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Lagrangian skeleta of hypersurfaces in $(\mathbb{C}^*)^n$

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Abstract

Let $W(z_1, \ldots, z_n) : (\mathbb{C}^*)^n \to \mathbb{C}$ be a Laurent polynomial in n variables, and let $\mathcal{H}$ be a generic smooth fiber of W. Ruddat et al. (Geom Topol 18:1343–1395, 2014) give a combinatorial recipe for a skeleton for $\mathcal{H}$. In this paper, we show that for a suitable exact symplectic structure on $\mathcal{H}$, the RSTZ-skeleton can be realized as the Liouville Lagrangian skeleton.

Mathematics Subject Classification 53D37

Let $(M, \omega = d\lambda)$ be an exact symplectic manifold, and let $X = X_\lambda$ be the Liouville vector field defined by $\iota_X \omega = -\lambda$. If (M, ω, λ, X) is a *Liouville manifold* (see [2, Chapter 11] for definition), then X shrinks M to a compact isotropic (possibly singular) submanifold Λ, called the *Liouville skeleton*. The Liouville skeleton is useful for symplectic topology, since the tubular neighborhood of the skeleton is symplectomorphic to the original manifold up to rescaling the symplectic form.

A large class of Liouville manifolds come from Stein manifolds, e.g. affine hypersurfaces $\mathcal{H}$ in $(\mathbb{C}^*)^n$. Given an exhausting pluri-subharmonic (psh) function φ on the Stein manifold, we can define the Liouville structure by setting $\omega = -dd^c\varphi$ and $\lambda = -d^c\varphi$. Ruddat et al. [22] give a combinatorial recipe for a topological skeleton in affine hypersurfaces. The RSTZ-skeleton depends on the Newton polytope Q of the defining polynomial for the hypersurface and a star triangulation of Q.

It is conjectured that the RSTZ-skeleton can be realized as a Liouville skeleton for a suitable choice of Liouville structure on the hypersurface. Here we construct such Liouville structure using tropicalization. The main idea is contained in the following example:

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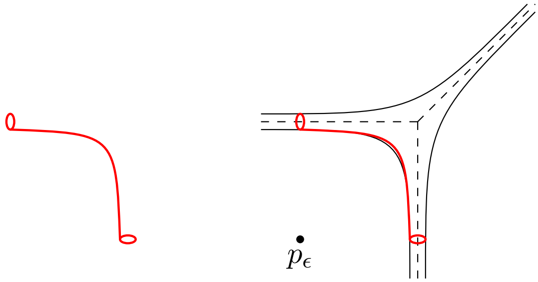

Fig. 1 RSTZ-skeleton (on the left) and its embedding for the pair-of-pants

0.1 Example: the pair-of-pants

Consider the hypersurface

$$\mathcal{H} = \{x + y = 1\}, \quad x, y \in \mathbb{C}^*.$$

The hypersurface can be identified as $\mathbb{C}\backslash\{0, 1\}$, a 'pair-of-pants'. A topological skeleton can be constructed as follows: fix an arbitrarily small positive number ϵ, and define the skeleton as

$$\Lambda = (\{|x| = \epsilon\} \cup \{|y| = \epsilon\} \cup \{x, y \in \mathbb{R}, x \geq \epsilon, y \geq \epsilon\}) \bigcap \{x + y = 1\}.$$

Thus Λ has the shape of two circles connected by an interval. To realize it as a Lagrangian skeleton, we need to choose an exact symplectic structure. Consider the following function φ on $(\mathbb{C}^*)^2$

$$\varphi(x, y) := \varphi_\epsilon(x, y) = (\log|x| - \log\epsilon)^2 + (\log|y| - \log\epsilon)^2.$$

It is easy to check that φ is a psh function on $(\mathbb{C}^*)^2$, and φ restricts to be a psh function on any complex submanifold of $(\mathbb{C}^*)^2$. Geometrically, φ is constructed by taking the projection map

$$\mathrm{Log} = \log|\cdot| : (\mathbb{C}^*)^2 \to \mathbb{R}^2$$

and then taking Euclidean distance on $\mathbb{R}^2$ to a point

$$\varphi(z) = |\,\mathrm{Log}(z) - p_\epsilon|^2, \quad p_\epsilon = (\log\epsilon, \log\epsilon).$$

The hypersurface $\{x + y = 1\}$ projects under $\log|\cdot|$ to an 'amoeba' shaped region in $\mathbb{R}^2$, with three tentacles asymptotic to the following three rays (drawn as dashed lines in Fig. 1)

$$\{\log|x| = 0, \log|y| \ll 0\}, \quad \{\log|x| \ll 0, \log|y| = 0\}, \quad \{\log|x| = \log|y| \gg 0\}.$$

The Liouville flow X_λ on $\mathcal{H}$ induced by $\lambda = -d^c\varphi$ is the same as the negative gradient flow $-\nabla_\omega(\varphi)$ of φ with respect to the Kähler metric $\omega = -dd^c\varphi$. The critical points of φ on $\mathcal{H}$ can be identified with the critical points on the amoeba $\mathrm{Log}(\mathcal{H})$ with respect to the distance function to point p_ϵ. The unstable manifold of $-\nabla\varphi$ on $\mathcal{H}$ is homeomorphic to the union of two circles and an interval.

0.2 Set-up and Summary of results

To state our main result precisely, we need some notations. See Sect. 1.1 for the background on triangulations.

Let M, N be dual lattices of rank n. Let $T = \mathbb{R}/2\pi\mathbb{Z}$. For any abelian group G, e.g. $G = \mathbb{C}^*, \mathbb{R}, T$, we define $M_G := M \otimes_{\mathbb{Z}} G$ and similarly for N_G. If we fix a basis of M, then $M \cong \mathbb{Z}^n$, and $M_{\mathbb{C}^*} \cong (\mathbb{C}^*)^n$, $M_{\mathbb{R}} \cong \mathbb{R}^n$, $M_T \cong T^n$.

Let $Q \subset N_{\mathbb{R}}$ be an integral convex polytope of full-dimension containing 0.[1] Let $\mathcal{T}$ be a coherent star triangulation of Q based at 0, and let $\partial\mathcal{T}$ be the subset of $\mathcal{T}$ consisting of simplices that do not contain 0. Let $\Sigma_{\mathcal{T}}$ be the simplicial fan spanned by the simplices in $\mathcal{T}$. Let A denote the vertices of $\mathcal{T}$, and ∂A that of $\partial\mathcal{T}$, so that $A = \partial A \cup \{0\}$.

We fix two functions

$$h : A \to \mathbb{R}, \quad \Theta : A \to T,$$

such that h induces the coherent star triangulation $\mathcal{T}$ (see Sect. 1.1 for the definition of "coherent star triangulation"). Without loss of generality, we let

$$h(0) = 0, \quad \Theta(0) = \pi.$$

We define a conical Lagrangian $\Lambda_{\mathcal{T},\Theta} \subset M_T \times N_{\mathbb{R}} \cong T^*M_T$ by

$$\Lambda_{\mathcal{T},\Theta} := \bigcup_{\tau\in\partial\mathcal{T}} \{\theta \in M_T : \langle\alpha, \theta\rangle = \Theta(\alpha) \text{ for all vertices } \alpha \in \tau\} \times \operatorname{cone}(\tau) \quad (0.1)$$

where we used the pairing $\langle -, -\rangle : M_T \times N_{\mathbb{R}} \to T$ induced by the canonical pairing between M, N, and $\operatorname{cone}(\tau) = \mathbb{R}_{\geq 0} \times \tau$. We also define the *generalized RSTZ-skeleton* [22] by

$$\Lambda^{\infty}_{\mathcal{T},\Theta} := \bigcup_{\tau\in\partial\mathcal{T}} \{\theta \in M_T : \langle\alpha, \theta\rangle = \Theta(\alpha) \text{ for all vertices } \alpha \in \tau\} \times \tau \quad (0.2)$$

Remark 0.1 If $\Theta|_{\partial\mathcal{A}} = 0$, $\Lambda_{\mathcal{T},\Theta}$ is the FLTZ skeleton [5] and $\Lambda^{\infty}_{\mathcal{T},\Theta}$ is the RSTZ-skeleton.

Remark 0.2 We will sometimes identify $|\partial\mathcal{T}|$ with its projection to $N^{\infty}_{\mathbb{R}} := (N_{\mathbb{R}}\backslash\{0\})/\mathbb{R}_{>0}$, then $\Lambda^{\infty}_{\mathcal{T},\Theta}$ is homeomorphic to the boundary-at-infinity of $\Lambda_{\mathcal{T},\Theta}$.

For all large enough $\beta > 0$, we define the *tropical polynomial* as

$$f_{\beta,h,\Theta}(z) = \sum_{\alpha\in A} e^{-i\Theta(\alpha)} e^{-\beta h(\alpha)} z^{\alpha}. \quad (0.3)$$

[1] The case where Q is not full-dimension can be reduced to this one, by defining $N'_{\mathbb{R}} = \operatorname{span}_{\mathbb{R}}(Q) \subset N_{\mathbb{R}}$, and $M_{\mathbb{R}} \twoheadrightarrow M'_{\mathbb{R}}$. The skeleton for $(M_{\mathbb{R}}, N_{\mathbb{R}}, Q)$ would be that of $(M'_{\mathbb{R}}, N'_{\mathbb{R}}, Q)$ times T^d where $d = \dim M_{\mathbb{R}} - \dim M'_{\mathbb{R}}$.

where z^{α} is a monomial on $M_{\mathbb{C}^*} \cong (\mathbb{C}^*)^n$. Let

$$\mathcal{H}_{\beta,h,\Theta} := \{z \in M_{\mathbb{C}^*} \mid f_{\beta,h,\Theta}(z) = 0\}$$

denote the complex hypersurface defined by $f_{\beta,h,\Theta}$.

Theorem [22] *If $\Theta|_{\partial\mathcal{A}} = 0$, then the skeleton $\Lambda^{\infty}_{\mathcal{T},\Theta}$ embeds into the hypersurface $\mathcal{H}_{\beta,h,\Theta}$ as a strong deformation retract.*

We prove the following theorem, for general Θ.

Main Theorem *The hypersurface $\mathcal{H}_{\beta,h,\Theta}$ admits a Liouville structure such that its Liouville skeleton is homeomorphic to $\Lambda^{\infty}_{\mathcal{T},\Theta}$.*

Remark 0.3 We explain the motivation for considering the general Θ. We know that the Fukaya-Seidel categories $FS(M_{\mathbb{C}^*}, f_{\beta,h,\Theta})$ is locally constant as one varies the coefficients in front of the monomials in the superpotential $f_{\beta,h,\Theta}$. However, on the constructible sheaf side, people usually only consider the category $Sh(T^n, \Lambda_{\mathcal{T}})$ with the FLTZ skeleton $\Lambda_{\mathcal{T}}$ whose boundary at infinity is the RSTZ skeleton. The point for considering the family of skeleta parametrized by Θ is to translate the flexibility of Fukaya category into the constructible/microlocal sheaf world. By a non-characteristic deformation result in [11,26], we can show that $Sh(T^n, \Lambda_{\mathcal{T},\Theta})$ forms a local system of category over the parameter space of Θ. And by Coherent-Constructible Correspondence, the monodromy action of this local system corresponds to tensoring by line bundle on $\mathrm{Coh}(X_{\Sigma(\mathcal{T})})$, where $X_{\Sigma(\mathcal{T})}$ is the smooth DM toric stack. See the recent result by Hanlon [13] about the effect of variation of Θ on the Fukaya-Seidel category.

0.3 Sketch of Proof

The idea of the proof is illustrated in the above example, that is, we project the hypersurface $\mathcal{H}_{\beta,h,\Theta}$ to $M_{\mathbb{R}}$, then use a distance function to a point to induce the psh function φ, which in turn induces a Liouville structure on $\mathcal{H}$. However, there are two technical modifications necessary.

(1) The first modification is "tropical localization", as introduced by Mikhalkin and Abouzaid [1,15]. In the defining equation of the hypersurface, $0 = f_{\beta,h,\Theta}$, we want to keep only the leading terms and drop the irrelevant terms. See Picture 6 in [1] for an illustration.
(2) The second modification is that we need a convex function φ on $M_{\mathbb{R}} \cong \mathbb{R}^n$ more general than a positive definite quadratic form. Concretely, let us consider the polytope

$$P = \{x \in M_{\mathbb{R}} \mid \langle x, a\rangle \le h(a), \text{ for all vertices } a \text{ of } \partial\mathcal{T}\}.$$

The key property of φ is that, for each face F of P (not just the facet of P), $\varphi|_{\mathrm{Int}(F)}$ has a minimum, where $\mathrm{Int}(F)$ means the interior of F. See Definition 2.8 for precise definition of 'Kähler potential adapted to a polytope'.

Given this potential function, we can describe the combinatorial skeleton over the boundary ∂P. For a d_F-dimensional face F of P, we put (several copies of) d_F-dimensional tori T_F over the minimum point x_F of $\varphi|_{\mathrm{Int}(F)}$. Then, we consider the unstable manifold W_F associated to x_F of downward gradient flow $\varphi|_{\partial P}$, where the metric is induced from Hess φ on $M_{\mathbb{R}}$. Then, W_F is $n - d_F$ dimensional. And the collection $\{W_F\}$ forms a cell-complex, dual to the face-complex of ∂P, and coincide with the complex $\partial \mathcal{T}$. The combinatorial skeleton is of the form $\cup_F T_F \times W_F$.

Since we are using a non-Euclidean metric on $M_{\mathbb{R}}$, the unstable manifold W_F is 'wiggly'. However, after the Legendre transformation from $M_{\mathbb{C}^*}$ to T^*M_T, the cone over the skeleton agrees with the piecewise linear FLTZ skeleton from the toric variety.

