Pointwise Weyl Law for Partial Bergman Kernels

Steve Zelditch and Peng Zhou

Abstract This article is a continuation of a series by the authors on partial Bergman kernels and their asymptotic expasions. We prove a 2-term pointwise Weyl law for semi-classical spectral projections onto sums of eigenspaces of spectral width $\hbar = k^{-1}$ of Toeplitz quantizations $\hat{H}_k$ of Hamiltonians on powers L^k of a positive Hermitian holomorphic line bundle $L \to M$ over a Kähler manifold. The first result is a complete asymptotic expansion for smoothed spectral projections in terms of periodic orbit data. When the orbit is 'strongly hyperbolic' the leading coefficient defines a uniformly continuous measure on $\mathbb{R}$ and a semi-classical Tauberian theorem implies the 2-term expansion. As in previous works in the series, we use scaling asymptotics of the Boutet-de-Monvel–Sjostrand parametrix and Taylor expansions to reduce the proof to the Bargmann–Fock case.

This article is part of a series [18, 19] devoted to partial Bergman kernels on polarized (mainly compact) Kähler manifolds $(L, h) \to (M^m, \omega, J)$, i.e. Kähler manifolds of (complex) dimension m equipped with a Hermitian holomorphic line bundle whose curvature form is $\omega_h = \omega$. Partial Bergman kernels

$$\Pi_{k,<E} : H^0(M, L^k) \to \mathcal{H}_{k,<E} \tag{1}$$

are orthogonal projections onto proper subspaces $\mathcal{H}_{k,<E} \subset H^0(M, L^k)$ of the space of holomorphic sections of L^k. Let $H \in C^\infty(M, \mathbb{R})$ denote a classical Hamiltonian, let $\xi = \xi_H$ denote the Hamilton vector field of H, let ∇ be the Chern connection. The quantization of H is the Toeplitz Hamiltonian

$$\hat{H}_k := \Pi_{h^k}\left(\frac{i}{k}\nabla_\xi + H\right)\Pi_{h^k} : H^0(M, L^k) \to H^0(M, L^k). \tag{2}$$

Research partially supported by NSF grant and DMS-1541126 and by the Stefan Bergman trust.

S. Zelditch (✉) · P. Zhou
Department of Mathematics, Northwestern University, Evanston, IL 60208, USA
e-mail: zelditch@math.northwestern.edu

M. Hitrik et al. (eds.), *Algebraic and Analytic Microlocal Analysis*,
Springer Proceedings in Mathematics & Statistics 269,
https://doi.org/10.1007/978-3-030-01588-6_13

Here, $\Pi_{h^k} : L^2(M, L^k) \to H^0(M, L^k)$ is the orthogonal (Szegö or Bergman) projection. Let $\{\mu_{k,j}\}_{j=1}^{d_k}$ denote the eigenvalues of $\hat{H}_k$ on the d_k-dimensional space $H^0(M, L^k)$ and denote the eigenspaces by

$$V_k(\mu_{k,j}) := \{s \in H^0(M, L^k) : \hat{H}_k s = \mu_{k,j} s\}.$$

Also, denote the eigenspace projections by

$$\Pi_{k,j} := \Pi_{\mu_{k,j}} : H^0(M, L^k) \to V_k(\mu_{k,j}).$$

Then the partial Bergman kernels (1) are the projections onto the spectral subspaces

$$\mathcal{H}_{k,<E} := \{\hat{H}_k < E\} := \{s \in H^0(M, L^k) : \langle \hat{H}_k s, s\rangle < E\langle s, s\rangle\}$$

of (2).

In this article, we study the pointwise semi-classical Weyl asymptotics of $\Pi_{k,<E}(z)$ (1) in the conventional semi-classical scaling by $h = \frac{1}{k}$. The main results give asymptotics for the scaled pointwise Weyl sums,

$$\Pi_{k,f}^E(z) = \sum_j f(k(\mu_{k,j} - E))\Pi_{k,j}(z, z)$$

for various types of test functions f. Equivalently, we consider a sequence of measures on $\mathbb{R}$,

$$d\mu_k^{z,1,E}(\lambda) = \sum_j \Pi_{k,j}(z)\delta_{k(\mu_{k,j-E})}(\lambda). \tag{3}$$

then $\Pi_{k,f}^E(z) = \int_{\mathbb{R}} f(\lambda)d\mu_k^{z,1,E}(\lambda)$. When $f \in \mathcal{S}(\mathbb{R})$ with $\hat{f} \in C_c^\infty(\mathbb{R})$, Theorem 2.2 gives a complete asymptotic expansion. When $f = \mathbf{1}_{[a,b]}$ (the indicator function) one has sharp Weyl sums, and Theorem 1.7 gives a pointwise Weyl formula with 2 term asymptotics.

The $\frac{1}{k}$ scaling originates in the Gutzwiller trace formula and has been studied in numerous articles in diverse settings. Two-term pointwise Weyl laws is a standard topic in spectral asymptotics. The pointwise asymptotics in the Kähler setting are quite analogous to Safarov's asymptotic results for spectral projections of the Laplacian of a compact Riemannian manifold [14, 15] and we use Safarov's notations to emphasize the similarity. For general Kähler manifolds, integrated Weyl laws and dual Gutzwiller trace expansions were studied in [17] using the Toeplitz calculus of [3]. Pointwise Weyl laws of the type studied in this article are given in Borthwick-Paul-Uribe [2], based on the Boutet-de-Monvel–Guillemin Hermite Toeplitz calculus [3].

The main purpose of this paper is to prove pointwise Weyl asymptotics using the techniques developed in [18, 19]. Existence of an asymptotic expansion for smoothed Weyl sums is a straightforward consequence of a parametrix construction and of the method of stationary phase, replacing the elaborate symplectic spinor symbol calculus of [3]. However, the coefficients are complicated to compute. In the Toeplitz theory of [2, 3] they are calculated using the symplectic spinor symbol calculus of Toeplitz operators, while we use scaling asymptotics of the quantized flow in the sense of [13, 16, 19]. It is shown that the leading coefficients depend only on the quadratic part of the Taylor expansions. Hence, the coefficients are the same as in the linear model of [5] once the flow is linearized at a period. Our approach gives a somewhat simpler formula for the leading term than in [2] and it is not completely obvious that the formulae agree; in Sect. 8 we show that the formulae do agree with those of [2]. Related calculations using the scaling approach of this article are also given in articles of Paoletti [10, 11].

