut-off function that equal to 1 on $[-R, R]$ and equals to zero for $|x| > R + 1$. Then

$$\left|\int f(d\mu_k - d\mu_\infty)\right| \leq \left|\int f\chi(x)(d\mu_k - d\mu_\infty)\right| + \left|\int f(1-\chi)d\mu_k\right| + \left|\int f(1-\chi)d\mu_\infty\right|.$$

The last two terms can be bounded by $2\epsilon\|f\|_{C^0}$, and the first term tends to 0 as $k \to \infty$ since $f\chi \in C_c(\mathbb{R})$. Thus

$$\lim_{k\to\infty} \left|\int f(d\mu_k - d\mu_\infty)\right| \leq 2\epsilon\|f\|_{C^0}$$

for all ϵ, and the left hand side has to be zero. This finishes the proof of the Lemma and hence the proof of Proposition 8.2. □

□

8.3. Proof by the Euler-MacLaurin method

In this section we use the Euler-MacLaurin method of Section 7.1 to obtain a remainder estimate for the weak convergence, as claimed in Theorem 4.

Define

$$(69) \quad I_{[-\frac{M}{\sqrt{k}}, \frac{M}{\sqrt{k}}]}(e^{\frac{\beta}{\sqrt{k}}} z_E) := k^{-m} \sum_{j:|\frac{j}{k}-E|\leq\frac{M}{\sqrt{k}}} \Pi_{k,j}(e^{\frac{\beta}{\sqrt{k}}} z_E) \simeq \mu_{k,z_E,\beta}[-M, M].$$

These are sums of the type (65) but with $f = \mathbf{1}_{[-M,M]}$. As above, we use that $\Pi_k(z_k, z_k) = k^m(1 + O(k^{-1/2}))$ to normalize by the simpler factor k^{-m}. The following Proposition (with trivial modification from $[-M, M]$ to $(-\infty, M]$) implies Theorem 4 (with the remainder estimate). For the sake of brevity, we omit further details.

Proposition 8.5. *Let $z_E \in H^{-1}(E)$ and fix real numbers $M > 0, \beta \in \mathbb{R}$. Then*

$$I_{[-\frac{M}{\sqrt{k}},\frac{M}{\sqrt{k}}]}(e^{\frac{\beta}{\sqrt{k}}} z_E) = \int_{-M}^{M} \sqrt{\frac{2}{\pi \partial_\rho^2 \varphi(z_E)}} e^{-\frac{(2y-\beta\partial_\rho^2\varphi(z_E))^2}{2\partial_\rho^2\varphi(z_E)}} (1+O(k^{-\frac{1}{2}}))dy,$$

Proof. We use Proposition 7.2 with $P = [E - \frac{M}{\sqrt{k}}, E + \frac{M}{\sqrt{k}}]$ and (58) with $z_k = e^{\frac{\beta}{\sqrt{k}}} z_E$ to get $I_{[-\frac{M}{\sqrt{k}},\frac{M}{\sqrt{k}}]}(z_k) := I_1 + I_2$ with

$$I_1 = e^{-k\varphi(e^{\frac{\beta}{\sqrt{k}}} z_E)} k \int_{-i\pi}^{i\pi} \int_{E-\frac{M}{\sqrt{k}}}^{E+\frac{M}{\sqrt{k}}} e^{k\varphi(e^{-w/2} e^{\frac{\beta}{\sqrt{k}}} z_E, e^{-\bar{w}/2} e^{\frac{\beta}{\sqrt{k}}} z_E) + kxw} L(w) A_k \frac{dxdw}{2\pi i}$$

where as above $A_k = (1 + O(k^{-1}))$ is a semi-classical symbol and where we omit the boundary term (62)

$$I_2 = \frac{1}{2}(K_{k,k[E-\frac{M}{\sqrt{k}}]}(e^{\frac{\beta}{\sqrt{k}}} z_E) + K_{k,k[E+\frac{M}{\sqrt{k}}]}(e^{\frac{\beta}{\sqrt{k}}} z_E))$$

since it has lower order, indeed the first sum I_1 having $O(\sqrt{k})$ terms of almost constant order and the boundary term I_2 having only two of the same order.