Remark 0.4 The use of 'non-standard' Kähler potential φ on $M_{\mathbb{C}^*}$ (non-canonically isomorphic to $(\mathbb{C}^*)^n$) may be unorthodox, but it is natural in some sense. (1) The often used 'standard' Kähler potential $\sum_i (\log |z_i|)^2$ on $M_{\mathbb{C}^*}$ is not standard in the first place, since it depends on the choice of basis for M. (2) To identify the RSTZ-skeleton (0.2) that lives in $M_T \times N_{\mathbb{R}}$ with the Liouville skeleton (3.1) that lives in $M_{\mathbb{C}^*} \cong M_T \times M_{\mathbb{R}}$, one needs to identify $N_{\mathbb{R}}$ with $M_{\mathbb{R}}$. Here this is done using the Legendre transformation induced by φ.

0.4 Related works

The study of skeleta for Liouville (or Weinstein) manifold was motivated largely by Homological Mirror Symmetry (HMS). It was Kontsevich's original proposal, to compute the Fukaya category of a Weinstein manifold W by taking global sections of a (co)sheaf of categories living on the skeleton. Following this approach, the (complex) 1-dimensional case has been studied by [3,21,23]. In general, the category under consideration has two versions, a microlocal sheaf version, and a Floer-theoretic version. The two versions are shown to agree by the work of Ganatra et al. [10–12]. The microlocal sheaf version, originating from the seminal work of Nadler and Zaslow [16,20], says the infinitesimally wrapped Fukaya category [12,20] on T^*M with asymptotic condition of non-compact Lagrangian given by a conical Lagrangian Λ, is equivalent to constructible sheaves on M with singular support in Λ. The wrapped Fukaya category also has a microlocal sheaf version, developed by Nadler [19].

The microlocal sheaf category for local Lagrangian singularities has been studied by Nadler. In [17], Nadler defined a class of 'simple' singularities, termed 'arboreal singularities', and proved that the microlocal sheaf category on arboreal singularity is equivalent to the category of representation of quivers. In [18], Nadler showed that one can deform an arbitrary Lagrangian singularity to an arboreal one, while keeping the microlocal sheaf category invariant. It is also expected that such arborealization can be induced by a perturbation of Weinstein structure [4,24]. The skeleta in this paper are not arboreal. They are the boundary-at-infinity of the FLTZ skeleta, so they should still be easy to work with.

The skeleta for n dimensional pair-of-pants $\mathcal{P}_n$ has been studied by Nadler[19], where a higher dimensional analog of Fig. 1 is constructued. A Σ_{n+2}-symmetric skeleton for $\mathcal{P}_n$ is constructed by Gammage–Nadler [7], where Σ_{n+2} is the symmetric group. With the technique of tropical phase variety of Kerr–Zharkov [14], we hope to find other Lagrangian skeleta Λ_k for $\mathcal{P}_{n-1}$, where $k = 1, \ldots, n$ indicating the number of dominant terms in the defining equation of $\mathcal{P}_{n-1}$.

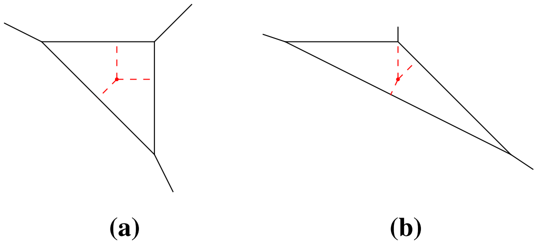

Fig. 2 **a** Tropical amoeba of $f = 1 + e^{-\beta}(x + y + 1/xy)$, **b** tropical amoeba of $f = 1 + e^{-\beta}(x + xy + 1/xy^2)$. The function $|x|^2$ on $\mathbb{R}^2$ is adapted to the amoeba polytope in **a** but not the one in **b**, since the minimum of $|x|^2$ on the top edge lies on the endpoint

In Gammage–Shende [8], as one ingredient in proving HMS for the toric boundary of a toric variety, they constructed the Liouville skeleton for the same hypersurface as considered here. However, their results [8, Theorem 3.4.2] depends on the following hypothesis that, *there exists some tropicalization function* $h : A \to \mathbb{R}$ *and some identification* $M \cong \mathbb{Z}^n$, *such that the tropical amoeba polytope* $P = \{x \in M_{\mathbb{R}} : \langle x, \alpha \rangle \leq h(\alpha), \forall \alpha \in \partial A\}$ *contains* 0 *as an interior point, and* $|x|^2$ *restricts to each face* F *of* P *has a minimum in the interior of* F. This hypothesis is true in two-dimension, and can be verified in certain examples, e.g. mirror to weighted projective spaces. But in general, one does not know if it is always true, it would be interesting to find a proof or construct a counter-example.[2] Our approach here does not rely on this hypothesis, which is equivalent to "$|x|^2$ *is adapted to the tropical amoeba polytope* P" *for some choice of* h. We avoid this by considering more flexible choices of Kähler potentials than $|x|^2$, and our approach works for any choice of h compatible with $\mathcal{T}$ (Fig. 2).

0.5 Outline

In Sect. 1, we review the tropical localization of Mikhalkin and Abouzaid. We are careful in picking the cut-off functions such that the inner boundary of the amoeba remains convex (Sect. 1.4). In Sect. 2, we review how to identify $M_{\mathbb{C}^*}$ with T^*M_T by choosing a Kähler potential, and we introduce the key concept of **Kähler potential adapted to a polytope** in Sect. 2.4. Then we state our main theorems in more detail in Sect. 3. The rest of the paper is devoted to proofs of these theorems.

1 Tropical geometry

1.1 Triangulation and amoeba

We follow [9, Chapter 7] and [15] to give background on coherent triangulations and tropical amoeba.

Let $A \subset N \cong \mathbb{Z}^n$ be a finite subset. Let $Q = \mathrm{conv}(A) \subset N_{\mathbb{R}}$ be the convex hull of A. Assume Q has dimension n. A *coherent triangulation* $(\mathcal{T}, h)$ for pair (Q, A) is a triangulation $\mathcal{T}$ of Q with vertices in A and a piecewise linear (PL) convex function $h : Q \to \mathbb{R}$, such that the maximal linear domains of h are exactly the maximal simplices of $\mathcal{T}$. Since h is determined by its restriction $h|_A$ on the vertices of the simplices, we sometimes abuse notation and write $h|_A$ as h.

[2] We thank Gammage and Shende for this clarification.

Let $(\mathcal{T}, h)$ be a coherent triangulation of (Q, A). We define the Legendre transformation of h as

$$L_h : M_{\mathbb{R}} \to \mathbb{R}, \quad L_h(y) = \max_{x \in A} \langle x, y \rangle - h(y),$$

where $\langle -, - \rangle$ is the dual pairing $M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$. One can show that L_h is a PL convex function on $M_{\mathbb{R}}$, inducing a cell-decomposition of $M_{\mathbb{R}}$ dual to the triangulation of $\mathcal{T}$ on Q. If $\tau \in \mathcal{T}$ is a k-simplex, then we use C_τ or $\tau^\vee$ to denote the dual cell of dimension $n-k$ in $M_{\mathbb{R}}$. In particular, C_α are the n-dimensional cells of $M_{\mathbb{R}}$. The cells and simplices are closed subsets in our convention.

Definition 1.1 The *tropical amoeba* $\Pi_h \subset M_{\mathbb{R}}$ is defined as the singular loci of L_h.

The tropical amoeba is the limit of amoeba, which we now define. Given a coherent triangulation (T, h) of (Q, A), $h : A \to \mathbb{R}$, we may define the patchworking polynomial

$$f_{\beta,h}(z) = \sum_{\alpha \in A} e^{-\beta h(\alpha)} z^\alpha : M^*_{\mathbb{C}} \to \mathbb{C}.$$

More generally, given a function $\Theta : A \to T$, we have

$$f_{\beta,h,\Theta}(z) = \sum_{\alpha \in A} e^{-\beta h(\alpha)} e^{-i\Theta(\alpha)} z^\alpha : M^*_{\mathbb{C}} \to \mathbb{C}.$$

Definition 1.2 The *Log amoeba* $\Pi_{\beta,h,\Theta}$ of $f = f_{\beta,h,\Theta}$ is defined as the image of $f^{-1}(0)$ under the (rescaled) log map

$$\mathrm{Log}_\beta : M \otimes_{\mathbb{Z}} \mathbb{C}^* \to M \otimes_{\mathbb{Z}} \mathbb{R}, \quad m \otimes z \mapsto m \otimes \beta^{-1} \log |z|.$$

Mikhalkin proved the following convergence theorem:

Theorem [15] *The tropical amoeba Π_h is the Hausdorff-limit of rescaled amoeba $\Pi_{\beta,h,\Theta}$ as $\beta \to \infty$.*

1.2 Monomial cut-off functions

The complements of the tropical amoeba has a one-to-one correspondence with the vertices of the triangulation $\mathcal{T}$,

$$M_{\mathbb{R}} \backslash \Pi_h = \bigsqcup_{\alpha \in A} C_\alpha, \quad M_{\mathbb{R}} \backslash \Pi_{\beta,h,\Theta} = \bigsqcup_{\alpha \in A} C_{\alpha,\beta,h,\Theta}.$$

C_α are convex polyhedra, and $C_{\alpha,\beta,h,\Theta}$ are smooth strictly convex domains [9, Chapter 6, Cor 1.6]. To simplify notation, we write $C_{\alpha,\beta,h,\Theta}$ as $C_{\alpha,\beta}$.

The purpose of introducing a monomial cut-off function $\chi_{\alpha,\beta}(z)$ is to turn off the term $e^{-\beta h(\alpha)} z^\alpha$ if it is much smaller than the other terms, thus straightening the

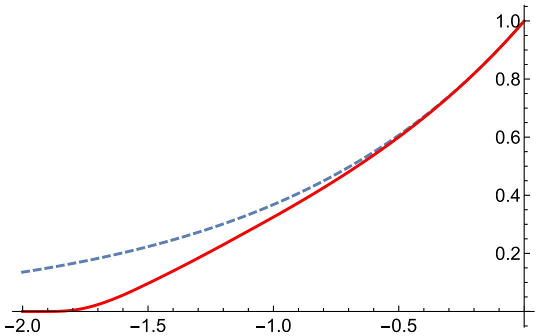

Fig. 3 We modify exponential function e^x (dashed line) to $e^x\chi(x)$ (solid line), such that $e^x\chi(x)$ remains convex

hypersurface. The idea is first used in Abouzaid [1] to control the symplectic geometry of the hypersurface. See Picture 6 in *loc.cit* for the effect on the hypersurface (and amoeba) by cutting-off irrelevant monomials.

We fix a cut-off function $\chi(x)$ on $\mathbb{R}$ with the following properties

- $\chi(x) = \begin{cases} 1 & x \in [0, \infty) \\ \in (0,1) & x \in (-2, 0) \\ 0 & x \in (-\infty, -2] \end{cases}$.
- $\chi(x)\exp(x)$ is convex.

Example 1.3 One can check that the following specification of $\chi(x)$ on $[-2, 0]$ gives a C^2 function on $\mathbb{R}$ with the desired convexity.

$$\chi(x) = e^{-1/(x+2)+1/2-x/4+x^2/8}.$$

See Fig. 3 for a plot of $\chi(x)e^x$. We have

$$\frac{(e^x\chi(x))''}{e^x\chi(x)} = 256 + 608x + 576x^2 + 288x^3 + 85x^4 + 14x^5 + x^6, \quad x \in (-2, 0).$$

which can be verified to be positive on $(-2, 0)$ The non-smooth point of $\chi(x)$ is at $x = 0$, and can be mollified if needed.

We use log coordinates $(\rho, \theta) \in M_{\mathbb{R}} \times M_T$ for a point $z \in M_{\mathbb{C}^*}$. We also use β-rescaled log coordinates $(u, \theta) = (\beta^{-1}\rho, \theta)$, thus $u = \mathrm{Log}_\beta(z)$. For each $\alpha \in A$, we define the linear function on $M_{\mathbb{R}}$ as follows

$$l_\alpha(u) := \langle u, \alpha\rangle - h(\alpha).$$

Definition 1.4 For any vertex $\alpha \in A$, we define the *monomial cut-off function* as

$$\chi_{\alpha,\beta}(u) = \prod_{\alpha' \text{ adjacent to } \alpha \text{ in } \mathcal{T}} \chi_{\alpha,\alpha',\beta}(u)$$

where

$$\chi_{\alpha,\alpha',\beta}(u) = \chi(\beta(l_\alpha(u) - l_{\alpha'}(u)) + \sqrt{\beta}).$$

We define a distance-like function to the region C_α,

$$r_\alpha(u) := L_{\hat{h}}(u) - l_\alpha(u) = \max_{\alpha' \in A}(l_{\alpha'}(u) - l_\alpha(u)).$$

Thus, $r_\alpha(u)$ is a non-negative PL convex function, vanishes only on C_α.