In the previous articles, we studied the scaling asymptotics of $\Pi_{k,<E}(z) := \Pi_{k,<E}(z,z)$ in a $\frac{1}{\sqrt{k}}$-tube around the interface $\partial\mathcal{A}$ between the allowed and forbidden regions,

$$\mathcal{A} := \{z : H(z) < E\}, \quad \mathcal{F} = \{z : H(z) > E\}.$$

This $\frac{1}{\sqrt{k}}$ scaling was the new feature of the Weyl asymptotics of [19] and is reminiscent of the scaling of the central limit theorem. The $\frac{1}{k}$-scaling was also studied in [19], but it was sufficient for the purposes of that article to obtain the crude asymptotics corresponding to the singularity of the Fourier transform $\widehat{d\mu_k^{z,1,E}}(t)$ at $t = 0$. Technically speaking, the main difference with respect to [19] is that the asymptotics of the $\frac{1}{\sqrt{k}}$ scaling only involve 'Heisenberg translations' while those of $d\mu_k^{z,1,E}$ involve the metaplectic representation. Although the notation and approach of this article have considerable overlap with [19] we give a rather detailed exposition for the sake of completeness.

1 Statement of Results

To state the results, we need some further notation. Given a Hermitian metric h on L, we denote by $X_h = \partial D_h^* \subset L^*$ the unit S^1 bundle $\pi : X_h \to M$ over M defined as the boundary of the unit co-disc bundle in the dual line bundle L^* to L. As reviewed in Sect. 3.5, X_h is a strictly pseudo-convex CR-manifold, and we denote the CR sub-bundle by $HX \subset TX_h$. As reviewed in Sect. 3.8, the Hamilton flow $g^t : M \to M$ lifts to a contact flow $\hat{g}^t : X_h \to X_h$ (Lemma 3.5) with respect to the contact structure α associated to the Kähler potential of ω. Then $HX = \ker \alpha$ and therefore $D\hat{g}^t : HX \to HX$. Moreover, HX inherits a complex structure J from that of M under the identification $\pi_* : H_x X \to T_{\pi(x)}M$, for all $x \in X$. Its complexification has a splitting $H_x X_{\mathbb{C}} = H_x X \otimes \mathbb{C} = H_x^{1,0} X \oplus H_x^{0,1} X$ into subspaces of types $(1,0)$ resp. $(0,1)$. In the generic case where $\hat{g}^t$ is non-holomorphic, it does not preserve this splitting.

At each point $x \in X$, the complexified CR subspace $HX_{\mathbb{C}}$ equipped with J_x together with the Hermitian metric h_x determines an *osculating Bargmann–Fock* *space* $\mathcal{H}_{J_x}$ (see Sects. 3.5 and 4.6 for background). Thus, $\mathcal{H}_{J_x}$ is the space of entire holomorphic functions on $H^{1,0}_x X$ which are square integrable with respect to the *ground state* Ω_{J_x} (defined in (19)). Symplectic transformations $T : H_x X \to H_x X$ resp. $T : T_z M \to T_z M$ may be quantized by the metaplectic representation as complex linear symplectic maps (see (29) and Sect. 4.4) on the osculating Bargmann–Fock space,

$$W_{J_x}(T) : \mathcal{H}_{J_x} \to \mathcal{H}_{J_x}. \tag{4}$$

The asymptotics of $\mu_k^{z,1,E}(f)$ depend on whether or not $z \in M$ is a periodic point for g^t.

Definition 1.1 Define periodic points of g^t, as follows:

$$\mathcal{P}_E := \{z \in H^{-1}(E) : \exists T > 0 : g^T z = z\}.$$

For $z \in \mathcal{P}_E$, let T_z denote the minimal period $T > 0$ of z.

It may occur that $z \in \mathcal{P}_E$ but the orbit $g_h^t(x)$ with $\pi(x) = z$ is not periodic, where g_h^t is the flow generated by the horizontal lift ξ_H^h of the Hamiltonian vector field ξ_H. This is due to holonomy effects: parallel translation of sections of L^k around the closed curve $t \mapsto g^t(z)$ may have non-trivial holonomy. We denote the holonomy by

$$e^{in\theta_z^h} := \text{the unique element } e^{i\theta} \in S^1 : g_h^{nT_z} x = r_\theta x.$$

Let $z \in \mathcal{P}_E$, $T = nT_z$ be a period for $n \in \mathbb{Z}$. Then Dg_z^T induces linear symplectic map

$$S := Dg_z^T : T_z M \to T_z M, \tag{5}$$

When working in the Kähler context it is better to conjugate to the complexifications,

$$T_z M \otimes \mathbb{C} = T_z^{1,0} M \oplus T_z^{0,1} M.$$

We denote the projection to the 'holomorphic component' by

$$\pi^{1,0} : T_z M \otimes \mathbb{C} \to T^{1,0} M.$$

The spaces $T^{1,0}M$, $T^{0,1}M$ are paired complex Lagrangian subspaces.

Relative to a symplectic basis $\{e_j, Je_k\}$ of $T_z M$ in which J assumes the standard form J_0, the matrix of Dg^{nT_z} has the form,

$$Dg_z^{nT_z} := S^n := \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \in Sp(m, \mathbb{R}). \tag{6}$$

If we conjugate to the complexification $T_z M \otimes \mathbb{C}$ by the natural map $\mathcal{W}$ defined in (27), then (6) conjugates to

$$\begin{pmatrix} P_n & Q_n \\ \bar{Q}_n & \bar{P}_n \end{pmatrix} \in Sp_c(m).$$

The holomorphic block

$$P_n = (A_n + D_n + i(-B_n + C_n)) = \pi^{1,0}\, \mathcal{W} S^n \mathcal{W}^{-1}\, \pi^{1,0} : T_z^{1,0}M \to T_z^{1,0}M \quad (7)$$

plays a particularly important role.