We change variables in the dx integral to $x = E + \frac{y}{\sqrt{k}}$ so that the $e^{kxw}dx$ integral becomes $\frac{e^{kwE}}{\sqrt{k}} e^{\sqrt{k}yw} dy$. The full ($k$-dependent) phase function becomes

$$\Psi(w, y) := k\varphi(e^{-w/2} e^{\frac{\beta}{\sqrt{k}}} z_E, e^{-\bar{w}/2} e^{\frac{\beta}{\sqrt{k}}} z_E) - k\varphi(e^{\frac{\beta}{\sqrt{k}}} z_E) + kEw + \sqrt{k}yw.$$

We then change variables to $t = i\sqrt{k}w$ to obtain a new phase resembling (67),

$$\begin{aligned} &\Psi(it, y) \\ &= iyt - it(\sqrt{k}E) \\ &\quad + k\left(\varphi(e^{(it/2+\beta)/\sqrt{k}} \cdot z_E, e^{(\bar{it}/2+\beta)/\sqrt{k}} \cdot z_E) - \varphi(e^{\beta/\sqrt{k}} \cdot z_E)\right) \end{aligned}$$

$$\begin{aligned}
&= iyt - it(\frac{1}{2}\sqrt{k}\partial_\rho\varphi(z_E)) \\
&\quad + k\left[\left(\frac{it/2+\beta}{\sqrt{k}}\right)\partial_\rho\varphi(z_E) + \frac{1}{2}\left(\frac{it/2+\beta}{\sqrt{k}}\right)^2\partial_\rho^2\varphi(z_E)\right. \\
&\quad \left. - \left(\frac{\beta}{\sqrt{k}}\right)\partial_\rho\varphi(z_E) - \frac{1}{2}\left(\frac{\beta}{\sqrt{k}}\right)^2\partial_\rho^2\varphi(z_E)\right] + g_3(it, z, \beta) \\
&= iyt + \frac{1}{2}((it/2+\beta)^2 - \beta^2)\partial_\rho^2\varphi(z_E) + g_4(z, it, \beta),
\end{aligned}$$

where

$$g_3 = O(k^{-1/2}(|\beta|^3 + |t|^3)), \quad g_4 = O(k^{-1/2}(|\beta|^3 + |t|^3)),$$

and the ranges are $t \in [-\pi\sqrt{k}, \pi\sqrt{k}]$ and $y \in [-M, M]$.

We further cutoff the integrand to a $(\log k)^2$-neighborhood of $t = 0$ using a smooth cutoff $\chi(\frac{t}{(\log k)^2})$ where $\chi \equiv 1$ in a neighborhood of $t = 0$ and $\chi \equiv 0$ outside a slightly larger neighborhood, and observe that the part of the integral with the cutoff $(1 - \chi(\frac{t}{(\log k)^2}))$ is rapidly decaying in k (c.f. the argument after (68)). Substituting $\tau = it$ and integrating dt gives,

$$I_1 \simeq \int_{-M}^{M}\int_{-\infty}^{\infty}\chi\left(\frac{t}{(\log k)^2}\right)e^{\frac{1}{2}((it/2+\beta)^2-\beta^2)\partial_\rho^2\varphi(z_E)}e^{iyt}(1+O(k^{-\frac{1}{2}}))\frac{dtdy}{2\pi}$$

$$\begin{aligned}
&\simeq \int_{-M}^{M}\int_{-\infty}^{\infty} e^{\frac{1}{2}((it/2+\beta)^2-\beta^2)\partial_\rho^2\varphi(z_E)}e^{iyt}(1+O(k^{-\frac{1}{2}}))\frac{dtdy}{2\pi} \\
&\simeq \int_{-M}^{M}\sqrt{\frac{2}{\pi\partial_\rho^2\varphi(z_E)}}e^{-\frac{(2y-\beta\partial_\rho^2\varphi(z_E))^2}{2\partial_\rho^2\varphi(z_E)}}(1+O(k^{-\frac{1}{2}}))dy,
\end{aligned}$$

where $\simeq$ denotes asymptotics as $k \to \infty$. □

9. Distribution of zero locus of a Random section: Proof of Theorem 5

First we recall a proposition that links the expectation of the $(1,1)$ current defined by a random section from the Hilbert subspace $\mathcal{S}_k \subset H^0(M, L^k)$. Recall our setup: Let $s = \sum_{j=1}^{\dim \mathcal{S}_k} a_{k,j}s_{k,j}$ where $a_{k,j}$ are i.i.d. complex $N(0,1)$ random variables and $\{s_{k,j}\}$ is an orthonormal basis of $\mathcal{S}_k$. Let Z_s be the zero set of s and let $[Z_s]$ be the current of integration over Z_s.