Proposition 1.5 *For all large enough β, $\chi_{\alpha,\beta}(u)$ satisfies the following property*

$$\chi_{\alpha,\beta}(u) = \begin{cases} 1 & r_\alpha(u) < \beta^{-1/2} \\ 0 & r_\alpha(u) > \beta^{-1/2} + 2\beta^{-1} \end{cases}$$

Proof If $r_\alpha(u) < \beta^{-1/2}$, then for all α' adjacent to α, we have

$$l_{\alpha'}(u) - l_\alpha(u) < \beta^{-1/2} \quad \Rightarrow \quad \beta(l_\alpha(u) - l_{\alpha'}(u)) + \sqrt{\beta} > 0$$

thus each factor in $\chi_{\alpha,\beta}(u)$ equals 1. The other case is similar to check, where β large enough means $r_\alpha(u)^{-1}(\beta^{-1/2} + 2\beta^{-1})$ intersects all the neighboring cells $C_{\alpha'}$ for C_α. □

Definition 1.6 For each $\alpha \in A$, we define the *bad region* as the open set

$$B_{\alpha,\beta} = \{u \in M_\mathbb{R} \mid \beta^{-1/2} + 2\beta^{-1} > r_\alpha(u) > \beta^{-1/2}\}.$$

The *(total) bad region* B_β is defined as the union of all $B_{\alpha,\beta}$. The *good region* is defined as the closed set $G_\beta := M_\mathbb{R} \backslash B_\beta$.

On the good regions, each $\chi_{\alpha,\beta}$ is either 0 or 1. Hence, we have a partition labeled by cells of $\mathcal{T}$:

$$G_\beta = \bigsqcup_{\tau \in \mathcal{T}} G_{\beta,\tau},$$

where $G_{\beta,\tau} = \{u \in G_\beta \mid \chi_{\alpha,\beta}(u) = 1 \iff \alpha \in \tau\}$ is a closed convex polyhedron with non-empty interior.

1.3 Tropical localized hypersurfaces

Following [1, Section 4], we define a family of hypersurfaces $\mathcal{H}_s$ as the zero-loci of $f_s(z)$:

$$f_s(z) := \sum_{\alpha \in A} f_\alpha(z)\chi_{\alpha,s}(z), \quad \mathcal{H}_s = f_s^{-1}(0),$$

where

$$f_\alpha = e^{-\beta h(\alpha) - i\Theta(\alpha)} z^\alpha, \quad \chi_{\alpha,s}(z) = s\chi_\alpha(z) + (1-s).$$

Here we have dropped the $\beta, h, \ldots$ subscripts from previous notations for clarity. These hypersurfaces $\mathcal{H}_s$ interpolates between the complex hypersurface $\mathcal{H}$ and tropical localized hypersurface $\widetilde{\mathcal{H}}$, where

$$\mathcal{H} := \mathcal{H}_0, \quad \widetilde{\mathcal{H}} := \mathcal{H}_1.$$

The following proposition is a modification of Proposition 4.2 in [1].

Proposition 1.7 *Fix any identification $M_{\mathbb{C}^*} \cong (\mathbb{C}^*)^n$. Let ω be any toric Kähler metric on $M_{\mathbb{C}^*}$, comparable with $\omega_0 = i\sum_i d\log z_i \wedge d\overline{\log z_i}$.* [3] *Then, for all large enough β, the family of hypersurfaces $\mathcal{H}_s$ are symplectic with induced symplectic form from ω.*

Proof We proceed as in Proposition 4.2 in [1]: to prove $f^{-1}(0)$ is symplectic, it suffices to prove $|\bar\partial f(z)|_\omega < |\partial f(z)|_\omega$ for $z \in f^{-1}(0)$.

Assume $s > 0$ and z is in the 'bad region' (see Definition 1.6) of the hypersurface $\mathcal{H}_s$, since otherwise the hypersurface is holomorphic at z and there is nothing to prove. Let $I'(z) = \{\alpha_0, \ldots, \alpha_k\} \subset A$ be subset of vertices where $r_{\alpha_i}(z) \le \beta^{-1/2}$, thus $I'(z)$ is vertex set for a simplex $\tau(z)$ of $\mathcal{T}$. Since $f_{\alpha_0}(z)^{-1} f_s(z)$ is an equally good defining equation for $\mathcal{H}_s$, hence without loss of generality, we may assume that $\alpha_0 = 0$. Let $I(z) = I'(z) \backslash \{\alpha_0\}$.

We claim that,

$$\frac{|\bar\partial f_s(z)|}{|\partial f_s(z)|} = O(e^{-\sqrt{\beta}}).$$

where $|-| := |-|_{\omega_0}$ indicate the norm given by ω_0. Given the claim, we have

$$\frac{|\bar\partial f_s(z)|_\omega}{|\partial f_s(z)|_\omega} < C^2 \frac{|\bar\partial f_s(z)|}{|\partial f_s(z)|} = O(e^{-\sqrt{\beta}}).$$

Thus for large enough β, $\mathcal{H}_s$ is symplectic near z.

Now we prove the above claim. Define

$$F(z) = \sum_{\alpha \in A} |f_\alpha(z)| = \sum_{\alpha \in A} e^{\beta l_\alpha(u)} =: e^{\beta \varphi_{\beta,h}(u)}.$$

[3] We say two Kähler metrics are comparable if the underlying Riemannian metric, denoted as g_1, g_2, satisfy

$$C^{-1} g_1 < g_2 < C g_1$$

for some positive constant C.

Then we have $\varphi_{\beta,h}(u) \geq L_h(u)$ for all $u \in M_{\mathbb{R}}$, and as $\beta \to \infty$ we have $\varphi_{\beta,h}(u) \to L_h(u)$ in C^0.

We first note that the derivatives for the cut-off functions $\chi_\alpha(z)$ have a uniform bound

$$\begin{aligned} |d\chi_\alpha(\rho)| &= \left| d\left(\prod_{\alpha'} \chi(\sqrt{\beta} + l_\alpha(\rho) - l_{\alpha'}(\rho)) \right) \right| \\ &\leq \sum_{\alpha'} |d(\chi(\sqrt{\beta} + l_\alpha(\rho) - l_{\alpha'}(\rho)))| \\ &\leq \|\chi'\|_\infty \sum_{\alpha'} |\alpha' - \alpha| < C \end{aligned}$$

where the product or sum are over α' adjacent to α in triangulation $\mathcal{T}$, and we used the bound $\chi \leq 1$.

We have for $\bar{\partial} f_s(z)$.

$$\begin{aligned} |F(z)^{-1}\bar{\partial} f_s(z)| &= |F(z)^{-1} \sum_\alpha f_\alpha(z)\bar{\partial}\chi_{\alpha,s}(z)| \leq s \sum_\alpha e^{\beta(l_\alpha(u) - \varphi_{\beta,h}(u))} |\bar{\partial}\chi_\alpha(z)| \\ &\leq \sum_\alpha e^{\beta(l_\alpha(u) - L_h(u))} |\bar{\partial}\chi_\alpha(z)| \leq \sum_\alpha e^{\beta(-\beta^{-1/2})} |\bar{\partial}\chi_\alpha(z)| \\ &\leq Ce^{-\sqrt{\beta}} \end{aligned}$$

where we used $\varphi_{\beta,h}(u) \geq L_h(u)$ and on the support of $d\chi_\alpha(z)$ we have $r_\alpha(z) = L_h(u) - l_\alpha(u) > 1/\sqrt{\beta}$.

Next, we compute $\partial f_s(z)$.

$$\begin{aligned} |F(z)^{-1}\partial f_s(z)| &= \left| F(z)^{-1} \sum_\alpha \partial f_\alpha(z)\chi_{\alpha,s}(z) + s f_\alpha(z)\partial\chi_\alpha(z) \right| \\ &= F(z)^{-1} \left| \sum_{i=1}^k \partial f_{\alpha_i}(z) \right| + O(e^{-\sqrt{\beta}}) \\ &= F(z)^{-1} \left| \sum_{i=1}^k f_{\alpha_i}(z)\langle \alpha_i, d(\rho + i\theta) \right| + O(e^{-\sqrt{\beta}}) \\ &= F(z)^{-1} \left[\sum_{i=1}^k \sum_{j=1}^k f_{\alpha_i}(z)\overline{f_{\alpha_j}(z)}\langle \alpha_i, \alpha_j \rangle \right]^{1/2} + O(e^{-\sqrt{\beta}}) \\ &> C_1 F(z)^{-1} \left[\sum_{i=1}^k |f_{\alpha_i}(z)|^2 \right]^{1/2} + O(e^{-\sqrt{\beta}}) \\ &> C_2 F(z)^{-1} \sum_{i=1}^k |f_{\alpha_i}(z)| + O(e^{-\sqrt{\beta}}) = C_2 + O(e^{-\sqrt{\beta}}) \end{aligned}$$

where $O(e^{-\sqrt{\beta}})$ represents a remainder bounded by $e^{-\sqrt{\beta}}$, C_1 is the smallest eigenvalue of the $k \times k$ matrix $M_{ij} = \langle \alpha_i, \alpha_j \rangle$, which is non-degenerate since $\{\alpha_0 = 0, \alpha_1, \ldots\}$ are vertices of a k-simplex in $\mathcal{T}$. We also used that all the l^p $(1 \leq p \leq \infty)$ norm on $\mathbb{R}^k$ are equivalent. □

Proposition 1.8 (Proposition 4.9 [1]). *The family of hypersurfaces $\mathcal{H}_s$ are symplectomorphic for all $s \in [0, 1]$.*

1.4 Tropical localization with convexity

For log amoeba $\Pi_{h,\beta}$ and tropical amoeba Π_h, their connected components of complement $C_{\alpha,\beta}$ and C_α are convex. Let $\widetilde{\Pi}_{\beta,h} = \mathrm{Log}_\beta(\widetilde{\mathcal{H}})$ be the amoeba of the tropical localized hypersurface, and let $\widetilde{C}_\alpha$ denote the complements dual to vertex $\alpha \in A$. We would like to show that $\widetilde{C}_\alpha$ are close to convex as well.

Proposition 1.9 *The defining equation for $\widetilde{C}_\alpha$ is*

$$1 = \sum_{\alpha' \text{ adjacent to } \alpha} e^{\beta(l_{\alpha'}(u) - l_\alpha(u))} \chi_{\alpha',\beta}(u) = F_\alpha.$$

and F_α is convex in the good region, i.e., where all $\chi_{\alpha,\beta}(u)$ are constant with value 0 *or* 1.

Proof The boundary of the complement $\widetilde{C}_\alpha$ is where the dominant term equals the sum of the other non-dominant term. By the tropical localization, there are at most n non-dominant terms for a point z on the boundary (thanks to $\mathcal{T}$ being a triangulation). Hence the θ_i can be chosen, such that the argument of the dominant and non-dominant terms are the same. For the second statement, we note that over the good region, F_α is a sum of convex functions. □

Definition 1.10 The convex model $\widehat{C}_\alpha$ for $\widetilde{C}_\alpha$ is defined by $\{u \in C_\alpha \mid \widehat{F}_\alpha(u) = 1\}$, where

$$\widehat{F}_\alpha = \sum_{\alpha' \text{ adjacent to } \alpha} e^{\beta(l_{\alpha'}(u) - l_\alpha(u))} \chi_{\alpha',\alpha,\beta}(u)$$

The two defining functions, F_α and $\widehat{F}_\alpha$, differ by the cut-off functions: F_α uses $\chi_{\alpha',\beta}$ which cuts along the boundary of $C_{\alpha'}$, whereas $\widehat{F}_\alpha$ uses $\chi_{\alpha',\alpha,\beta}$ which cuts along the hyperplane separating C_α and $C_{\alpha'}$. However, on C_α the two functions and the hypersurfaces are very close, as the following two propositions show.

Proposition 1.11 *For all $k \geq 1$, there are constant c_k, c'_k, such that*

$$\|F\|_{C^k(C_\alpha)} + \|\widehat{F}\|_{C^k(C_\alpha)} \leq c'_k \beta^k$$

and

$$\|F - \widehat{F}\|_{C^k(C_\alpha)} < c_k \beta^k e^{-\sqrt{\beta}}.$$

Proof First, we note that since all $\chi_{\alpha_1,\alpha_2,\beta} \leq 1$, we have $\chi_{\alpha'}(u) < \chi_{\alpha',\alpha}(u)$, thus

$$\widehat{F}_\alpha(u) > F_\alpha(u), \quad \widehat{C}_\alpha \subset \widetilde{C}_\alpha.$$

For $u \in \overline{C}_\alpha$ and α' adjacent to α, if $\chi_{\alpha'}(u) - \chi_{\alpha',\alpha}(u) \neq 0$, then $l_{\alpha'}(u) - l_\alpha(u) + \sqrt{\beta} \in (-2, 0)$. Hence

$$\widehat{F}_\alpha(u) - F_\alpha(u) = \sum_{\alpha' \text{ adjacent to } \alpha} e^{\beta(l_{\alpha'}(u) - l_\alpha(u))}(\chi_\alpha(u) - \chi_{\alpha',\alpha}(u)) < Ce^{-\sqrt{\beta}}.$$

Similarly, taking k-th derivative, we have

$$|\partial_u^k \widehat{F}_\alpha(u) - \partial_u^k F_\alpha(u)| < C_k \beta^k e^{-\sqrt{\beta}}.$$

where the norm are taken with respect to Euclidean norm on $\mathbb{R}^n$, after choosing an identification $M_\mathbb{R} \cong \mathbb{R}^n$. This finishes the proof. $\square$

Fix $M_\mathbb{R} \cong \mathbb{R}^n$ and equip $\mathbb{R}^n$ with Euclidean metric. Let $S^*\mathbb{R}^n$ denote the unit cosphere bundle. If C is a domain with smooth boundary, we define

$$\Lambda_C = \{(p, \xi) \in S^*\mathbb{R}^n \mid p \in \partial C, \xi \in (T_p \partial C)^\perp \text{ and points outward } \}$$

Proposition 1.12 *We have the following convergence results:*

(1) *In the good region in C_α, $\partial \widetilde{C}_\alpha = \partial \widehat{C}_\alpha$.*
(2) *The Hausdorff distance between $\partial \widetilde{C}_\alpha$ and $\partial \widehat{C}_\alpha$ is $O(\beta^{-1} e^{-\sqrt{\beta}})$.*
(3) *The Hausdorff distance between $\Lambda_{\widetilde{C}_\alpha}$ and $\Lambda_{\widehat{C}_\alpha}$ is $O(\beta e^{-\sqrt{\beta}})$.*

Proof We will write $F = F_\alpha$, $\widehat{F} = \widehat{F}_\alpha$ and so on, omitting the α subscript when it is not confusing.