The symplectic map (5) is quantized by the metaplectic representation W_{J_z} (4) (see Sect. 4.4) on the osculating Bargmann–Fock space $\mathcal{H}_{J_z}$ of square integrable holomorphic functions on $T_z^{1,0}M$, that is, the metaplectic representation defines a unitary operator

$$W_{J_z}\, (Dg_z^{nT_z}) : \mathcal{H}_{J_z}(T_z^{1,0}M) \to \mathcal{H}_{J_z}(T_z^{1,0}M). \quad (8)$$

The two-term Weyl law is stated in terms of certain data associated to Dg^{nT_z} and $W_{J_z}(Dg^{nT_z})$ (8). First, we let $\mathcal{W}\xi_H$ be the image of the Hamilton vector field ξ_H in $T_zM \otimes \mathbb{C}$. Let $\alpha = \pi^{1,0}\mathcal{W}\xi_H$, let $\bar{\alpha} \in \pi^{0,1}\mathcal{W}\xi_H$, and let P_n be as in (7). Set,

$$\mathcal{G}_n(z) := (\det P_n)^{-\frac{1}{2}} \cdot (\bar{\alpha} \cdot P_n^{-1}\alpha)^{-\frac{1}{2}}. \quad (9)$$

The factor $(\det P)^{-\frac{1}{2}}$ has an interpretation,

$$(\det P_n)^{-1/2} = \langle W_{J_x}\, (Dg_x^{nT(x)})\, \Omega_{J_z}, \Omega_{J_z} \rangle \quad (10)$$

as the matrix element of (8) relative to the ground state Ω_{J_z} in $\mathcal{H}_{J_z}$. This relation is essentially proved by Bargmann and by Daubechies [5]. It can be proved by comparing the Bargmann–Fock metaplectic representation of Sect. 4.4 with Daubechies' Toeplitz construction of metaplectic representation in Sect. 4.5. Daubechies did not explicitly use the conjugation $\mathcal{W}$ to the complexification, and therefore did not record the identity (10).

Also let $e^{in\theta_x^h}$ denote the holonomy of the horizontal lift of the orbit $t \to g^t(z)$ at $t = nT_z$. We define the function $Q_{z,k}^E(s)$ by:

Definition 1.2

$$Q_{z,k}^E(s) = \begin{cases} \mathcal{G}_0(z) & z \notin \mathcal{P}_E \\ \sum_{n\in\mathbb{Z}} (2\pi)^{-1} e^{-inT_z s} e^{-ink\theta_z^h}\, \mathcal{G}_n(z) & z \in \mathcal{P}_E. \end{cases} \quad (11)$$

Definition 1.3 For $z \in \mathcal{P}_E$, define the distributions $d\nu_k^z$ on $f \in \mathcal{S}(\mathbb{R})$ by

$$\int_{\mathbb{R}} f(\lambda) d\nu_k^z(\lambda) = \sum_{n\in\mathbb{Z}} \hat{f}(nT_z)\, \mathcal{G}_n(z) e^{-ink\theta_z^h} = \int_{\mathbb{R}} f(s) Q_{z,k}^E(s) ds$$

The nature of $Q_{z,k}(s)$ and ν_k^z depends on the type of periodic orbit of $z \in \mathcal{P}_E$. In this article we confine ourselves to the case where the orbit of z is 'real positive definite symmetric' in the following sense:

Definition 1.4 Let $z \in \mathcal{P}_E$, with $T_z = T$, and let $(T_z M, J_z, \omega_z)$ be the tangent space equipped with its complex structure and symplectic structure. Let $\{e_j, f_k\}_{j,k=1}^m$ be a symplectic basis of $T_z M$ in which $J = J_0$ and $\omega = \omega_0$ take the standard forms. We say that DG_z^T is *positive definite symmetric symplectic* if its matrix $S \in Sp(m, \mathbb{R})$ in the basis $\{e_j, f_k\}_{j,k=1}^m$ is a symmetric positive definite symplectic matrix.

Positive definite symplectic matrices are discussed in Sects. 3.1 and 3.2 and in Sect. 6.1. They are diagonalizable by orthogonal matrices in $O(2n)$ and by unitary matrices in $U(n)$. In invariant terms, $O(2n)$ is the orthogonal group of $(T_z M, g_{J_z})$ where $g_{J_z}(X, Y) = \omega_z(X, J_z Y)$. Unitary matrices commute with J_z. The eigenvalues of DG_z^T are real and to come in inverse pairs. The eigenvalue 1 corresponds to the Hamilton vector field ξ_H of H and there is a second eigenvector of eigenvalue 1 coming from the fact that periodic orbits come in 1-parameter families (symplectic cylinders) as the energy level E is varied (see [1]). The eigenvalues in the symplectic orthogonal complement of the eigenspace $V(1)$ of eigenvalue 1 come in unequal real inverse pairs λ, λ^{-1}. For expository simplicity, we omit the case where eigenvalues are complex of modulus $\neq 1$ and arise in 4-tuples $\lambda, \lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1}$ (sometimes called loxodromic). We do discuss the elliptic case where $S \in U(n)$, and thus all of the eigenvalues have modulus 1 and come in complex conjugate pairs.

We refer to [6] for background on positive definite symmetric symplectic matrices and to [8] for types of periodic orbits of Hamiltonian flows.

Definition 1.5 We say that z satisfies the *strong hyperbolicity hypothesis* if $Dg_z^T : (T_z M, J_z) \to (T_z M, J_z)$ is a positive symplectic map, with a 2-dimensional symplectic eigenspace $V(1)$ for the eigenvalue 1.