Proposition 9.1 ([ShZ], Proposition 4.1).

$$\frac{1}{k}\mathbb{E}([Z_s]) = \frac{\sqrt{-1}}{2\pi k}\partial\bar{\partial}\log \Pi_{\mathcal{S}_k}(z) + c_1(L,h) \tag{70}$$

where $\Pi_{\mathcal{S}_k}(z)$ is the partial Bergman density function.

The random zero locus in the interior of the allowed region is uniformly distributed, as if $\mathcal{S}_k = H^0(M, L^k)$, indeed $\Pi_{\mathcal{S}_k}(z) = 1 + O(k^{-1})$ is approximately constant. Our main interest is the distribution in the forbidden region.

As before, we take $\mathcal{A} = \{z \mid H(z) < E\}$ and $\mathcal{F} = \{z \mid H(z) > E\}$ for some regular value E of H. Let $\mathcal{F}^E_{\max}$ be the open dense subset of $\mathcal{F}$ where the S^1-action acts freely, and where the $\mathbb{R}_+$-orbit of z intersect $H^{-1}(E)$, say at $q_E(z)$. We define $\pi_E : H^{-1}(E) \cap M_{\max} \to X_E$ to be the Hamiltonian reduction of $H^{-1}(E) \cap M_{\max}$ by the S^1-action. Then we define another projection map

$$\overline{q}_E : \mathcal{F}^E_{\max} \to X_E, \quad \overline{q}_E(z) = \pi_E \circ q_E(z).$$

The complex structure on the quotient is defined as the quotient of the semi-stable points by the $\mathbb{C}^*$ action, and $\overline{q}_E$ is the restriction of this quotient map to $\mathcal{F}^E_{\max}$. Hence $\overline{q}_E$ is holomorphic by definition.

Let φ_E be the restriction of φ to $H^{-1}(E)$. Since it is S^1-invariant, it descends to a Kähler potential on X_E as well and is the Kähler potential of the reduced Kähler form ω_E on X_E. The following is a somewhat more precise version of Theorem 5:

Proposition 9.2. *Let $H : M \to \mathbb{R}$ be a smooth function generating a holomorphic S^1-action, and E a regular value of H. Let $\mathcal{S}_k = \mathcal{S}_{k,E} \subset H^0(M, L^k)$ be the subspace spanned by eigensections of $\hat{H}_k$ (see* (2)*) with eigenvalues less than E. For any compact subset $K \subset \mathcal{F}^E_{\max}$, we have the following weak* convergence*

$$\lim_{k\to\infty} \frac{1}{k}\mathbb{E}([Z_s]) = \overline{q}_E^{-1}(\omega_E) + 2E\frac{\sqrt{-1}}{2\pi}(\partial\bar{\partial}\tau(z,E)). \tag{71}$$

In particular, the right hand side of (71) *is a smooth* $(1,1)$*-form of rank* $(n-1)$ *in $\mathcal{F}^E_{\max}$.* [9]

[9] In the toric case the leaves (orbits) of the $\mathbb{C}^*$ action vary holomorphically and $\partial\bar{\partial}\tau = 0$.

Proof. Using the expansion of the partial Bergman density in the forbidden region, we have for $z \in \mathcal{F}$,

$$\Pi_{kP}(z) = k^{m-1/2}\sqrt{\frac{2}{\pi\partial_\rho^2\varphi(z_E)}}\frac{e^{-kb(z,E_k)}}{1-e^{-|2\tau(z,E)|}}(1+O(k^{-1})).$$