(1) Since in the good region in C_α, all the cut-off functions $\chi_{\alpha'}$ and $\chi_{\alpha',\alpha}$ are equal.

(2) Since $\widehat{F} \leq F$, hence the domain $\widehat{C} \subset \widetilde{C} \subset C$. If $u \in \partial \widehat{C} \backslash \partial \widetilde{C}$, we take gradient flow of F, starting from u and ending on $u' \in \partial \widetilde{C}$. Let $\gamma : [0, t] \to C_\alpha$ denote this integral curve. Since around $\partial \widetilde{C}$ and $\partial \widehat{C}$, we have uniform lower bound for $|dF|$ and $|d\widehat{F}|$ by some constant $c\beta$, hence

$$c\beta \operatorname{dist}(u, u') \leq c\beta t < \int_0^t |\nabla F(\gamma(s))| ds$$

and

$$\int_0^t |\nabla F(\gamma(s))| ds = F(\gamma(t)) - F(\gamma(0)) = 1 - F(u) = \widehat{F}(u) - F(u) < Ce^{-\sqrt{\beta}}$$

hence

$$\operatorname{dist}(u, u') = O(\beta^{-1} e^{-\sqrt{\beta}}) \tag{1.1}$$

Similarly, if we start from $u' \in \partial \widetilde{C} \backslash \partial \widehat{C}$ we may find $u \in \widehat{C}$ using gradient flow of $\widehat{F}$, with the same bound as above. This establishes the bound on Hausdorff distance

(3) Let $u \in \partial \widehat{C} \backslash \partial \widetilde{C}$, and $u' \in \partial \widetilde{C}$ constructed as in (2). We have

$$\begin{aligned} |dF(u') - d\widehat{F}(u)| &\leq |dF(u') - dF(u)| + |dF(u) - d\widehat{F}(u)| \\ &\leq \|F\|_{C^2} \operatorname{dist}(u, u') + O(\beta e^{-\sqrt{\beta}}) = O(\beta e^{-\sqrt{\beta}}) \end{aligned}$$

where we used the C^2 bound of F in Proposition 1.11 and the distance bound in (1.1). ◻

Let Λ_{C_α} denote the Legendrian of the unit exterior conormal to C_α.

Proposition 1.13 *Let $\alpha \in A$, $\widehat{C}_\alpha$ be the tropical localized amoeba's complement. We have the following convergence of the boundary of $\widehat{C}_\alpha$, and its Legendrian lifts:*

(1) *The Hausdorff distance between $\partial \widehat{C}_\alpha$ and ∂C_α is $O(1/\sqrt{\beta})$.*
(2) *The Hausdorff distance between $\Lambda_{\widehat{C}_\alpha}$ and Λ_{C_α} is $O(1/\sqrt{\beta})$.*

Proof Consider the simplices around vertex α in $\mathcal{T}$. Let τ be such a k-dimensional simplex, with vertices $\alpha, \alpha_1, \dots, \alpha_k$. Denote the dual face by $\tau^\vee$ on the polytope C_α. We also define a locally closed subset $U_\tau \subset \partial \widehat{C}_\alpha$, such that $z \in U_\tau$ iff the set $I_\alpha(z) = \{\alpha_1, \dots, \alpha_k\}$.

$$I_\alpha(z) := \{\alpha' \text{ adjacent to } \alpha,\ \chi_{\alpha', \alpha, \beta}(z) > 0\}.$$

Define the orthogonal projection map

$$\pi_\tau : U_\tau \to \tau^\vee.$$

If $u \in U_\tau$, $u' = \pi_\tau(u)$, then since

$$-1/\sqrt{\beta} - 2/\beta < l_{\alpha_i}(u) - l_\alpha(u) < 0, \quad \text{and} \quad l_{\alpha_i}(u') - l_\alpha(u') = 0$$

we have then

$$dist(u, u') < O\left(\sum_{i=1}^k |l_{\alpha_i}(u) - l_\alpha(u)|\right) = O(1/\sqrt{\beta}).$$

Also $\tau^\vee$ is in $O(1/\sqrt{\beta})$ neighborhood of $Im(\pi_\tau)$, thus the Hausdorff distance between $\tau^\vee$ and U_τ is $O(1/\sqrt{\beta})$. Considering all faces $\tau^\vee$ of C_α proves the first statement.

For the second statement, we further note that, for any $u \in U_\tau$, the exterior unit conormal ξ of $\widehat{C}_\alpha$ at u is contained in $\operatorname{cone}(\alpha_1 - \alpha, \dots, \alpha_k - \alpha) = \mathbb{R}_+ \tau$. Define

$$\Lambda_{\tau^\vee} := \tau^\vee \times (\mathbb{R}_+ \cdot \tau) \ \cap\ S^*\mathbb{R}^n.$$

Then the projection map π_τ, lifts to

$$\widetilde{\pi}_\tau : \Lambda_{\widetilde{C}_\alpha}|_{U_\tau} \to \Lambda_{\tau^\vee}, \quad (u, \xi) \mapsto (\pi_\tau(u), \xi)$$

Since the fiber direction has distance zero, a similar argument as (1) proves the second statement. □

2 Legendre transformation and toric Kähler potential

In this section we use Legendre transform to define a diffeomorphism between $(\mathbb{C}^*)^n$ and T^*T^n, and define a Kähler structure on $(\mathbb{C}^*)^n$.

2.1 Legendre transformation

Let V be a real vector space of dimension n, and $V^\vee$ be its dual space. There is a natural identification of symplectic space

$$T^*V \cong V \times V^\vee \cong T^*V^\vee.$$

Let π_V and $\pi_{V^\vee}$ denote the projection of $V \times V^\vee$ to its first and second factor, respectively.

Let φ be a smooth strictly convex function on V. The Legendre transformation for φ is defined as

$$\Phi_\varphi : V \to V^\vee, \quad x \mapsto d\varphi(x).$$

We will always assume φ satisfies some growth condition such that the Legendre transformation is surjective. The Legendre dual ψ of φ is also a convex function defined as

$$\psi : V^\vee \to \mathbb{R}, \quad y \mapsto \sup_{x \in V} \langle x, y\rangle - \varphi(x) = \langle \Phi_\varphi^{-1}(y), y\rangle - \varphi(\Phi_\varphi^{-1}(y)).$$

If we fix a linear coordinate $\rho = (\rho_1, \ldots, \rho_n)$ on V and dual coordinate $p = (p_1, \ldots, p_n)$ on $V^\vee$, then the Legendre transformation can be written as

$$p_i = \partial_{\rho_i}\varphi(\rho).$$

If $p = d\varphi(p)$, then Legendre dual function

$$\psi(p) = \sum_i \rho_i p_i - \varphi(\rho).$$

And the two matrices Hess $\varphi(\rho) = \partial_{ij}\varphi(\rho)$ and Hess $\psi(p) = \partial_{ij}\psi(p)$ are inverse of each other. There is a metric on V induced by φ:

$$g_\varphi = \partial_{ij}\varphi(\rho) d\rho_i \otimes d\rho_j.$$

The above construction can be interpreted symplectically. Consider the graph Lagrangian $\Gamma_{d\varphi}$ in T^*V

$$\Gamma_{d\varphi} = \{(x, y) \in V \times V^\vee \mid y = d\varphi(x)\}.$$

Let $L = \Gamma_{d\varphi}$. Then the Legendre transform is $\Phi_\varphi = \pi_{V^\vee}|_L \circ \pi_V|_L^{-1}$

$$\begin{tikzcd} & L \arrow[dl, "\pi_V"'] \arrow[dr, "\pi_{V^\vee}"] & \\ V & & V^\vee \end{tikzcd}.$$

L as a section in $T^*V^\vee$ is the graph of $\Gamma_{d\psi}$ for the Legendre dual function ψ of φ.

The following lemma says, gradient vector field on V and differential one form are related by Legendre transformation.

Lemma 2.1 *Let φ be any smooth convex function on V, and let $f : V \to \mathbb{R}$ be any smooth function. For any $\rho \in V$, and $p = \Phi_\varphi(\rho) \in V^\vee$, then $(\Phi_\varphi)_*(\nabla f|_\rho) \in T_pV^\vee \cong V^\vee$ and $df(\rho) \in T^*_\rho V \cong V^\vee$ are equal.*

Proof We work with linear coordinates $(\rho_1, \dots, \rho_n)$ on V and dual coordinate $(p_1, \dots, p_n)$ on $V^\vee$. Let $g_{ij} = (g_\varphi)_{ij} = \partial_{ij}\varphi$ and g^{ij} be the matrix inverse of g_{ij}.

$$\begin{aligned} (\Phi_\varphi)_*\nabla f(\rho) &= \sum_{i,j,k} \partial_{\rho_k} f \cdot g^{jk} \cdot \frac{\partial p_i(\rho)}{\partial \rho_j} \cdot \partial_{p_i} \\ &= \sum_{i,j,k} \partial_{\rho_k} f \cdot g^{jk} \cdot g_{ij} \cdot \partial_{p_i} \\ &= \sum_{i,k} \partial_{\rho_k} f \cdot \delta^i_k \cdot \partial_{p_i} = df. \end{aligned}$$

□

2.2 Identification between $M_{\mathbb{C}^*}$ and T^*M_T

There is a canonical complex structure on $M_{\mathbb{C}^*} \cong M_\mathbb{R} \times M_T$, coming from

$$\mathbb{C}^* \cong \mathbb{R} \times T, \quad z = e^{\rho + i\theta} \mapsto (\rho, \theta).$$

And there is a canonical symplectic structure on $T^*M_T \cong N_\mathbb{R} \times M_T$. We will use notation $\theta \in M_T$, $\rho \in M_\mathbb{R}$ and $p \in N_\mathbb{R}$. If we fix a $\mathbb{Z}$-basis for M, then we have $M_{\mathbb{C}^*} \cong (\mathbb{C}^*)^n = \{(e^{\rho_i + i\theta_i})_i\}$ and $T^*M_T \cong T^*T^n = \{(\theta_i, p_i)_i\}$.

Let $\varphi : M_\mathbb{R} \to \mathbb{R}$ be a smooth strictly convex function such that the Legendre transformation $\Phi_\varphi : M_\mathbb{R} \to N_\mathbb{R}$ is surjective. We abuse notation and also denote by φ the pullback via $M_{\mathbb{C}^*} \to M_\mathbb{R}$, and call φ a Kähler potential on $M_{\mathbb{C}^*}$. Then we may define Liouville one-form and symplectic two-form on $M_{\mathbb{C}^*}$

$$\lambda = -d^c\varphi, \quad \omega = -dd^c\varphi.$$

In coordinate form, we have

$$\lambda_\varphi = \sum_i \partial_i\varphi(\rho)d\theta_i, \quad \omega_\varphi = \sum_{i,j} \partial_{ij}\varphi(\rho)\, d\rho_i \wedge d\theta_j.$$

The Riemannian metric can also be obtained by $g_\varphi(X, Y) = \omega_\varphi(X, JY)$, where $J\partial_{\rho_i} = \partial_{\theta_i}$, $J\partial_{\theta_i} = -\partial_{\rho_i}$, or in coordinate form

$$g = \sum_{i,j} \partial_{ij}\varphi(\rho)(d\rho_i \otimes d\rho_j + d\theta_i \otimes d\theta_j).$$

If we equip T^*M_T with the standard exact symplectic structure (ω, λ):

$$\lambda_{std} = \sum_i p_i d\theta_i, \quad \omega_{std} = \sum_i dp_i \wedge d\theta_i,$$

then by Legendre transformation $\Phi_\varphi \times \mathrm{id} : M_{\mathbb{C}^*} = M_\mathbb{R} \times M_T \to N_\mathbb{R} \times M_T = T^*M_T$, we have

$$(\Phi_\varphi \times \mathrm{id})^*(\lambda_{std}) = \lambda_\varphi, \quad (\Phi_\varphi \times \mathrm{id})^*(\omega_{std}) = \omega_\varphi.$$

2.3 Homogeneous Kähler potential

Next we will restrict ourselves to homogeneous convex functions as Kähler potential.

Definition 2.2 A convex function φ on $M_\mathbb{R}$ is said to be *homogeneous of degree d* for some $d \geq 1$, if for any $0 \neq x \in M_\mathbb{R}$ and any $\lambda > 0$, we have

$$\varphi(\lambda x) = \lambda^d \varphi(x), \tag{2.1}$$

and $\Omega = \{x : \varphi(x) \leq 1\}$ is a bounded strictly convex closed set with smooth boundary.

Remark 2.3 Any positive definite quadratic form on $M_\mathbb{R}$ is a homogeneous degree two convex function. Moreover, for any $d \geq 1$, and for any bounded strictly convex subset $\Omega \subset M_\mathbb{R}$ with smooth boundary and with 0 as an interior point, there exists an unique a homogeneous degree d convex function $\varphi_{\Omega,d}$ such that $\Omega = \{x : \varphi(x) \leq 1\}$.