The main motivation for this hypothesis is that we can explicitly compute (9) in this case (see Proposition 6.1). Almost the same computation works if Dg_z^T is unitary (the elliptic case) However, in the strong hyperbolic case, we can prove that the infinite series defining (11) converges absolutely and uniformly, and therefore:

Proposition 1.6 *If z satisfies the strong hyperbolicity hypothesis, then ν_k^z is an absolutely continuous measure.*

The main result is a sharp 2-term Weyl law in this case:

Theorem 1.7 *Assume that $z \in H^{-1}(E)$ and that z satisfies the strong hyperbolicity hypothesis. Then,*

$$\int_a^b d\mu_k^{z,1,E} = \begin{cases} \left(\frac{k}{2\pi}\right)^{m-1/2} \mathcal{G}_0(z)(b-a)(1+o(1)).,\ z \in H^{-1}(E),\ z \notin \mathcal{P}_E \\[2ex] \left(\frac{k}{2\pi}\right)^{m-1/2} \nu_k^z(a,b)(1+o(1)), \qquad z \in H^{-1}(E), z \in \mathcal{P}_E, \end{cases}.$$

Theorem 1.7 is a Kähler Toeplitz analogue of [15, Theorem 1.8.14] (originally proved in [14]). The difference between $z \notin \mathcal{P}_E$ and $z \in \mathcal{P}_E$ is that in the former case, there is a contribution only from the $t = 0$ times of g^t (the identity map) and in the latter case there are contributions from all iterates of g^{T_z}.

It may be expected that Theorem 1.7 extends in some suitable way to any type of periodic orbit. In the somewhat analogous Riemannian setting studied in [15], the pointwise Weyl law involves first return maps on the set of geodesic loop directions $\xi \in S^*_x M$ at a point $x \in M$ rather than closed orbits. In some cases (such as where x is a focus of an ellipsoid), the corresponding measures or Q-functions are calculated in [15, Example 1.8.20]. Otherwise, the authors say simply that it is difficult to determine when the "Q" function of [15, (1.8.11)] is uniformly continuous. It is likely that Theorem 1.7 can be extended to any orbit for which none of the eigenvalues on the symplectic orthogonal complement of the $V(1)$-eigenspace of S have modulus one. This is certainly the case, by the same proof as in Proposition 1.6, if S is diagonalizable by a unitary matrix.

2 Outline of the Proof

The proof is a continuation of that in [19], adding information on the remainder term and its relation to periodic orbits of periods $T > 0$. Given a function $f \in \mathcal{S}(\mathbb{R})$ (Schwartz space) one defines

$$f(k\hat{H}_k) = \int_{\mathbb{R}} \hat{f}(\tau) e^{ik\tau\hat{H}_k} d\tau = \int_{\mathbb{R}} \hat{f}(t) U_k(t) dt, \tag{12}$$

where

$$U_k(t) = \exp itk\hat{H}_k. \tag{13}$$

is the unitary group on $H^0(M, L^k)$ generated by $k\hat{H}_k$. Note that $f(k\hat{H}_k)$ is the operator on $H^0(M, L^k)$ with the same eigensections as $\hat{H}_k$ and with eigenvalues $f(k\mu_{k,j})$. The metric contraction of the Schwarz kernel on the diagonal is given by,

$$\Pi^E_{k,f}(z) = \int_{\mathbb{R}} \hat{f}(t) e^{-iktE} e^{ikt\hat{H}_k}(z, z) dt = \int_{\mathbb{R}} \hat{f}(t) e^{-iktE} U_k(t, z) dt. \tag{14}$$

Here, and henceforth, the metric contraction of a kernel $K_k(z, w)$ is denoted by $K(z)$.

Definition 2.1 The metric contraction of a kernel $M_k(z, w) := \sum_{j=1}^{d_k} \mu_{k,j} s_{k,j}(z) \overline{s_{k,j}(w)}$ expressed in an orthonormal basis $\{s_{k,j}\}_{j=1}^{d_k}$ of $H^0(M, L^k)$ is defined by

$$M_k(z) := \sum_{j=1}^{d_k} \mu_{k,j} |s_{k,j}(z)|^2_{h^k}, \quad (d_k = \dim H^0(M, L^k))$$

In Sect. 3.8 below, we lift sections and kernels to the associated $U(1)$ frame bundle of L^*; then metric contractions are the same as values of the lifts along the diagonal.

In [19] it is shown that $U_k(t)$ is a semi-classical Toeplitz Fourier integral operator of a type defined in [17]. As in [19] we construct a parametrix of the form,

$$\hat{\Pi}_{h^k}\sigma_{k,t}(\hat{g}^{-t})^*\hat{\Pi}_{h^k} \tag{15}$$

where $(\hat{g}^{-t})^*$ is the pullback of functions on X_h by $\hat{g}^t$ and where $\sigma_{k,t}$ is a semi-classical symbol originally calculated in [17, Unitarization Lemma 1 (2b.5) and (3.10)]. In fact, to leading order in k, and up to a phase factor,

$$\sigma_{kt}(z) = \langle \Omega_{Dg_z^T J_z}, \Omega_{J_{g^t z}}\rangle^{-\frac{1}{2}}. \tag{16}$$

Here, $Dg^T J_z$ is the image of the complex structure at z and $J_{g^t z}$ is the complex structure of $T_{g^t z}M$ and Ω_J denotes the ground state in the Bargmann–Fock Hilbert space with complex structure J. It was proved in [5, 17] that (16) equals $(\det P)^{-\frac{1}{2}}$ by calculating the inner product of the two Gaussians.

Combining (3) and (14) shows that

$$\mu_k^{z,1,E}(f) := \int_{\mathbb{R}} f(x)d\mu_k^{z,1,E} = \int_{\mathbb{R}} \hat{f}(t)e^{-iEkt}\hat{\Pi}_{h^k}\sigma_{kt}(\hat{g}^t)^*\hat{\Pi}_{h^k}(z)dt, \tag{17}$$

or equivalently

$$\widehat{\mu_k^{z,1,E}}(t) = e^{-iEkt}U_k(t,z,z). \tag{18}$$

Using a semi-classical Tauberian theorem, it is proved in Sect. 7 that the singularities of (18) determine the 2-term asymptotics of $\mu_k^{z,1,E}[a,b]$ for any interval. Proposition 1.6 follows because the singularities are of a different type depending on the convergence of $Q_z(k)$.