Using Proposition 9.1, we get

$$\begin{aligned}\frac{1}{k}\mathbb{E}([Z_s]) &= \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(-b(z,E_k)+\varphi(z))+O(k^{-1})\\ &= \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(-b(z,E)+\varphi(z))+O(k^{-1})\\ &= \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}[\varphi(q_E(z))+2E\tau(z,E)]+O(k^{-1})\end{aligned}$$

where we have used $c_1(L,h) = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi(z)$ and expression of $b(z,E)$ from (24). Since φ_E is the Kähler potential on the symplectic reduction $H^{-1}(E)/S^1$, we have $\varphi(q_E(z)) = \varphi_E(\overline{q}_E(z))$, and since $\overline{q}_E$ is holomorphic, we can commute $\partial\bar{\partial}$ with the pullback. This gives the desired result (71). For the rank statement, we note that the $(1,1)$-form on the right-hand-side of (71) vanishes on any $\mathbb{C}^*$-leaf inside $\mathcal{F}^E_{\max}$ hence is of rank $(n-1)$. □

10. Example: The Bargmann-Fock model

In this section we illustrate the results in the Bargmann-Fock model of the line bundle $\mathbb{C}^m \times \mathbb{C} \to \mathbb{C}^m$ with Kähler potential $\varphi = \|z\|^2$.

$$\mathcal{H}_k^2 = \mathcal{H}^2_{h^k_{BF}} = \left\{ f \in \mathcal{O}(\mathbb{C}^m) : \int_{\mathbb{C}^m} |f(z)|^2 e^{-k\|z\|^2} dm(z) < \infty \right\}$$

where $dm = (\omega)^m/m!$ is Lebesgue measure, and $\omega = \frac{i}{2\pi}\partial\bar{\partial}\varphi = \frac{i}{2\pi}dz \wedge d\bar{z} = \pi^{-1}\sum_j dx_j \wedge dy_j$. As mentioned in §2.8, the linear S^1 actions on $\mathbb{C}^m$ have the form,

$$e^{i\theta}\cdot(z_1,\ldots,z_m) = (e^{ib_1\theta}z_1,\ldots,e^{ib_m\theta}z_m), \quad b_j \in \mathbb{Z},$$

with Hamiltonians $H = \frac{1}{2}\partial_\rho|_{\rho=0}\sum_{j=1}^m(|e^{b_j\rho}z_j|^2) = \sum_j b_j|z_j|^2$. We only consider the diagonal $\mathbf{T}$ action and Hamiltonian $H(z) = \|z\|^2$, i.e. the isotropic harmonic oscillator in the Bargmann-Fock representation.

The usual quantum Hamiltonian for the harmonic oscillator is $\hbar\left(\hat{N}+\frac{m}{2}\right)$ where $\hbar = 1/k$ where $\hat{N} = Z \cdot \frac{\partial}{\partial Z}$ is the number or Euler operator with eigenvalues/eigenfunctions

$$\hat{N} z^\alpha = |\alpha| z^\alpha.$$

where $|\alpha| := \sum_i \alpha_i$. Since we chose our normalization of H to have minimum 0, we will drop the $m/2$ constant, and define $H_k = \frac{1}{k}\hat{N}$. It is an elliptic S^1 action in the sense that its moment map H is proper and all weight spaces

$$\mathcal{H}_{k,j} = \mathrm{Span}\{z^\alpha = z_1^{\alpha_1} \cdots z_m^{\alpha_m}, |\alpha| = \alpha_1 + \cdots + \alpha_m = j\}$$

are finite dimensional.

We will fix the constant section $1 \in \Gamma(\mathbb{C}^m, \mathbb{C})$ as the holomorphic reference frame, then an orthonormal basis is given by

$$c_\alpha z^\alpha = \prod_{i=1}^m \sqrt{\frac{k^{\alpha_i+1}}{(\alpha_i)!}} z_i^{\alpha_i}$$

thus the full Bergman kernel

$$\Pi_k(z,w) = K_k(z,w), \quad K_k(z,w) := \sum_{\alpha \in \mathbb{Z}^m_{\geq 0}} \frac{k^{|\alpha|+m} z^\alpha \overline{w}^\alpha}{\alpha!} = k^m e^{kz\cdot\bar{w}}.$$

and the equivariant Bergman kernels are

$$\Pi_{k,j}(z,w) = K_{k,j}(z,w), \quad K_{k,j}(z,w) := \sum_{\alpha:|\alpha|=j} \frac{k^{|\alpha|+m} z^\alpha \overline{w}^\alpha}{\alpha!}.$$

The equivariant kernel is obtained from the full kernel by

$$K_{k,j}(z,w) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ij\theta} K(e^{i\theta}z, w) d\theta. \tag{72}$$