Proposition 2.4 *For any homogeneous convex function φ of degree d with $d \in [0, \infty)$, we have*

(1) *φ is smooth on $M_\mathbb{R}\backslash\{0\}$.*
(2) *φ is C^k at 0 where k is the largest integer less than d.*
(3) *If $d > 1$, then φ is strictly convex.*

Proof (1) and (3) are easy to verify. We only prove (2). Fix a linear coordinate $x_1, \ldots, x_n$ on $M_{\mathbb{R}}$. For multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, any point $0 \neq x \in M_{\mathbb{R}}$ and $\lambda > 0$, we have $\partial_x^\alpha \varphi(\lambda x) = \lambda^{d-|\alpha|}\partial_x^\alpha \varphi(x)$. Hence if in addition $|\alpha| \leq k < d$, then $\lim_{\lambda \to 0} \partial_x^\alpha \varphi(\lambda x) = 0$. Hence all k-th derivative can be continuated to $x = 0$. □

Lemma 2.5 *If φ is a homogeneous degree d convex function, then for $\lambda > 0$*

$$\Phi_\varphi(\lambda\rho) = \lambda^{d-1}\Phi_\varphi(\rho).$$

Definition 2.6 Let $M_{\mathbb{R}}^\infty := (M_{\mathbb{R}}\backslash 0)/\mathbb{R}_{>0}$ and $N_{\mathbb{R}}^\infty := (N_{\mathbb{R}}\backslash 0)/\mathbb{R}_{>0}$. Then we define the projective Legendre transformation

$$\Phi_\varphi^\infty : M_{\mathbb{R}}^\infty \to N_{\mathbb{R}}^\infty.$$

It is easy to check that Φ_φ^∞ is an orientation preserving diffeomorphism from S^{n-1} to itself. Geometrically, if we take the level set $S = \varphi^{-1}(1)$, then each element in $M_{\mathbb{R}}^\infty$ corresponds to a point on S, and the outward conormal of S at the point is the element in $N_{\mathbb{R}}^\infty$ obtained by Φ_φ^∞.

Proposition 2.7 *Let φ be any homogeneous convex function on $M_{\mathbb{R}}$ of degree $k > 1$, and equip $M_{\mathbb{R}}$ with metric g_φ induced from Hessian of φ. Then the integral curves in $M_{\mathbb{R}}\backslash\{0\}$ of the gradient of φ are rays. Equivalently,*

$$\nabla\varphi(\rho) = C(\rho)\sum_i \rho_i \partial_{\rho_i}, \quad C(\rho) > 0.$$

Proof For any nonzero $\rho \in M_{\mathbb{R}}$, we have $\Phi_\varphi(\rho) = d\varphi(\rho)$, also by Proposition 2.1 we have $(\Phi_\varphi)_*(\nabla\rho) = d\varphi(\rho)$, hence the gradient vector field $\nabla\varphi$ on $M_{\mathbb{R}}$ when pushed-forward to $N_{\mathbb{R}}$ is exactly the radial vector field $p\partial_p$ whose integral curves are rays. Since φ is homogeneous, hence Φ_φ takes ray to ray, hence the integral curve of $\nabla\varphi$ is the pull-back of integral curve of $p\partial_p$, i.e. rays. □

2.4 Kähler potentials adapted to a polytope

This is one of the key construction in this paper. We replace the Kähler potential $\sum_i u_i^2$ on $(\mathbb{C}^*)^n$ where $u_i = \log|z_i|$ by any homogeneous degree two Kähler potential $\varphi(u)$.

Let P be a convex polytope (possibly unbounded) in $M_{\mathbb{R}}$ containing 0 as an interior point. We define a notion of convexity with respect to P.

Definition 2.8 A homogeneous convex function φ on $M_{\mathbb{R}}$ is *convex with respect to P*, if for each face F of P of positive dimension, the restriction $\varphi|_F$ has a unique minimum point in the interior of F. A *Kähler potential adapted to P* is a homogeneous degree two convex function $\varphi : M_{\mathbb{R}} \to \mathbb{R}$ that is convex with respect to P.

Remark 2.9 As noted in [13] and pointed out by a referee, the "degree two" condition can be relaxed to degree $d > 1$. The results in this section still holds, but proofs needs modification, e.g., $\Gamma_{d\varphi}$ is homogeneous under $\mathbb{R}_+$ rescaling of $M_\mathbb{R} \times N_\mathbb{R}$, where the weights on the two factors are 1 and $d-1$. We leave the generalization for the interested reader.

Remark 2.10 A homogeneous convex function φ on $M_\mathbb{R}$ is convex with respect to P, if the increasing sequence of level sets $\{\varphi(\rho) < c\}$ meet the faces of P in the interior first.

Proposition 2.11 *For any convex polytope P in $M_\mathbb{R}$ containing* 0 *as an interior point, there exists a non-empty contractible set of Kähler potential adapted to P.*

Proof First, we show the existence of such potential φ. We will build the level set $S = \{\varphi = 1\}$, and show that as we rescale S to λS, for λ from 0 to ∞, S will meet the interior of each face F first. We will proceed by first build a polyhedral approximation of S, then smooth it.

For each face F of P, we pick a point x_F in the interior of F if $\dim F > 0$, or $x_F = F$ if F is a point. Let T be the simplicial triangulation of P with vertices of F, then T is also a barycentric subdivision of P. Let $\phi_T : P \to \mathbb{R}$ a piecewise linear convex function on P, with maximal convex domain the top-dimensional simplices of T, and such that for any $0 \leq d \leq n-1$, and any face x_F of dimension d, $\phi_T(x_F) = c_d$ are the same for all such F. Such ϕ_T can be constructed inductively from x_F with $\dim F$ from 0 to $n-1$. Let ϕ_T be extended to $M_\mathbb{R}$ by linearity. Thus ϕ_T has a unique minium point in each face F.

Let $\eta \in C_c^\infty(\mathbb{R}^n)$ be a bump function with $\int \eta = 1$, and let $\eta_\epsilon(x) = \eta(x/\epsilon)/\epsilon^n$. Let $\phi_{T,\epsilon} = \eta_\epsilon \star \phi_T + \epsilon |x|^2$, where $|x|$ is taken with respect to any fixed inner product on $\mathbb{R}^n$, then $\phi_{T,\epsilon}$ is a linear combination of convex function hence still convex. Since $\phi_{T,\epsilon} \to \phi_T$ as $\epsilon \to 0$, for ϵ small enough, $\phi_{T,\epsilon}$ still has a unique minimum point in each face F. And $S_{T,\epsilon} = \{\phi_{T,\epsilon} = 1\}$ is a convex smooth boundary, such that $S_{T,\epsilon} \to S_T = \{\phi_T = 1\}$ as $\epsilon \to 0$. Then, for small enough ϵ, we can use $S_{T,\epsilon}$ as the contour of the homogeneous degree two convex function $\{\varphi(x) = 1\}$.

(2) Let $\mathcal{K}$ be the set of homogeneous degree two potential adapted to P. Then there is a surjective continuous map $\pi : \mathcal{K} \to \prod_{F, \dim F > 0} \mathrm{Int}(F)$, defined by sending φ to its critical points on each face. Since if two convex functions φ_1, φ_2 have the same critical points, then their convex linear combination $t\varphi_1 + (1-t)\varphi_2$ for $t \in [0, 1]$ are still homogeneous degree two and with the same critical points, we see the fiber of map π is convex hence contractible. Since the base of the fibration Cr is contractible as well, we see $\mathcal{K}$ is contractible. □

Let P be a convex polytope in $M_\mathbb{R}$ containing 0 as an interior point. Recall the definition of the dual polytope $P^\vee \subset N_\mathbb{R}$

$$P^\vee = \{p \in N_\mathbb{R} \mid \langle p, x\rangle \leq 1 \ \forall x \in P\}. \tag{2.2}$$

For any face $F \subset P$, there is dual face $F^\vee \subset P^\vee$, and $\dim_\mathbb{R} F + \dim_\mathbb{R} F^\vee = n-1$. We define three subsets of $M_\mathbb{R} \times N_\mathbb{R}$

$$L_P = \bigcup_F \mathrm{cone}(F) \times F^\vee, \quad L_{P^\vee} = \bigcup_F F \times \mathrm{cone}(F^\vee), \quad \Lambda_P = \bigcup_F F \times F^\vee, \tag{2.3}$$

where F runs over the faces of P, and $\mathrm{cone}(F) = \mathbb{R}_{>0} \cdot F$.

Remark 2.12 L_P and $L_{P^\vee}$ are piecewise Lagrangians, and $\Lambda_P = L_P \cap L_{P^\vee}$ is piecewise isotropic. L_P is the exterior conormal of $P^\vee$ in $T^*N_\mathbb{R}$, and $L_P^\vee$ is the exterior conormal of P in $T^*M_\mathbb{R}$. If we let $\varphi_{P,1}$ be the piecewise linear function on $M_\mathbb{R}$, such that $P = \{x : \varphi_{P,1}(x) \leq 1\}$, then L_P morally is $\Gamma_{d\varphi_{P,1}}$.

Lemma 2.13 *Let φ be a homogeneous degree two convex function on $M_\mathbb{R}$. P, $P^\vee$ be dual convex polytopes in $M_\mathbb{R}$ and $N_\mathbb{R}$ as above. Let F be a face of P. Then there is a natural bijection*

$$\mathrm{cone}(F) \times F^\vee \cap \Gamma_{d\varphi} \leftrightarrow F \times \mathrm{cone}(F^\vee) \cap \Gamma_{d\varphi}. \tag{2.4}$$

Proof If $(\lambda x, p) \in \mathrm{cone}(F) \times F^\vee \cap \Gamma_{d\varphi}$, where $\lambda > 0$ and $x \in F$, $p \in F^\vee$, then by conic invariance of $\Gamma_{d\varphi}$, we have

$$(x, p/\lambda) = \frac{1}{\lambda}(\lambda x, p) \in F \times \mathrm{cone}(F^\vee) \cap \Gamma_{d\varphi}. \tag{2.5}$$

Sending $(\lambda x, p)$ to $(x, p/\lambda)$ is the desired bijection. □

Next, we give some equivalent characterization for convexity with respect to a polytope.

Proposition 2.14 *Let P be a convex polytope in $M_\mathbb{R}$ containing* 0 *as an interior point. Let φ be a homogeneous degree two convex function on $M_\mathbb{R}$. The following conditions are equivalent:*

(1) *φ is adapted to P.*
(2) *For each face F of P, the smooth component* $\mathrm{Int}(F \times \mathrm{cone}(F^\vee))$ *of $L_P^\vee$ has a unique intersection with $\Gamma_{d\varphi}$.*
(3) *For each face F of P, the smooth component* $\mathrm{Int}(\mathrm{cone}(F) \times F^\vee)$ *of L_P has a unique intersection with $\Gamma_{d\varphi}$.*

Proof (2) is equivalent to (3) by Lemma 2.13.

(2) $\Rightarrow$ (1): since $\varphi|_F$ is still convex, hence as at most one minimum point in the interior, and any interior critical point is a minimum. Since

$$\emptyset \neq F \times \mathrm{cone}(F^\vee) \cap \Gamma_{d\varphi} \subset T_F^* M_\mathbb{R} \cap \Gamma_{d\varphi} \tag{2.6}$$

we see $\varphi|_F$ has a critical point.

(1) $\Rightarrow$ (2): for each face F of P, let x_F be the critical point of $\varphi|_F$, and let $H_F \subset M_\mathbb{R}$ be the affine hyperplane tangent to the contour of φ at x_F. We claim that H_F is a supporting hyperplane for P, and $P \cap H_F = F$. Then $p = d\varphi|_{x_F} \in T^*_{x_F} M_\mathbb{R} \cong N_\mathbb{R}$ is in the exterior conormal of H_F (exterior with respect to P), hence $p \in \mathrm{cone}(F^\vee)$. Thus, $(x, p) \in F \times \mathrm{cone}(F^\vee)$. □

A consequence of the proposition is the compatibility of the 'adaptedness' with Legendre transformation.

Corollary 2.15 *Let P be a convex polytope in $M_{\mathbb{R}}$ containing 0 as an interior point and $P^\vee$ the dual polytope. Let φ be homogeneous degree two convex function, and ψ the Legendrian dual of φ. Then φ is adapted to P if and only if ψ is adapted to $P^\vee$.*

3 Main results

Let Q be a convex lattice polytope in $N_{\mathbb{R}}$ containing $0 = \alpha_0$. Let $\mathcal{T}$ be a coherent star triangulation of Q based at 0 with integral vertices, and $\partial\mathcal{T}$ be the subset of simplices not containing 0. Let A be the set of vertices of $\mathcal{T}$, and let $h : A \to \mathbb{R}$ induce $\mathcal{T}$ with $h(0) = 0$. Fix a $\Theta : A \to T$ with $\Theta(\alpha_0) = \pi$.

Let $\Pi_h \subset M_{\mathbb{R}}$ be the tropical amoeba of $(\mathcal{T}, h)$, and $P = C_{\alpha_0}$ be the connected component in $M_{\mathbb{R}} \backslash \Pi_h$ corresponding to α_0.

Let $\varphi : M_{\mathbb{R}} \to \mathbb{R}$ be a homogeneous degree two convex function (i.e. $\varphi(\lambda x) = \lambda^2\varphi(x)$ for all $\lambda > 0$). [4] We assume φ is adapted to P, i.e, every face of P contains a minimum of φ in its interior.

Then the tropical localized polynomial is

$$\widetilde{f}(z) := \sum_{\alpha \in A} e^{-\beta h(\alpha) - i\Theta(\alpha)} \chi_{\alpha,\beta}(z) z^\alpha$$

where the monomial cut-off function $\chi_{\alpha,\beta}$ defined as in Definition 1.4. Denote the localized hypersurface and amoeba as $\widetilde{\mathcal{H}} := \widetilde{f}^{-1}(0)$ and $\widetilde{\Pi} := \mathrm{Log}_\beta(\widetilde{\mathcal{H}})$ respectively. We use $\widetilde{C} = \widetilde{C}_{\alpha_0}$ to denote the complement labeled by α_0.