To prove the two-term Weyl law, we begin by obtaining asymptotics for the smoothed partial density of states (17). In the first case where $z \notin \mathcal{P}_E$, the only singularity occurs at $t = 0$ and so the expansion is the same as in [19, Theorem 3] (recalled here as Theorem 7.1). The time interval $[-\epsilon, \epsilon]$ is assumed to be so short that it contains no non-zero periods of periodic orbits. When $z \notin H^{-1}(E)$ the expansion is rapidly decaying. Thus, the new aspect is the second case where $z \in \mathcal{P}_E$.

Theorem 2.2 *For $f \in \mathcal{S}(\mathbb{R})$ with $\hat{f} \in C_c^\infty(\mathbb{R})$, we have (see Definitions 1.4 and 2.1)*

$$\Pi_{k,f}(z) := \int_{\mathbb{R}} f d\mu_k^{z,1,E} = \begin{cases} \left(\frac{k}{2\pi}\right)^{m-1/2} \hat{f}(0)\mathcal{G}_0(z)(1+O(k^{-1}))., & z \in H^{-1}(E),\ z \notin \mathcal{P}_E \\ \left(\frac{k}{2\pi}\right)^{m-1/2} \sum_{n\in\mathbb{Z}} \hat{f}(nT_z)\,\mathcal{G}_n(z)e^{-ikn\theta_z^h} + O(k^{m-3/2}), & z \in H^{-1}(E), z \in \mathcal{P}_E, \\ O(k^{-\infty}), & z \notin H^{-1}(E) \end{cases}$$

To prove Theorem 2.2 we use the Boutet de Monvel–Sjöstrand parametrix for $\hat{\Pi}_{h^k}$. This gives a parametrix for (12) and (17) as semi-classical oscillatory integrals with complex phases. The phase has no critical points when the orbit does not lie in $H^{-1}(E)$ and no critical points for $t \neq 0$ when $z \notin \mathcal{P}_E$. The main difficulty is to evaluate or interpret the phases and the Hessian determinant (and other invariants that arise) dynamically, and to determine whether or not they are invariants of $D\hat{g}^T$ or invariants of the full orbit. One phase factor is a holonomy integral around the periodic orbit $\hat{g}^t(x)$. In Proposition 5.6 it is shown that although the holonomy is apriori a 'global invariant' of the orbit rather than an invariant of the first return map, in fact the Hessian of the holonomy can be expressed as an invariant of the first return map.

To evaluate the Hessian determinants, we first do so in the linear Bargmann–Fock setting, where H is a quadratic Hamiltonian on the Kähler manifold $\mathbb{C}^m$, equipped with a general complex structure J and a Hermitian metric h.

Proposition 2.3 *Let H be a quadratic Hamiltonian in the Bargmann–Fock setting. Assume that H has compact level sets and non-degenerate periodic orbits on level E. Then, in the notation of Definition 2.1,*

$$\int_{\mathbb{R}} \hat{f}(t)U_k(t,z)e^{-itEk}dt \simeq \left(\frac{k}{2\pi}\right)^{m-\frac{1}{2}} \sum_{n\in\mathbb{Z}} \hat{f}(nT_z)e^{-ik\theta_{znT_z}}(\bar{\alpha}P_n^{-1}\alpha)^{-1/2}(\det P_n)^{-1/2},$$

where P_n is the holomorphic block of Dg^{nT_z} (7) and $\pi^{1,0}\mathcal{W}\xi_H = \alpha$.

We give a detailed proof in Sect. 5.4 because the general case is reduced to the Bargmann–Fock case. It is shown in this article that the linearized calculation is the principal symbol of non-linear problem (17), hence that Theorem 1.7 can be reduced to Proposition 2.3. The proof consists of nothing more than Taylor expansions of the phase in suitable Kähler normal coordinates and stationary phase.

3 Background

The background to this article is largely the same as in [19], and we refer there for many details. Here we give a quick review to setup the notation. First we introduce co-circle bundle $X \subset L^*$ for a positive Hermitian line bundle (L, h), so that holomorphic sections of L^k for different k can all be represented in the same space of CR-holomorphic functions on X, $\mathcal{H}(X) = \oplus_k \mathcal{H}_k(X)$. The Hamiltonian flow g^t generated by ξ_H on (M, ω) will be lifted to a contact flow $\hat{g}^t$ generated by $\hat{\xi}_H$ on X. Then we review the Toeplitz quantization for a contact flow on X following [13, 17].

3.1 Symplectic Linear Algebra

Let (V, σ) be a real symplectic vector space of dimension $2n$ and let J be a compatible complex structure on V. There exists a symplectic basis in which $V \simeq \mathbb{R}^{2m}$, σ takes the standard form $\omega = 2\sum_{j=1}^m dx_j \wedge dy_j$ and J has the standard form, $J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. Let $H_J^{1,0}$ resp. $H_J^{0,1}$, denote the $\pm i$ eigenspaces of J in $V \otimes \mathbb{C}$. The projections onto these subspaces are denoted by

$$P_J = \frac{1}{2}(I - iJ) : V \otimes \mathbb{C} \to H_J^{1,0}, \quad \bar{P}_J = \frac{1}{2}(I + iJ) : V \otimes \mathbb{C} \to H_J^{0,1}.$$

Let $S \in Sp(m, \mathbb{R})$ be a real symplectic matrix. Then its transpose $S^t = JS^{-1}J^{-1}$ also lies in $Sp(m, \mathbb{R})$ and $SJ = J(S^t)^{-1}$.