And Bergman density $B_k(z) = k^m$, and the equivariant Bergman density is

$$\Pi_{k,j}(z) = K_{k,j}(z,z)\|1^k(z)\|^2_{h^k(z)} = e^{-k\|z\|^2} \sum_{\alpha:\|\alpha\|=j} \frac{k^{m+|\alpha|} z^\alpha \overline{w}^\alpha}{\alpha!}.$$

Lemma 10.1. *As $k \to \infty$, and $E = j/k$, the equivariant Bergman kernel is*

$$K_{k,j}(z,z) = k^m \fint_{\mathbf{T}} e^{-ij\theta} e^{ke^{i\theta}\|z\|^2} d\theta$$
$$= k^m \frac{k^j}{j!}\|z\|^{2j} \simeq k^{m-1/2}\left(\frac{e\cdot\|z\|^2}{E}\right)^{kE}(2\pi E)^{-1/2}$$

and the equivariant Bergman kernel is

$$B_{k,j}(z) = K_{k,j}(z,z)e^{-k\|z\|^2} = k^{m-1/2}(2\pi E)^{-1/2}\left(\frac{\|z\|^2}{E}\right)^{kE} e^{-k(\|z\|^2-E)}.$$

The maximum of $B_{k,j}(z)$ is obtained, when $\|z\|^2 = E$.

Proof. In fact, $K_{k,j}(z,w)$ is $U(m)$-invariant and so $K_{k,j}(z,w)$ is a function of $z\cdot\overline{w}$. It is also homogeneous of degree $2j$ so it is a constant multiple $C = C_{k,j,m}$ of $(z\cdot\overline{w})^k$. The constant may be determined from the fact that

$$\dim V_k(j) = \int_{\mathbb{C}^m} B_{k,j}(z)dm(z) = \pi^{-m}\int_0^\infty e^{-kr^2}\cdot Cr^{2j}\cdot r^{2m-1}\omega_{2m-1}dr$$

where $\dim V_k(j) = \binom{j+m-1}{m-1}$ is the number of partitions of j in m parts, $\omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the volume of $S^{d-1}\subset\mathbb{R}^d$. A straightforward computation gives $C_{k,j,m} = \frac{k^{m+j}}{j!}$. Hence $K_{k,j}(z,z)$ is the j-th term of the Taylor expansion $k^m e^{k\|z\|^2}$. But it is useful to compute the integral using the general method, which we will explain next.

Let $E := E_{k,j} = \frac{j}{k}$. The first equality follows from (72). We then change θ to $\theta + i\tau$ so that the complex phase is

$$\Psi_{z,\tau}(\theta) = -iE(\theta + i\tau) + e^{i(\theta+i\tau)}\|z\|^2.$$

The critical point equation is

$$\frac{\partial}{i\partial\theta}\Psi_{z,\tau}(\theta) = -E + e^{i(\theta+i\tau)}\|z\|^2 = 0 \iff Ee^{-i\theta} = e^{-\tau}\|z\|^2.$$

Since the right side is positive real, the only possible solution is $\theta = 0$ and for this we need to choose τ so that $e^\tau = \|z\|^2/E = H(z)/E$. With this choice of τ, and by deforming the contour to this $|w| = e^\tau \in \mathbb{C}$, the phase becomes $-iE(\theta + i\log(\|z\|^2/E)) + Ee^{i\theta}$ and we have a non-degenerate critical point at $\theta = 0$ and an asymptotic expansion,

$$\begin{aligned} K_{k,j}(z,z) &= k^m \fint_{\mathbf{T}} e^{-ikE(\theta + i\log(\|z\|^2/E))} e^{kEe^{i\theta}} d\theta \\ &\simeq k^{m-1/2} \left(\frac{e \cdot \|z\|^2}{E}\right)^{kE} (2\pi E)^{-1/2}. \end{aligned}$$

where we used the stationary phase formula for $d\theta$ integral. The result agrees with the exact one after applying Stirling formula.