Theorem 1 *The critical points for $\varphi|_{\partial\widetilde{C}}$ on the boundary of amoeba $\partial\widetilde{C}$ are indexed by simplices $\tau \in \partial\mathcal{T}$. The critical point $\widetilde{\rho}_\tau$ for τ in $\partial\mathcal{T}$ has Morse index* $\dim|\tau|$. *The unstable manifold (downward flowing) $\widetilde{W}_\tau$ of $\widetilde{\rho}_\tau$ contains $\widetilde{\rho}_{\tau'}$ in its closure, if and only if $\overline{\tau} \supset \tau'$.*

The Liouville structure of $\widetilde{\mathcal{H}}$ are induced from $(M_{\mathbb{C}^*}, \omega, \lambda)$ (see Sect. 2.2), where in coordinates

$$\omega = \partial_{ij}\varphi(\rho) d\rho_i \wedge d\theta_j, \quad \lambda = \partial_i\varphi(\rho) d\theta_i.$$

The (candidate for) Liouville skeleton is defined as

$$\mathcal{S}_{\beta,h,\Theta} = \bigcup_{\tau \in \mathcal{T}} (\beta \cdot \widetilde{W}_\tau) \times T_{\tau,\Theta} \subset M_{\mathbb{R}} \times M_T \cong M_{\mathbb{C}^*}, \tag{3.1}$$

[4] We may smooth φ at a small neighborhood around $0 \in M_{\mathbb{R}}$, but this is irrelevant since we will use φ only as $\varphi(\beta u)$ for $\beta \gg 1$ and u in a neighborhood of ∂P.

where $\widetilde{W}_\tau$ is the unstable manifold from $\widetilde{\rho}_\tau$ and $T_{\tau,\Theta}$ is the subtorus of M_T defined by

$$T_{\tau,\Theta} = \{\theta \in M_T : \langle\theta, \alpha\rangle = \Theta(\alpha), \text{for each vertex } \alpha \text{ in } \tau.\} \tag{3.2}$$

Theorem 2 *$\mathcal{S}_{\beta,h,\Theta}$ is the Lagrangian Liouville skeleton for $(\widetilde{\mathcal{H}}, \omega|_{\widetilde{\mathcal{H}}}, \lambda|_{\widetilde{\mathcal{H}}})$*

The Lagrangian skeleton defined here can be related with the RSTZ skeleton via the 'projective' Legendre transformation $\Phi_\varphi^\infty : M_\mathbb{R}^\infty \xrightarrow{\sim} N_\mathbb{R}^\infty$, which is induced by homogeneous Legendre transformation $\Phi_\varphi : M_\mathbb{R} \to N_\mathbb{R}$. Let $q_M : M_\mathbb{R}\backslash\{0\} \to M_\mathbb{R}^\infty$ and $q_N : N_\mathbb{R}\backslash\{0\} \to N_\mathbb{R}^\infty$ be quotient by $\mathbb{R}_+$. Recall the RSTZ-skeleton $\Lambda_{\mathcal{T},\Theta}^\infty$ is defined in the introduction (0.2). Let id denote the identity map on M_T, then we have:

Theorem 3

$$\Phi_\varphi^\infty \times id : M_\mathbb{R}^\infty \times M_T \to N_\mathbb{R}^\infty \times M_T$$

induces a homeomorphism between $\mathcal{S}_{\beta,h,\Theta}$ identified as $(q_M \times id)(\mathcal{S}_{\beta,h,\Theta})$ and $\Lambda_{\mathcal{T},\Theta}^\infty$ identified as $(q_N \times id)(\Lambda_{\mathcal{T},\Theta}^\infty)$.

Recall from introduction that our main theorem is as follows.

Main Theorem *The hypersurface $\mathcal{H}_{\beta,h,\Theta}$ admits a Liouville structure such that its Liouville skeleton is homeomorphic to $\Lambda_{\mathcal{T},\Theta}^\infty$.*

The main theorem then follows from Theorems 2 and 3, and the diffeomorphism of $\mathcal{H}$ with $\widetilde{\mathcal{H}}$ from Proposition 1.8.

4 Gradient flow: Proof of Theorem 1

Notation: we will index the critical points by simplices in $\partial\mathcal{T} = \mathcal{T} \cap \partial Q$, e.g. $\rho_\tau, \widetilde{\rho}_\tau$. For a given simplex $\tau \in \partial\mathcal{T}$, we denote the simplex $\mathrm{conv}(\{0\} \cup \tau)$ by τ_0.

We will sometimes omit β from the subscript to unclutter the notation.

4.1 Convergence of smooth convex domain and critical points

We fix an identification of $V \cong \mathbb{R}^n$ and take Euclidean metric on V and the induced metric on T^*V and S^*V. We identify the sphere compactification boundary $T^\infty V = (T^*V - V)/\mathbb{R}_{>0}$ with the unit cosphere bundle S^*V. If $U \subset V$ is an open subset with smooth boundary, then S_U^*V is the one-sided unit conormal bundle of ∂U with covectors pointing outward. The generalization to open convex set U with piecewise smooth boundary is also straightforward.

Proposition 4.1 *Let $V \cong \mathbb{R}^n$ be a real vector space of dimension n, $P \subset V$ a convex polytope containing the origin, $\varphi : V \to \mathbb{R}$ a potential adapted to P. Let $\{P_j\}$ be a sequence of convex bounded domains with smooth boundaries, such that the exterior* *conormal $L_j := S^*_{P_j} V$ converges to $L := S^*_P V$ in the cosphere bundle $S^* V$ in the Hausdorff metric. Then for all large enough j, there is a one-to-one correspondence between faces F of P and critical points of φ on ∂P_j, denoted as $\widetilde{\rho}_F$, such that*

(1) *$\widetilde{\rho}_F$ has Morse index $n - 1 - \dim F$.*
(2) *As $\beta \to \infty$, $\widetilde{\rho}_F$ tends to the ρ_F, the minimum of φ on the face F of P.*

Proof (1) We express the critical point condition in terms of Legendrian intersection. Define the projection image of $\Gamma_{d\varphi}$ in $T^\infty V$ as

$$\Gamma^\infty_{d\varphi} = (\mathbb{R}_{>0} \cdot \Gamma_{d\varphi})/\mathbb{R}_{>0} \subset T^\infty V. \tag{4.1}$$

Then $\Gamma^\infty_{d\varphi}$ is also the union of unit conormal for level sets of φ:

$$\Gamma^\infty_{d\varphi} = \bigcup_{c \in \mathbb{R}} S^*_{\{\varphi(\rho) \le c\}} V. \tag{4.2}$$

The Legendrian $L = S^*_P V$ is a piecewise smooth C^1 manifold, where the smooth components L_F are labeled by faces F of P. If ρ_F is a critical point of φ on F, then there is a unique unit covector $p_F \in L_F$, such that $x_F = (\rho_F, p_F) \in L \pitchfork \Gamma^\infty_{d\varphi}$, and the intersection is transversal.

(2) Consider the unit speed geodesic flow Φ^t_R on the unit cosphere bundle $S^* V$. Fix any small flow time $1 \gg \epsilon > 0$, since $\Phi^\epsilon_R : S^* V \to S^* V$ is a diffeomorphism, $\Phi^\epsilon_R(L_j)$ still converges to $\Phi^\epsilon_R(L)$ in Hausdorff metric. For any subset $A \subset V$, define

$$A^\epsilon := \{x : \operatorname{dist}(x, A) < \epsilon\}$$

to be the ϵ-fattening of A. If A is a convex set, we have $\Phi^\epsilon_R(S^*_A V) = S^*_{A^\epsilon} V$. Hence ∂P^ϵ is a C^1 hypersurface, and $\partial P^\epsilon_j \to \partial P^\epsilon$ in Hausdorff metric as $j \to \infty$. Define

$$L^t = \Phi^t_R(L), \quad L^t_j = \Phi^t_R(L_j).$$

The geodesic flow applied to $\Gamma^\infty_{d\varphi}$ can be understood as follows

$$\Phi^\epsilon_R(\Gamma^\infty_{d\varphi}) = \bigcup_{c \in \mathbb{R}} \Phi^\epsilon_R(S^*_{\{\varphi(\rho) \le c\}} V) = \bigcup_{c \in \mathbb{R}} S^*_{\{\varphi(\rho) \le c\}^\epsilon} V.$$

Define function $\widetilde{\varphi}^\epsilon$, such that $\{\widetilde{\varphi}^\epsilon(\rho) < c\} = \{\varphi(\rho) \le c\}^\epsilon$, then $\widetilde{\varphi}^\epsilon$ is a levelset convex function. By Lemma 2.7 of [2], there exists a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$, such that $\varphi^\epsilon = f \circ \widetilde{\varphi}^\epsilon$ is a convex function. Thus, we have

$$\Phi^\epsilon_R(\Gamma^\infty_{d\varphi}) = \Gamma^\infty_{d\varphi^\epsilon} \quad \varphi^\epsilon \text{ is convex}.$$

Let $x^\epsilon_F = \Phi^\epsilon_R(x_F)$, $\rho^\epsilon_F = \pi(x^\epsilon_F)$ in the expanded face $F^\epsilon = \pi(\Phi^\epsilon_R(L_F))$. Then x^ϵ_F is still the intersection of $\Gamma^\infty_{d\varphi^\epsilon}$ and $S^*_{P^\epsilon} V$, and ρ^ϵ_F is the unique Morse critical points of φ^ϵ restricted on F^ϵ, and ρ_F^ϵ is in the interior of F^ϵ. One may easily check that the Morse index of ρ_F^ϵ is $n-1-\dim F$.

(3) We now prove that for large enough j, for each F, there is a unique critical points $\rho_{F,j}^\epsilon$ of φ^ϵ on ∂P_j^ϵ approaching ρ_F^ϵ.

Fix a small neighborhood $\mathcal{W}_F \subset \partial P^\epsilon$ near ρ_F^ϵ, and for small enough δ, let $\widetilde{\mathcal{W}}_F \cong \mathcal{W}_F \times (-\delta, \delta)$ be the flow-out of $\mathcal{W}_F$ under the Reeb flow for time in $(-\delta, \delta)$, with projection map $\pi_W : \widetilde{\mathcal{W}}_F \to \mathcal{W}_F$. We claim that for large enough j, $\partial P_j^\epsilon \cap \widetilde{\mathcal{W}}_F$ projects bijectively to $\mathcal{W}_F$, since otherwise this contradicts with P_j^ϵ being convex and the fiber of π_W being straight-line segments Reeb trajectories. Thus, we have a sequence of smooth sections $\iota_j : \mathcal{W}_F \to \widetilde{\mathcal{W}}_F$ for large enough j, such that ι_j converges to the zero section in C^1.

Let $f_j = \iota_j^* \varphi_\epsilon|_{\widetilde{\mathcal{W}}_F} \in C^\infty(\mathcal{W}_F, \mathbb{R})$, and a smooth function $f_\infty = \iota_\infty^* \varphi_\epsilon|_{\widetilde{\mathcal{W}}_F}$, where $\iota_\infty : \mathcal{W}_F \hookrightarrow \widetilde{\mathcal{W}}_F$ is the identity map of zero section. Since $\iota_j \to \iota_\infty$ in C^1, $f_j \to f_\infty$ in C^1. Since f_∞ has a non-degenerate critical point, by stability of critical points under C^1-perturbation, f_j has a unique critical point of the same index as f_∞.

(4) Finally, we show that there are no other critical points. Let U_F be the preimage of $\widetilde{W}_F$ under $S^*V \to V$. Let U be the union of all such $\widetilde{U}_F$. Take $\delta > 0$ small enough such that

$$\operatorname{dist}(\Gamma_{d\varphi^\epsilon}^\infty \backslash U, L^\epsilon) > 3\delta.$$

By the assumption that L_j^ϵ converges to L^ϵ in Hausdorff metric, there exists j_0 large enough, such that

$$\forall j > j_0, \text{ and } x \in L_j^\epsilon, \quad \text{we have } \operatorname{dist}(x, L^\epsilon) < \delta.$$

This shows

$$\operatorname{dist}(\Gamma_{d\varphi^\epsilon}^\infty \backslash U, L_j^\epsilon) \geq \operatorname{dist}(\Gamma_{d\varphi^\epsilon}^\infty \backslash U, L^\epsilon) - \operatorname{dist}(L_j^\epsilon, L^\epsilon) > 2\delta,$$

hence there is no intersection between L_j^ϵ and $\Gamma_{d\varphi^\epsilon}^\infty$ away from U.

(5) Since Φ_R^ϵ is a diffeomorphism, the result about $L_j^\epsilon \cap \Gamma_{d\varphi^\epsilon}^\infty$ implies the same result about $L_j \cap \Gamma_{d\varphi}^\infty$, and we finish the proof of the proposition. □

4.2 Proof of Theorem 1: critical points and unstable manifolds

Proposition 4.2 *For large enough β, there is a one-to-one correspondence between simplices $\tau \in \partial\mathcal{T}$ and critical points of φ on $\partial\widetilde{C}$, denoted as $\widetilde{\rho}_\tau$, such that*

(1) *The Morse index of $\widetilde{\rho}_\tau$ equals* $\dim \tau$.
(2) *As $\beta \to \infty$, $\widetilde{\rho}_\tau$ tends to the ρ_τ, which is the minimum of φ on the face $\tau_0^\vee$ of P.*

Proof First we approximate $\partial\widetilde{C}$ by its convex model $\widehat{C}$ (see Definition 1.10). Then by Proposition 4.1, we have critical points $\{\widehat{\rho}_\tau\}$ on $\widehat{C}$ indexed by $\tau \in \partial\mathcal{T}$. Then, a perturbation argument shows φ has critical points on $\partial\widetilde{C}$ as $\{\widetilde{\rho}_\tau = \widehat{\rho}_\tau\}$. □

Next, we prove that the unstable manifold $\widetilde{W}_\tau$ for critical point $\widetilde{\rho}_\tau$ are cells of a dual polyhedral decomposition of ∂P. This is true not only in the combinatorial sense, but in a more refined geometrical sense.