3.2 Symmetric Symplectic Matrices

A matrix S is called a symmetric symplectic matrix if $S \in Sp(n, \mathbb{R})$ and $S^t = S$. For such S it follows that $SJ = JS^{-1}$. A good reference for positive definite symplectic matrices is [8, p. 6] and [8, p. 52]. For the following see [6, Proposition 22]. Let $U(n) = Sp(n) \cap O(2n, \mathbb{R})$. Then $UJ = JU$ and

$$U = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}, \quad AB^t = B^tA, \quad AA^t + BB^t = I, \quad U^{-1} = \begin{pmatrix} A^t & B^t \\ -B^t & A^t \end{pmatrix} = U^t.$$

Proposition 3.1 *If S is a positive definite symmetric symplectic matrix and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n; \lambda_1^{-1}, \ldots, \lambda_n^{-1})$ is the given diagonal matrix, then there exists $U \in U(n)$ so that $S = U^t \Lambda U$.*

The following is [6, Proposition 26].

Proposition 3.2 *A symplectic matrix S is symmetric positive definite if and only if $S = e^X$ with $X \in \mathfrak{sp}(n)$ and $X = X^t$. The map $\exp : \mathfrak{sp}(n) \cap Sym(2n, \mathbb{R}) \to Sp(n) \cap Sym_+(2n, \mathbb{R})$ is a diffeomorphism.*

If $e_1, \ldots, e_n$ are orthonormal eigenvectors of S corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$ then since $SJ = JS^{-1}$,

$$SJe_k = JS^{-1}e_k = \frac{1}{\lambda_j} Je_k.$$

Hence $\pm Je_1, \ldots, \pm Je_n$ are orthonormal eigenvectors of U corresponding to eigenvalues $\lambda_1^{-1}, \ldots, \lambda_n^{-1}$ and $\begin{pmatrix} A \\ B \end{pmatrix} = [e_1, \ldots, e_n]$.

3.3 The Bargmann–Fock Space of a Complex Hermitian Vector Space

The Bargmann–Fock spaces can be defined more generally for any complex structure J on $\mathbb{R}^{2n}$ and any Hermitian metric on $\mathbb{C}^n$.

Let (V, ω) be a real symplectic vector space. Define

$$\mathcal{J} = \{J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \quad J^2 = -I, \quad \omega(JX, JY) = \omega(X, Y), \quad \omega(X, JX) >> 0\}$$

to be the space of complex structures on $\mathbb{R}^n$ compatible with ω. The Bargmann–Fock space of a symplectic vector space (V, σ) with compatible complex structure $J \in \mathcal{J}$ is the Hilbert space,

$$\mathcal{H}_J = \{fe^{-\frac{1}{2}\sigma(v, Jv)} \in L^2(V, dL),\ f \text{ is entire J-holomorphic}\}.$$

Here,

$$\Omega_J(v) := e^{-\frac{1}{2}\sigma(v, Jv)} \tag{19}$$

is the 'vacuum state' and dL is normalized Lebesgue measure (normalized so that square of the symplectic Fourier transform is the identity). The orthogonal projection onto $\mathcal{H}_J$ is denoted by P_J in [5] but we denote it by Π_J in this article. Its Schwartz kernel relative to $dL(w)$ is denoted by $\Pi_J(z, w)$.

Remark: The Bargmann–Fock space with $J = i$ the standard complex structure is often defined instead as the weighted Hilbert space of entire holomorphic functions with Gaussian weight $C_n e^{-|z|^2} dL(z)$ where C_n is a dimensional constant. In this definition the vacuum state is 1. There is a natural isometric 'ground states' isomorphism to $\mathcal{H}_J$ defined by multiplying by $\sqrt{\Omega_J}$. With the Gaussian measure, the Bergman kernel is $B(z, w) = e^{z \cdot \bar{w}}$. When $V = \mathbb{C}^n$ we write $v = Z$, $JZ = iZ$, and $\sigma(Z, W) = \mathrm{Im}\overline{Z} \cdot W$. Then $\Omega_J(Z) = e^{-\frac{1}{2}|Z|^2}$.

3.4 Bargmann–Fock Bergman Kernels

For BF model, we have $\Pi_k : L^2(M, L^k) \to H^0(M, L^k)$ the Bergman projection operator. And $\hat{\Pi}_k : L^2(X) \to \mathcal{H}_k(X)$, the Szego projection operator on X to Hardy space's Fourier component. Let H also denote its pull back on X.

The semi-classical Bargmann–Fock Bergman kernels (23) on $\mathbb{C}^n$ are given by

$$\Pi^{\mathbb{C}^m}_{k,h_0,i}(z,w) = \left(\frac{k}{2\pi}\right)^m e^{k(z\bar{w}-|z|^2/2-|w|^2/2)}.$$

Their lifts to X are given by

$$\hat{\Pi}^{\mathbb{C}^m}_{k,h_0,i}(\hat{z},\hat{w}) = \left(\frac{k}{2\pi}\right)^m e^{k\psi(\hat{z},\hat{w})}$$

where

$$\hat{\psi}(\hat{z},\hat{w}) = i(\theta_z - \theta_w) + \psi(z,w) = i(\theta_z - \theta_w) + z\cdot\bar{w} - |z|^2/2 - |w|^2/2.$$

where $\hat{z} = (\theta_z, z) \in S^1 \times M \cong X$ denotes a lift of z.[1]

In the general case, by (3.1) of [5], one has

$$\Pi_J\psi(z) = \langle \Omega^z_J, \psi\rangle = \int_{\mathbb{C}^n} \psi(v)\overline{\Omega^z_J}(v)dv,$$

i.e.

$$\Pi_J(z,w) = \overline{\Omega^z_J}(w) = e^{i\sigma(z,w)}e^{-\frac{1}{2}\sigma(z-w,J(z-w))} \tag{20}$$

which reduces to $e^{i\mathrm{Im}z\bar{w}}e^{-\frac{1}{2}(|z-w|^2)} = e^{z\bar{w}}e^{-\frac{1}{2}(|z|^2+|w|^2)}$ in the case $J=i$, $h=h_0$.

3.5 Holomorphic Sections in L^k and CR-Holomorphic Functions on X

Let $(L,h)\to(M,\omega)$ be a positive Hermitian line bundle, L^* the dual line bundle. Let

$$X := \{p\in L^* \mid \|p\|_h = 1\},\quad \pi : X\to M$$

be the unit circle bundle over M.