The statement about the maximum of $B_{k,j}(z)$ can be obtained by solving

$$\frac{d}{d|z|^2}\left(\frac{1}{k}\log B_{k,j}(z)\right) = -1 + E/\|z\|^2 = 0.$$

Indeed, the maximum of $B_{k,j}(z)$ occurs when $\|z\|^2 = E$. □

Now we scale the equivariant Bargmann-Fock kernels around $H^{-1}(E)$ and prove Theorem 2 in this case. Let $z_0 \in H^{-1}(E)$, i.e. $\|z_0\|^2 = E$ and fix $u \in \mathbb{R}$.

$$\begin{aligned} &\Pi_{k,j}\left(z_0\left(1 + \frac{u}{\sqrt{k}}\right), z_0\left(1 + \frac{u}{\sqrt{k}}\right)\right) \\ &= k^m \fint_{\mathbf{T}} e^{-ikE\theta} e^{k\left(e^{i\theta}\|z_0\|^2(1+\frac{u}{\sqrt{k}})^2 - \|z_0\|^2(1+\frac{u}{\sqrt{k}})^2\right)} d\theta. \end{aligned} \tag{73}$$

As $k \to \infty$,

$$\begin{aligned} &e^{i\theta}\|z_0\|^2\left(1 + \frac{u}{\sqrt{k}}\right)^2 - \|z_0\|^2\left(1 + \frac{u}{\sqrt{k}}\right)^2 \\ &= E(i\theta - \theta^2/2 + e_3(\theta))(1 + 2u/\sqrt{k} + u^2/k)) \end{aligned}$$

so the phase has the form $kE\Psi$ with

$$\begin{aligned} \Psi = i\theta\left(2u/\sqrt{k} + u^2/k\right) - \frac{\theta^2}{2}\left(1 + 2u/\sqrt{k} + u^2/k\right) \\ + e_3(i\theta)\left(1 + 2u/\sqrt{k} + u^2/k\right), \end{aligned}$$

where $e^x = 1 + x + x^2/2! + e_3(x)$. We localize around $\theta = 0$ using a cutoff $\chi \in C_0^\infty(-1,1)$ and change variables $\theta \to k^{-1/2}\theta$ to get

$$k^{-1/2}(2\pi)^{-1}\int_{\mathbb{R}} \chi(\theta/\sqrt{k})\, e^{i\theta(2uE) - E\frac{\theta^2}{2}} A_k(\theta,u) d\theta,$$

where A is a semi-classical symbol of order zero. Here, we absorbed the other terms,

$$e^{E(\frac{i\theta|u|^2}{\sqrt{k}}-\frac{\theta^2 u}{\sqrt{k}}-\frac{i\theta^3}{3!\sqrt{k}})+O(1/k)}$$

into A. Since $A_0(\theta, u) = 1$ as $k \to \infty$ the integral tends to

$$k^{-1/2}(2\pi)^{-1}\int_{\mathbb{R}} e^{i\theta(2Eu)-E\frac{\theta^2}{2}}\,d\theta = (2\pi kE)^{-1/2}e^{-2u^2E}(1+O(k^{-1/2})).$$

Thus

$$\Pi_{k,j}\left(z_0\left(1+\frac{u}{\sqrt{k}}\right), z_0\left(1+\frac{u}{\sqrt{k}}\right)\right) = k^{m-1/2}\left(\frac{e^{-2Eu^2}}{\sqrt{2\pi E}}+O(k^{-1/2})\right)$$

proving Theorem 2 in this case.

Appendix A. Stationary phase on a half-space

Lemma A.1. *Let* $\Psi(\xi, y), A(\xi, y) \in \mathcal{C}^\infty(\mathbb{R}^k_{\geq 0}\times\mathbb{R}^s)$ *such that* $d_y\Psi(0,0) = 0$, *A has compact support and*

1) $\frac{\partial\Psi}{\partial\xi_j} \neq 0$ *on* $\operatorname{supp}(A)$, *for* $1 \leq j \leq k$,
2) $d_y\Psi(0, y) \neq 0$ *for* $(0, y) \in \operatorname{supp}(A)\setminus\{0\}$,
3) $\det \mathcal{H}_y\Psi(0,0) \neq 0$ *(where $\mathcal{H}_y$ denotes the Hessian with respect to y)*,
4) $Re\Psi \leq Re\Psi(0,0)$ *on* $\operatorname{supp}(A)$.