Proposition 4.3 *For large enough β, and for any $\tau \in \partial\mathcal{T}$, the unstable manifold $\widetilde{W}_\tau$ is a smooth manifold of dimension* $\dim\tau$. *Furthermore, another critical point $\widetilde{\rho}_{\tau'}$ is contained in the boundary of the closure of $\widetilde{W}_\tau$ if and only if $\tau' \subset \tau$.*

Proof The statement of $\dim \widetilde{W}_\tau$ follows from the Morse index of $\widetilde{\rho}_\tau$. For any critical point ρ_τ, take a small enough ball B of radius ϵ around it, then B can be stratified by the limit of gradient flow. For each facet $\sigma^\vee$ of the polytope P adjacent to $\tau^\vee$, there is an open ball U_σ in ∂B whose points flow to critical point ρ_σ. If a face $\tau'^\vee$ adjacent to $\tau^\vee$ can be written as $\tau'^\vee = \sigma_1^\vee \cap \cdots \cap \sigma_k^\vee$ for facets $\sigma_i^\vee$, then points in the relative interior of $\cap_{i=1}^k \overline{U_{\sigma_i}}$ will flow to $\rho_{\tau'}$. □

Now we give an explicit description of the unstable manifold. Let Φ_φ^∞, q_N and q_M be as in the statement of Theorem 3.

Proposition 4.4 *For all large enough β, and $\tau \in \partial\mathcal{T}$, Φ_φ^∞ induces a homeomorphism*

$$\Phi_\varphi^\infty : q_M(\widetilde{W}_\tau) \xrightarrow{\sim} q_N(\tau).$$

Proof Without loss of generality, we set $h(0) = 0$. We will first do the computation on the convex model $\widehat{C}$, then state the necessary modifications for $\widetilde{C}$.

The defining function for $\widehat{C}$ is

$$1 = \sum_{\alpha \in A\backslash\{0\}} e^{\beta l_\alpha(u)} \chi(\beta l_\alpha(u) + \sqrt{\beta}) =: \widehat{F}(u).$$

For any point $u \in \widehat{C}$, we define the simplex

$$\tau(u) = \mathrm{conv}\{\alpha \in A\backslash\{0\} : \chi(\beta l_\alpha(u) + \sqrt{\beta}) > 0\}.$$

Then the gradient of φ on $\widehat{C}$ can be expressed as

$$\nabla(\varphi|_{\widehat{C}}) = \nabla\varphi - c_1 \nabla\widehat{F}$$

where $c_1 = \frac{\langle\nabla\varphi, d\widehat{F}\rangle}{\langle\nabla\widehat{F}, d\widehat{F}\rangle}$. Since by Proposition 2.7, $\nabla\varphi$ is in the outward radial direction, and $\widehat{F}$ is a convex function with bounded sub-level set, hence $\langle\nabla\varphi, d\widehat{F}\rangle > 0$. Combining $\langle\nabla\widehat{F}, d\widehat{F}\rangle > 0$, we have $c_1 > 0$.

For u in the unstable manifold $\widehat{W}_\tau$, we have $\tau(u) \subset \tau$. The defining function $\widehat{F}$ for a neighborhood of u can be written as

$$\widehat{F}_\tau(u) = \sum_{\alpha\in\tau} e^{\beta l_\alpha(u)} \chi(\beta l_\alpha(u) + \sqrt{\beta})$$

thus

$$d\widehat{F}_\tau = \sum_{\alpha\in\tau}(e^x\chi(x+\sqrt{\beta}))'|_{x=\beta l_\alpha(u)}\cdot\alpha \in \operatorname{Int}\operatorname{cone}(\tau).$$

And at the critical point

$$d\varphi(\widehat{\rho}_\tau) = c_1 d\widehat{F}_\tau \in \operatorname{Int}\operatorname{cone}(\tau).$$

If $\gamma : (-\infty,+\infty) \to \widehat{C}$ is an integral curve for $-\nabla(\varphi|_{\widetilde{C}})$ with $\lim_{t\to-\infty}\gamma(t) = \widehat{\rho}_\tau$, then under Legendre transformation we have a curve $\eta(t)$, such that

$$\lim_{t\to-\infty}\eta(t) = d\varphi(\widetilde{\rho}_\tau) \in \operatorname{Int}\operatorname{cone}(\tau)$$

and using Lemma 2.1 and Proposition 2.7

$$\frac{d}{dt}\eta(t) = (\Phi_\varphi)_*(-\nabla(\varphi|_{\widetilde{C}})) = (\Phi_\varphi)_*(-\nabla\varphi + c_1\nabla F) \in \mathbb{R}(p\partial_p) + \operatorname{Int}\operatorname{cone}(\tau),$$

where $p\partial_p$ is the radial vector field on $N_\mathbb{R}$. Thus $\eta(t)$ is within the cone $\operatorname{Int}\operatorname{cone}(\tau)$ for all $t\in\mathbb{R}$. This shows that

$$\Phi_\varphi^\infty(q_M(\widehat{W}_\tau)) \subset q_N(\tau).$$

Using induction on dimension of τ from 0 to $n-1$, we can show the image is onto.

Now consider $\widetilde{C}$. One need to replace $\chi(\beta l_\alpha(u)+\sqrt{\beta})$ by $\chi_{\alpha,0}(u)$ in defining $\widetilde{F}$. And one can still show that $d\widetilde{F}_\tau(u) \in \operatorname{Int}\operatorname{cone}(\tau)$, the rest is the same as $\widehat{C}$. □

5 Liouville flow: Proof of Theorem 2 and 3

First we find all the critical points (manifolds) of the Liouville vector field on $\widetilde{\mathcal{H}}$. We show that they are exactly the preimage of critical points of $\varphi|_{\partial\widetilde{C}}$ under Log_β, which are tori of various dimensions. The more difficult part is to show there are no other critical points.

Then, we study the Liouville flow trajectory from these critical manifolds. There are two key points:

(1) We write the fiberwise Liouville vector field as the ambient Liouville vector field in $M_{\mathbb{C}^*}$ subtract its symplectic orthogonal component, then show that on the 'positive loci' (Definition 5.1), the symplectic orthogonal component is proportional to the Hamiltonian vector field $X_{\operatorname{Im}\widetilde{f}}$.

(2) We show that the unstable manifold correponding to the critical manifold indexed by $\tau\in\partial\mathcal{T}$, is geometrically identified with the simplex τ under the projective Legendre transformation $\Phi_\varphi^\infty : M_\mathbb{R}^\infty \to N_\mathbb{R}^\infty$. This determines the unstable manifolds.

Recall our notations:

- The complex hypersurface $\mathcal{H}$ defined by $f(z) = 0$:

$$f(z) = \sum_{\alpha \in A} f_\alpha(z) = \sum_{\alpha \in A} z^\alpha e^{-\beta h(\alpha) - i\Theta(\alpha)}.$$

- The tropical localized hypersurface $\widetilde{\mathcal{H}}$, defined by $\widetilde{f}(z) = 0$:

$$\widetilde{f}(z) = \sum_{\alpha \in A} \widetilde{f_\alpha}(z) = \sum_{\alpha \in A} f_\alpha(z)\chi_\alpha(u),$$

- The real-valued functions, $F_\alpha(z) = |f_\alpha(z)|$ and

$$\widetilde{F}(z) = -1 + \sum_{0 \neq \alpha \in A} F_\alpha(u)\chi_\alpha(u).$$

The proof of Theorem 2 follows from Propositions 5.7, 5.8 and 5.9. The proof of Theorem 3 follows from Propositions 4.4 and 5.7.

5.1 Liouville vector field

Recall that λ is the Liouville 1-form on $M_{\mathbb{C}^*}$, and $\lambda_{\widetilde{\mathcal{H}}}$ is the restriction of λ on the (tropicalized) hypersurface $\widetilde{\mathcal{H}}$.

Take any point $z \in \widetilde{\mathcal{H}}$, we have

$$X_\lambda(z) = X_\lambda^{\|}(z) + X_\lambda^{\perp}(z).$$

where $X_\lambda^{\perp}(z)$ is symplectically orthogonal to $T_z\widetilde{\mathcal{H}}$. We note that $X_\lambda^{\|}(z) = X_{\lambda_{\widetilde{\mathcal{H}}}}(z)$, since for any $v \in T_z\widetilde{\mathcal{H}}$,

$$\omega_{\widetilde{\mathcal{H}}}(X_\lambda^{\|}(z), v) = \omega(X_\lambda(z) - X_\lambda^{\perp}(z), v) = \omega(X_\lambda(z) - X_\lambda^{\perp}(z), v) = \lambda(v) = \lambda_{\mathcal{H}}(v).$$

And $X_\lambda^{\perp}(z)$ is the symplectic horizontal lift of $\widetilde{f}_*(X_\lambda(z)) \in T_0\mathbb{C}$.

Definition 5.1 The *positive loci* $\widetilde{\mathcal{H}}^+$ is the subset of $\widetilde{\mathcal{H}}$ where $\widetilde{f_0} = -1$ and $\widetilde{f_\alpha} \geq 0$ for all $\alpha \neq 0$.

Remark 5.2 An equivalent definition is that $\widetilde{\mathcal{H}}^+ = \mathrm{Log}_\beta^{-1}(\partial\widetilde{C})$.

Proposition 5.3 *For all $z \in \widetilde{\mathcal{H}}^+$, we have $X_{Im\widetilde{f}}(z)$ positively proportional to $X_\lambda^{\perp}(z)$.*

Proof Since both vectors are symplectic orthogonal to $\widetilde{\mathcal{H}}$, we only need to check their image under $\widetilde{f}_*$ are positively proportional to each other.

First we study $X_{\mathrm{Im}\widetilde{f}}(z)$. On $\widetilde{\mathcal{H}}^+$, we have $\widetilde{f} = \widetilde{F}$. We also have

$$\begin{aligned} d\widetilde{f} &= \sum_{\alpha\in A} \widetilde{F}_\alpha(u)\langle\alpha, d(\rho + i\theta)\rangle + \sum_{\alpha\in A} f_\alpha(z) d\chi_\alpha(u) \\ &= d\widetilde{F}(u) + i\sum_{\alpha\in A} \widetilde{F}_\alpha(z)\langle\alpha, d\theta\rangle \end{aligned}$$

Hence

$$d(\mathrm{Im} f) = \mathrm{Im} df = \sum_{\alpha\in A} \widetilde{F}_\alpha(u)\langle\alpha, d\theta\rangle$$

Thus

$$X_{\mathrm{Im} f} = \sum_{\alpha\in\partial A} \widetilde{F}_\alpha(u)\sum_{i,j} \alpha_i g^{ij}(\rho)\partial_{\rho_j} \tag{5.1}$$

compare with

$$\nabla(\widetilde{F}) = g^{-1}(d\widetilde{F}) = g^{-1}\left(\sum_{\alpha\in\partial A} F_\alpha(u)\chi_\alpha(u)\langle\alpha, d\rho\rangle + F_\alpha d\chi_\alpha\right) = X_{\mathrm{Im} f} + O(e^{-\sqrt{\beta}}).$$

We thus have

$$\langle d\widetilde{f}, X_{\mathrm{Im}\widetilde{f}}\rangle = \langle d\widetilde{F}, X_{\mathrm{Im}\widetilde{f}}\rangle = \|\nabla\widetilde{F}\|^2 + O(e^{-\sqrt{\beta}}) > 0.$$

Next, we study $X_\lambda^\perp(z)$. We have

$$\langle d\widetilde{f}, X_\lambda^\perp(z)\rangle = \langle d\widetilde{f}, X_\lambda(z)\rangle = \langle d\widetilde{f}, \nabla\varphi\rangle = \langle d\widetilde{F}, \nabla\varphi\rangle$$

Since $\nabla\varphi$ is positively proportional to the radial vector field $u\partial_u$ by Proposition 2.7, and $\langle d\widetilde{F}, u\partial_u\rangle > 0$. We have also $\langle d\widetilde{f}, X_\lambda^\perp(z)\rangle > 0$.

Since $\widetilde{f}_*(X_\lambda^\perp(z))$ and $\widetilde{f}_*(X_{\mathrm{Im} f})$ are both in the positive direction of $T_0\mathbb{C}$, $X_\lambda^\perp(z)$ is positively proportional to $X_{\mathrm{Im}\widetilde{f}}(z)$. □

5.2 Critical manifolds

Recall from the previous section, that on the boundary of the amoeba $\partial\widetilde{C}$, the critical points of φ are indexed by $\tau \in \partial\mathcal{T}$ as $\widetilde{\rho}_\tau$.