Let $e_L \in \Gamma(U,L)$ be a non-vanishing holomorphic section of L over U, $\varphi = -\log\|e_L\|^2$ and $\omega = i\partial\bar{\partial}\varphi$. We also have the following trivialization of X:

$$U\times S^1 \cong X|_U,\ (z;\theta)\mapsto e^{i\theta}\frac{e^*_L|_z}{\|e^*_L|_z\|}. \tag{21}$$

[1] We also use the notation $x = (z,\theta_z)$, $y = (w,\theta_w)$.

X has a structure of a contact manifold. Let ρ be a smooth function in a neighborhood of X in L^*, such that $\rho > 0$ in the open unit disk bundle, $\rho|_X = 0$ and $d\rho|_X \neq 0$. Then we have a contact one-form on X

$$\alpha = -\mathrm{Re}(i\bar{\partial}\rho)|_X,$$

well defined up to multiplication by a positive smooth function. We fix a choice of ρ by

$$\rho(x) = -\log \|x\|_h^2, \quad x \in L^*,$$

then in local trivialization of X (21), we have

$$\alpha = d\theta - \frac{1}{2}d^c\varphi(z). \tag{22}$$

X is also a strictly pseudoconvex CR manifold. The *CR structure* on X is defined as follows: The kernel of α defines a horizontal hyperplane bundle

$$HX := \ker \alpha \subset TX,$$

invariant under J since $\ker \alpha = \ker d\rho \cap \ker d^c\rho$. Thus we have a splitting

$$TX \otimes \mathbb{C} \cong H^{1,0}X \oplus H^{0,1}X \oplus \mathbb{C}R.$$

A function $f : X \to \mathbb{C}$ is CR-holomorphic, if $df|_{H^{0,1}X} = 0$.

A holomorphic section s_k of L^k determines a CR-function $\hat{s}_k$ on X by

$$\hat{s}_k(x) := \langle x^{\otimes k}, s_k\rangle, \quad x \in X \subset L^*.$$

Furthermore $\hat{s}_k$ is of degree k under the canonical S^1 action r_θ on X, $\hat{s}_k(r_\theta x) = e^{ik\theta}\hat{s}_k(x)$. The inner product on $L^2(M, L^k)$ is given by

$$\langle s_1, s_2\rangle := \int_M h^k(s_1(z), s_2(z))d\,\mathrm{Vol}_M(z), \quad d\,\mathrm{Vol}_M = \frac{\omega^m}{m!},$$

and inner product on $L^2(X)$ is given by

$$\langle f_1, f_2\rangle := \int_X f_1(x)\overline{f_2(x)}d\,\mathrm{Vol}_X(x), \quad d\,\mathrm{Vol}_X = \frac{\alpha}{2\pi} \wedge \frac{(d\alpha)^m}{m!}.$$

Thus, sending $s_k \mapsto \hat{s}_k$ is an isometry.

3.6 Szegö Kernel on X

On the circle bundle X over M, we define the orthogonal projection from $L^2(X)$ to the CR-holomorphic subspace $\mathcal{H}(X) = \hat{\bigoplus}_{k\geq 0}\mathcal{H}_k(X)$, and degree-$k$ subspace $\mathcal{H}_k(X)$:

$$\hat{\Pi} : L^2(X) \to \mathcal{H}(X), \quad \hat{\Pi}_k : L^2(X) \to \mathcal{H}_k(X), \quad \hat{\Pi} = \sum_{k\geq 0} \hat{\Pi}_k.$$

The Schwarz kernels $\hat{\Pi}_k(x, y)$ of $\hat{\Pi}_k$ is called the degree-k Szegö kernel, i.e.

$$(\hat{\Pi}_k F)(x) = \int_X \hat{\Pi}_k(x, y)F(y)d\,\mathrm{Vol}_X(y), \quad \forall F \in L^2(X).$$

If we have an orthonormal basis $\{\hat{s}_{k,j}\}_j$ of $\mathcal{H}_k(X)$, then

$$\hat{\Pi}_k(x, y) = \sum_j \hat{s}_{k,j}(x)\overline{\hat{s}_{k,j}(y)}.$$

The degree-k kernel can be extracted as the Fourier coefficient of $\hat{\Pi}(x, y)$

$$\hat{\Pi}_k(x, y) = \frac{1}{2\pi}\int_0^{2\pi} \hat{\Pi}(r_\theta x, y)e^{-ik\theta}d\theta. \tag{23}$$

We refer to (23) as the *semi-classical Bergman kernels*.

3.7 Boutet de Monvel–Sjöstrand Parametrix for the Szegö Kernel

Near the diagonal in $X \times X$, there exists a parametrix due to Boutet de Monvel–Sjöstrand [4] for the Szegö kernel of the form,

$$\hat{\Pi}(x, y) = \int_{\mathbb{R}^+} e^{\sigma\hat{\psi}(x,y)}s(x, y, \sigma)d\sigma + \hat{R}(x, y). \tag{24}$$

where $\hat{\psi}(x, y)$ is the almost-CR-analytic extension of $\hat{\psi}(x, x) = -\rho(x) = \log \|x\|^2$, and $s(x, y, \sigma) = \sigma^m s_m(x, y) + \sigma^{m-1}s_{m-1}(x, y) + \cdots$ has a complete asymptotic expansion. In local trivialization (21),

$$\hat{\psi}(x, y) = i(\theta_x - \theta_y) + \psi(z, w) - \frac{1}{2}\varphi(z) - \frac{1}{2}\varphi(w),$$

where $\psi(z, w)$ is the almost analytic extension of $\varphi(z)$.

3.8 Lifting the Hamiltonian Flow to a Contact Flow on X_h

In this section we review the definition of the lifting of a Hamiltonian flow to a contact flow, following [19, Section 3.1]. Let $H : M \to \mathbb{R}$ be a Hamiltonian function on (M, ω). Let ξ_H be the Hamiltonian vector field associated to H, such that $dH = \iota_{\xi_H}\omega$. The purpose of this section is to lift ξ_H to a contact vector field $\hat{\xi}_H$ on X. Let α denote the contact 1-form (22) on X, and R the corresponding Reeb vector field determined by $\langle \alpha, R\rangle = 1$ and $\iota_R d\alpha = 0$. One can check that $R = \partial_\theta$.