Then

$$\int_{\mathbb{R}^s}\int_0^\infty e^{N\Psi(\xi,y)}A(\xi,y)\,d\xi\,dy$$
$$= N^{-k-s/2}e^{N\Psi(0,0)}[c_0 + c_1N^{-1} + c_2N^{-2} + \cdots + c_lN^{-l} + O(N^{-l-1})]$$

for $l = 1, 2, 3, \ldots,$ *where*

$$c_0 = \left.\frac{(2\pi)^{s/2}\,A}{\sqrt{\det(-\mathcal{H}_y\Psi)}\;\prod_{j=1}^k \partial\psi/\partial\xi_j}\right|_{\xi=y=0}.$$

Proof. Integrating by parts,

$$\int_{\mathbb{R}^s}\int_0^{+\infty} e^{N\Psi} A\, d\xi_1\, dy = \frac{1}{N}\int_{\mathbb{R}^s} e^{N\Psi} \frac{A}{\partial\Psi/\partial\xi_1}\Big|_{\xi_1=0} dy - \frac{1}{N}\int_{\mathbb{R}^s}\int_0^{+\infty} e^{N\Psi}\frac{\partial}{\partial\xi_1}\left[\frac{A}{\partial\Psi/\partial\xi_1}\right] d\xi_1\, dy. \tag{A.1}$$

Applying the stationary phase expansion [Hö, Th. 7.7.5] to the first term of (A.1) and iterating, we obtain the desired expansion. □

Appendix B. Off-diagonal decay estimates

Theorem B.1. *(See Theorem 2 of [Del] and Proposition 9 of [L])] Let M be a compact Kähler manifold, and let $(L, h) \to M$ be a positive Hermitian line bundle. Then the exists a constant $\beta = \beta(M, L, h) > 0$ such that*

$$|\tilde{\Pi}_N(x, y)|_{\tilde{h}^N} \le CN^m e^{-\beta\sqrt{N}d(x,y)}$$

where $d(x, y)$ is the Riemannian distance with respect to the Kähler metric $\tilde{\omega}$.

The theorem is stated for strictly pseudo-convex domains in $\mathbb{C}^n$ but applies with no essential change to unit codisc bundles of positive Hermitian line bundles.

References

[Ber] R. J. Berman, *Bergman kernels and equilibrium measures for line bundles over projective manifolds*, American Journal of Mathematics **131** (2009), no. 5, 1485–1524.

[BBSj] R. J. Berman, B. Berndtsson and J. Sjöstrand , *A direct approach to Bergman kernel asymptotics for positive line bundles*, Ark. Mat. **46** (2008), no. 2, 197–217.

[BB] A. Bialynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. of Math. (2) **98** (1973), 480–497.

[BBS] A. Bialynicki-Birula and A. Sommese, *Quotients by $\mathbb{C}^*$ and $SL(2,\mathbb{C})$ actions*, Trans. Amer. Math. Soc. **279** (1983), no. 2, 773–800.

[BSj] L. Boutet de Monvel and J. Sjöstrand, *Sur la singularité des noyaux de Bergman et de Szegö*, Asterisque **34-35** (1976), 123–164.

[BGL] D. Burns, V. Guillemin and E. Lerman, *Kähler cuts*, `arXiv:math/0212062`.

[CS] J. B. Carrell and A. J. Sommese, $\mathbb{C}^*$ *-actions*, Math. Scand. **43** (1978/79), no. 1, 49–59.

[Ch] M. I. Chlodovsky, *Sur la représentation des fonctions discontinues par les polynômes de M. S. Bernstein*, Fund. Math. **13** (1929), 62–72.

[CM] D. Coman and G. Marinescu, *On the first order asymptotics of partial Bergman kernels*, Ann. Fac. Sci. Toulouse Math. (6) **26** (2017), no. 5, 1193–1210.

[Del] H. Delin, *Pointwise estimates for the weighted Bergman projection kernel in* $\mathbf{C}^n$*, using a weighted* L^2 *estimate for the* $\overline{\partial}$ *equation*, Ann. Inst. Fourier (Grenoble) **48** (1998), no. 4, 967–997.

[Fe] R. Feng, *Szasz analytic functions and noncompact Kähler toric manifolds*, J. Geom. Anal. **22** (2012), no. 1, 107–131.

[F] T. Frankel, *Fixed points and torsion on Kähler manifolds*, Ann. of Math. (2) **70** (1959), 1–8.