Proposition 5.4 *The preimages* $\mathrm{Crit}_\tau := \mathrm{Log}_\beta|_{\widetilde{\mathcal{H}}}^{-1}(\widetilde{\rho}_\tau)$ *are critical manifolds.*

Proof Since the critical points $\widetilde{\rho}_\tau$ are in the 'good' region $\widetilde{\mathcal{H}}^{good} \subset \widetilde{\mathcal{H}}$, where the monomial cut-off functions χ_α are either zero or one, hence the hypersurface $\widetilde{\mathcal{H}}^{good}$ is holomorphic. Thus, zero of $d\varphi|_{\widetilde{\mathcal{H}}}$ is also zero of $d^c\varphi|_{\widetilde{\mathcal{H}}}$. □

Proposition 5.5 *For each $\tau \in \partial\mathcal{T}$, $\mathrm{Log}_\beta|_{\widetilde{\mathcal{H}}}^{-1}(\widetilde{\rho}_\tau) = \{\beta\widetilde{\rho}_\tau\} \times T_{\tau,\Theta}$, where $T_{\tau,\Theta}$ is defined in* (3.2).

Proof Since $\widetilde{\rho}_\tau$ is on the boundary $\partial\widetilde{C}$, we have $1 = \sum_{\alpha\in\tau} F_\alpha(z)$. Comparing with the defining equation of $\widetilde{\mathcal{H}}$ in a neighborhood of $\mathrm{Log}_\beta|_{\widetilde{\mathcal{H}}}^{-1}(\widetilde{\rho}_\tau)$, we have $1 = \sum_{\alpha\in\tau} f_\alpha(z)$. Hence $0 = \arg(f_\alpha(z)) = \langle\alpha,\theta\rangle - \Theta(\alpha)$ for each vertex α in τ. Thus the fiber is the torus $T_{\tau,\Theta}$. □

5.3 Unstable manifolds

Proposition 5.6 *The Liouville vector field $X_{\lambda_{\widetilde{\mathcal{H}}}}$ on the positive loci $\widetilde{\mathcal{H}}^+$ does not change the θ coordinate. In particular, the positive loci $\widetilde{\mathcal{H}}^+$ is preserved under the Liouville flow.*

Proof Since $X_{\lambda_{\mathcal{H}}} = X_\lambda^{\|} = X_\lambda - X_\lambda^{\perp}$, suffice to check that X_λ and $X_\lambda^{\perp}$ does not change θ coordinates. We have $X_\lambda \propto \rho\partial_\rho$, and $X_\lambda^{\perp} \propto X_{\mathrm{Im}\widetilde{f}}$. From Eq. (5.1), we see $X_{\mathrm{Im}\widetilde{f}}$ has no θ-component. Hence $\langle X_{\lambda_{\mathcal{H}}}, d\theta\rangle = 0$. □

Proposition 5.7 *For any $\tau \in \partial\mathcal{T}$, the unstable manifold for* Crit_τ *is $\widetilde{W}_\tau \times T_{\tau,\Theta}$.*

Proof From Proposition 5.6, we see the flowout of Crit_τ by the Liouville flow does not affect the M_T component. Thus Liouville flow $X_{\lambda,\widetilde{\mathcal{H}}}$ on $\widetilde{\mathcal{H}}$ induces a flow on $\widetilde{\mathcal{H}}^+$, and it descends to $\partial\widetilde{C}$, for which we denote as $X_{\lambda,\partial\widetilde{C}}$.

On $\partial\widetilde{C}^{good}$, $X_{\lambda,\partial\widetilde{C}}$ agrees with $\nabla(\varphi|_{\partial\widetilde{C}})$. And they have the same critical points set. On $\partial\widetilde{C}^{bad}$, we have

$$\|X_{\lambda,\partial\widetilde{C}} - \nabla(\varphi|_{\partial\widetilde{C}})\| = O(e^{-\sqrt{\beta}}).$$

Despite individual flowlines for the two vector fields with the same starting point in the good region may be split after flow through a bad region, we claim that for each critical point $\widetilde{\rho}_\tau$, the unstable manifolds $\widetilde{W}_\tau^{X_\lambda}$ and $\widetilde{W}_\tau^{\nabla\varphi}$ for the two flows are the same.

Let $\tau \in \partial\mathcal{T}$ have vertices $\{\alpha_1, \ldots, \alpha_k\}$. Then

$$\nabla(\varphi|_{\widetilde{C}}) = \nabla\varphi - c_1\nabla\widetilde{F} \in \mathbb{R}\cdot\rho\partial\rho + (\Phi_\varphi)_*^{-1}(\mathrm{Int\,cone}\,\tau)$$

and

$$X_{\lambda,\partial\widetilde{C}} = X_\lambda - X_\lambda^{\perp} = X_\lambda - c(u)X_{\mathrm{Im}\widetilde{f}} \in \mathbb{R}\cdot\rho\partial\rho + (\Phi_\varphi)_*^{-1}(\mathrm{Int\,cone}\,\tau)$$

where we used $X_\lambda^{\perp}$ positively proportional to $X_{\mathrm{Im}\widetilde{f}}$, and $X_{\mathrm{Im}\widetilde{f}}$ is given by Eq. (5.1). By similar argument in Proposition 4.4 that $W_\tau^{\nabla\varphi}$ is dual to τ via Φ_φ^∞, we have $W_\tau^{X_\lambda}$ is dual to τ via Φ_φ^∞. Thus $\widetilde{W}_\tau^{\nabla\varphi}$ and $\widetilde{W}_\tau^{\nabla\varphi}$ has to be the same. We drop the superscripts and denote both as $\widetilde{W}_\tau$. □

Proposition 5.8 *For each $\tau \in \partial\mathcal{T}$, the unstable manifold $\widetilde{W}_\tau \times T_{\tau,\Theta}$ is a Lagrangian in $\widetilde{\mathcal{H}}$.*

Proof One can use the property of the Liouville flow to show the unstable manifold is isotropic, and then counting dimension

$$\dim_{\mathbb{R}} \widetilde{W}_\tau \times T_{\tau,\Theta} = (\dim \tau) + n - (\dim \tau + 1) = n - 1 = \frac{1}{2} \dim_{\mathbb{R}} \widetilde{\mathcal{H}}.$$

We give an alternative proof. By Proposition 4.4, we have

$$\Phi_\varphi(\text{cone } \widetilde{W}_\tau) \times T_{\tau,\Theta} = \text{cone } \tau \times T_{\tau,\Theta}.$$

However, cone $\tau \times T_{\tau,\Theta}$ is part of the conormal Lagrangian $T^*_{T_{\tau,\Theta}} M_T$ for the submanifold $T_{\tau,\Theta}$ in M_T. Since $\Phi_\varphi \times id$ is a symplectomorphism between $M_{\mathbb{C}^*}$ and T^*M_T, we get cone $\widetilde{W}_\tau \times T_{\tau,\Theta}$ is a conical Lagrangian in $M^*_{\mathbb{C}^*}$. Finally, a Lagrangian restricts to a symplectic submanifold is isotropic. Thus by dimension counting,

$$\widetilde{W}_\tau \times T_{\tau,\Theta} = \left(\text{cone } \widetilde{W}_\tau \times T_{\tau,\Theta}\right) \bigcap \widetilde{\mathcal{H}}$$

is a Lagrangian in $\widetilde{\mathcal{H}}$. □

5.4 No other critical points

Proposition 5.9 *There are no other zeros of the Liouville vector field away from* $\{\text{Crit}_\tau\}$.

Proof It suffices to prove that there are no zeros of the Liouville vector field outside of the positive loci $\widetilde{\mathcal{H}}^+$. Here we only give the sketch the proof. We look at the good region first. Then $d^c\varphi|_{\mathcal{H}} = 0$ is equivalent to $d\varphi|_{\mathcal{H}} = 0$. Hence, we only need to check that there are no critical points for φ.

Suppose there is a critical point of φ at $z \in \widetilde{\mathcal{H}}^{good}$, the terms labeled by $\alpha_1, \ldots, \alpha_k$ are non-zero, i.e., near z, $\widetilde{\mathcal{H}}$ is defined by

$$\sum_{i=1}^{k} f_{\alpha_i}(z) = 0.$$

Let $\tau \in \mathcal{T}$ be the simplex with vertices $\{\alpha_1, \ldots, \alpha_k\}$. Let $\tau^\vee$ be the cell in the tropical amoeba Π, and $U_\tau \subset \widetilde{\mathcal{H}}^{good}$ where the defining equation is as above. We split into two cases below. Recall P is the polytope corresponding to vertex $0 \in \mathcal{T}$. Let g_0 denote the Euclidean metric on $M_{\mathbb{R}}$ after identification $M_{\mathbb{R}} \cong \mathbb{R}^n$.

(1) The case $0 \notin \tau$**.** Then $\tau^\vee$ is a non-compact cell in Π, and intersects the amoeba polytope P at face $F_\tau = P \cap \tau^\vee$.

Let $u = \text{Log}_\beta(z)$, and let u' denote the orthogonal projection to the cell $\tau_0^\vee$ with respect to g_0. Then $dist_{g_0}(u, u') = O(1/\sqrt{\beta})$. Let u'' denote the minimum of φ on F_τ. We claim that

$$\varphi(u'') < \varphi(u'),$$

since the family of increasing level sets of φ meet the convex cell $\tau^\vee$ first at u''.

Let $v = u'' - u' \in M_\mathbb{R}$. If we view v as a tangent vector at u'', then $\langle d\varphi(u'), v\rangle < 0$. Since u and u' are $O(1/\sqrt{\beta})$ close, we also have $\langle d\varphi(u'), v\rangle < 0$. Finally, one can check v can be lifted as a tangent vector to $T_z\widetilde{\mathcal{H}}$, hence $d\varphi \neq 0$ at z.

(2) The case $0 \in \tau$**.** Without loss of generality, we may assume $\Theta(0) = \pi$, $h(0) = 0$, and $\alpha_k = 0$. Thus, the defining equation of $\widetilde{\mathcal{H}}$ near z can be written as

$$1 = \sum_{i=1}^{k-1} f_{\alpha_i} = \sum_{i=1}^{k-1} e^{-i\Theta(\alpha_i)-\beta h(\alpha_i)} e^{\beta\langle\alpha_i,u\rangle+i\langle\alpha_i,\theta\rangle} =: F(u,\theta)$$

Suppose z is a critical point of $\varphi|_{\{F=1\}}$, then there exists c_1, c_2, such that

$$d\varphi(\rho) = c_1 d\mathrm{Re}F(\rho,\theta) + c_2 d\mathrm{Im}F(\rho,\theta).$$

However, since $d\varphi(\rho)$ has no $d\theta$ component, hence the coefficients of $d\theta$ on the RHS need to cancel out. Since all the α_is are linearly independent, we see this is possible only if all $\arg(f_{\alpha_i})$ are equal or differ by π. Since $\sum_i f_{\alpha_i} = 1$, we get all $f_{\alpha_i} \in \mathbb{R}$, and at least one is positive.

If all of $f_{\alpha_i}(z)$ are positive, then there is nothing to show, since our goal is to show that all the critical points lie on the positive loci.

If not all of $f_{\alpha_i}(z)$ are positive, say for $i = 1, \dots, m$, $f_{\alpha_i}(z) < 0$, then u lies on the real hypersurface

$$1 = -e^{\beta l_{\alpha_1}(u)} - \cdots - e^{\beta l_{\alpha_m}(u)} + \cdots + e^{\beta l_{\alpha_{k+1}}(u)} =: H(u).$$

near the face $\tau^\vee$ on P. If we further require $d\varphi$ to be in the $\mathbb{R}$-span of $\alpha_1, \dots, \alpha_{k-1}$, then u has to be near the critical point of φ on face $\tau^\vee$. One can show that $d\varphi$ has to be in the $\mathbb{R}_+$-span of $\alpha_1, \dots, \alpha_{k-1}$. Hence, there does not exist $c \in \mathbb{R}$, such that $d\varphi(u) = c\,dH(u)$.

This concludes the discussion for z in the good region. If z is in the bad region, where at least one $0 < \chi_\alpha(z) < 1$, we will approximate the bad region using good region in the following way. Define a different set of cut-off functions, by changing the cut-off threshold from $-\sqrt{\beta}$ to $-2\sqrt{\beta}$, ie. redefine

$$\chi_{\alpha,\alpha',\beta}(u) = \chi(\beta(l_\alpha(u) - l_{\alpha'}(u)) + 10\sqrt{\beta})$$

in Definition 1.4. Denote the new tropical localized hypersurface $\widetilde{\mathcal{H}}_{10}$. We claim the Hausdorff distance between $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{H}}_{10}$ in $M_{\mathbb{C}^*}$ is $O(e^{-c\sqrt{\beta}})$ for some $c > 0$. Furthermore, their unit conormal bundles $S^*_{\widetilde{\mathcal{H}}}M_{\mathbb{C}^*}$ and $S^*_{\widetilde{\mathcal{H}}_{10}}M_{\mathbb{C}^*}$ should have distance $O(e^{-c\sqrt{\beta}})$ as well. A zero of $d^c(\varphi|_{\widetilde{\mathcal{H}}})$ corresponds to an intersection of $\Gamma^\infty_{d^c\varphi} \subset S^*M_{\mathbb{C}^*}$ with $S^*_{\widetilde{\mathcal{H}}}M_{\mathbb{C}^*}$, where

$$\Gamma^\infty_{d^c\varphi} = (\Gamma_{d^c\varphi} \cap \dot{T}^*(M_{\mathbb{C}^*}))/\mathbb{R}_+ \subset T^\infty(M_{\mathbb{C}^*}) \cong S^*(M_{\mathbb{C}^*}).$$

[cf Definition (4.1) and (4.2)]. Thus the bad region of $\widetilde{\mathcal{H}}$ can be approximately by part of good regions in $\widetilde{\mathcal{H}}_{10}$, where we know there does not exist critical points of φ away from the positive loci, hence there are no critical points in the bad region of $\widetilde{\mathcal{H}}$ away from the positive loci. □

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