[GS] V. Guillemin and S. Sternberg, *Geometric quantization and multiplicities of group representations*, Invent. Math. **67** (1982), no. 3, 515–538.

[Hö] L. Hörmander, The Analysis of Linear Partial Differential Operators, I, Springer Verlag, N.Y., (1983).

[HS] A. D. Hwang and M. A. Singer, *A momentum construction for circle-invariant Kähler metrics*, Trans. Amer. Math. Soc. **354** (2002), no. 6, 2285–2325.

[J] J. K. Jain, Composite Fermions, Cambridge Univ. Press (2007), and Composite Fermions And The Fractional Quantum Hall Effect: A Tutorial, online at `www.phys.psu.edu/~jain/cftutorial.pdf`.

[KSW] Y. Karshon, S. Sternberg, and J. Weitsman, *Euler-MacLaurin with remainder for a simple integral polytope*, Duke Math. J. **130** (2005), no. 3, 401–434.

[Ki] F. C. Kirwan, *Intersection homology and torus actions*, J. Amer. Math. Soc. **1** (1988), no. 2, 385–400.

[KN] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry. Vol. I. Interscience Publishers, John Wiley & Sons, New York-London, (1963).

[Ko] B. Kostant, *Quantization and unitary representations. I. Prequantization*, Lectures in Modern Analysis and Applications, III, pp. 87–208, Lecture Notes in Math., Vol. 170, Springer, Berlin, (1970).

[Lev] B. Levikson, *On the behavior of a certain class of approximation operators for discontinuous functions*, Acta Math. Acad. Sci. Hungar. **33** (1979), no. 3-4, 299–306.

[L] N. Lindholm, *Sampling in weighted L^p spaces of entire functions in $\mathbb{C}^n$ and estimates of the Bergman kernel*, J. Funct. Anal. **182** (2001), no. 2, 390–426.

[Lo] G. G. Lorentz, Bernstein Polynomials, second edition, Chelsea Publishing Co., New York, (1986).

[P] R. Paoletti, *Lower-order asymptotics for Szegö and Toeplitz kernels under Hamiltonian circle actions*, Recent Advances in Algebraic Geometry, pp. 321–369, London Math. Soc. Lecture Note Ser. **417**, Cambridge Univ. Press, Cambridge, (2015).

[P2] R. Paoletti, *Local scaling asymptotics for the Gutzwiller trace formula in Berezin-Toeplitz quantization*, J. Geom. Anal. **28** (2018), no. 2, 1548–1596.

[PS] F. Pokorny and M. Singer, *Toric partial density functions and stability of toric varieties*, Math. Ann. **358** (2014), no. 3-4, 879–923.

[RS] J. Ross and M. Singer, *Asymptotics of partial density functions for divisors*, J. Geom. Anal. **27** (2017), no. 3, 1803–1854.

[ShZ] B Shiffman and S. Zelditch, *Random polynomials with prescribed Newton polytope*, J. Am. Math. Soc. **17**, no. 1, 49–108.

[STZ] B. Shiffman, T. Tate, and S. Zelditch, *Distribution laws for integrable eigenfunctions*, Ann. Inst. Fourier (Grenoble) **54** (2004), no. 5, 1497–1546.

[W] X.-G. Wen, Quantum Field Theory of Many-Body Systems, Oxford Grad Texts (2004).

[Y] Q. Yang, *Morse and semistable stratifications of Kähler spaces by $\mathbb{C}^*$-actions*, Monatsh. Math. **155** (2008), no. 1, 79–95.

[Ze] S. Zelditch, *Bernstein polynomials, Bergman kernels and toric Kähler varieties*, J. Symplectic Geom. **7** (2009), 1–26.

[ZZ17] S. Zelditch and P. Zhou, *Central limit theorem for spectral partial Bergman kernels*, Geometry & Topology **23** (2019), 1961–2004.

Department of Mathematics, Northwestern University
2033 Sheridan Road, Evanston, IL 60208-2730, USA
E-mail address: `s-zelditch@northwestern.edu`

Institut des Hautes Études Scientifiques
35 Route de Chartres, 91440 Bures-sur-Yvette, France
E-mail address: `pzhou.math@gmail.com`

Received May 3, 2016
Accepted September 13, 2017