Definition 3.3 (1) The horizontal lift of ξ_H is a vector field on X denoted by ξ_H^h. It is determined by

$$\pi_*\xi_H^h = \xi_H, \quad \langle \alpha, \xi_H^h\rangle = 0.$$

(2) The contact lift of ξ_H is a vector field on X denoted by $\hat{\xi}_H$. It is determined by

$$\pi_*\hat{\xi}_H = \xi_H, \quad \mathcal{L}_{\hat{\xi}_H}\alpha = 0.$$

Lemma 3.4 *The contact lift $\hat{\xi}_H$ is given by*

$$\hat{\xi}_H = \xi_H^h - HR.$$

The Hamiltonian flow on M generated by ξ_H is denoted by g^t

$$g^t : M \to M, \quad g^t = \exp(t\xi_H).$$

The contact flow on X generated by $\hat{\xi}_H$ is denoted by $\hat{g}^t$

$$\hat{g}^t : X \to X, \quad \hat{g}^t = \exp(t\hat{\xi}_H).$$

Lemma 3.5 *In local trivialization (21), we have a useful formula for the flow, $\hat{g}^t$ has the form (see [19, Lemma 3.2]):*

$$\hat{g}^t(z,\theta) = \left(g^t(z), \quad \theta + \int_0^t \frac{1}{2}\langle d^c\varphi, \xi_H\rangle(g^s(z))ds - tH(z)\right).$$

Since $\hat{g}^t$ preserves α it preserves the horizontal distribution $H(X_h) = \ker\alpha$, i.e.

$$D\hat{g}^t : H(X)_x \to H(X)_{\hat{g}^t(x)}.$$

It also preserves the vertical (fiber) direction and therefore preserves the splitting $V \oplus H$ of TX. Its action in the vertical direction is determined by Lemma 3.5. When g^t is non-holomorphic, $\hat{g}^t$ is not CR holomorphic, i.e. does not preserve the horizontal complex structure J or the splitting of $H(X) \otimes \mathbb{C}$ into its $\pm i$ eigenspaces.

3.9 Toeplitz Quantum Dynamics

Here we consider quantization for the Hamiltonian flow g^t on holomorphic sections of L^k, or CR-functions of degree k on X. An operator $T : C^\infty(X) \to C^\infty(X)$ is called a *Toeplitz operator of order* k, denoted as $T \in \mathcal{T}^k$, if it can be written as $T = \hat{\Pi} \circ Q \circ \hat{\Pi}$, where Q is a pseudo-differential operator on X. Its principal symbol $\sigma(T)$ is the restriction of the principal symbol of Q to the symplectic cone

$$\Sigma = \{(x, r\alpha(x)) \mid r > 0\} \cong X \times \mathbb{R}_+ \subset T^*X.$$

The symbol satisfies the following properties

$$\begin{cases} \sigma(T_1 T_2) = \sigma(T_1)\sigma(T_2); \\ \sigma([T_1, T_2]) = \{\sigma(T_1), \sigma(T_2)\}; \\ \text{If } T \in \mathcal{T}^k, \text{ and } \sigma(T) = 0, \text{ then } T \in \mathcal{T}^{k-1}. \end{cases}$$

The choice of the pseudodifferential operator Q in the definition of $T = \hat{\Pi}\, Q\, \hat{\Pi}$ is not unique. However, there exists some particularly nice choices.

Lemma 3.6 ([3] Proposition 2.13) *Let T be a Toeplitz operator on* Σ *of order p, then there exists a pseudodifferential operator Q of order p on X, such that* $[Q, \hat{\Pi}] = 0$ *and* $T = \hat{\Pi}\, Q\, \hat{\Pi}$.

Now we specialize to the setup here, following closely [13]. Consider an order one self-adjoint Toeplitz operator

$$T = \hat{\Pi} \circ (H \cdot \mathbf{D}) \circ \hat{\Pi},$$

where $\mathbf{D} = (-i\partial_\theta)$ and ∂_θ is the fiberwise rotation vector field on X, and H is multiplication by $\pi^{-1}(H)$, where we abuse notation and identify H downstairs with its pullback upstairs $\pi^{-1}(H)$. We note that $\mathbf{D}$ decompose $L^2(X)$ into eigenspaces $\oplus_{k\in\mathbb{Z}} L^2(X)_k$ with eigenvalue $k \in \mathbb{Z}$. The symbol of T is a function on $\Sigma \cong X \times \mathbb{R}_+$, given by

$$\sigma(T)(x, r) = (\sigma(H)\sigma(\mathbf{D})|_\Sigma)(x, r) = H(x)r, \quad \forall (x, r) \in \Sigma.$$

Definition 3.7 ([13], Definition 5.1) Let $\hat{U}(t)$ denote the one-parameter subgroup of unitary operators on $L^2(X)$, given by

$$\hat{U}(t) := \hat{\Pi}\, e^{it\hat{\Pi}(\mathbf{D}H)\hat{\Pi}}\, \hat{\Pi} : \mathcal{H}(X) \to \mathcal{H}(X),$$

and let $\hat{U}_k(t)$ (13) denote the Fourier component acting on $L^2(X)_k$:

$$\hat{U}_k(t) := \hat{\Pi}_k \, e^{it\hat{\Pi}(kH)\hat{\Pi}} \, \hat{\Pi}_k : \mathcal{H}_k(X) \to \mathcal{H}_k(X) \tag{25}$$

We use $U_k(t)$ to denote the corresponding operator on $H^0(M, L^k)$.

Proposition 3.8 ([13], Proposition 5.2) *$\hat{U}(t)$ is a group of Toeplitz Fourier integral operators on $L^2(X)$, whose underlying canonical relation is the graph of the time t Hamiltonian flow of rH on the symplectic cone Σ of the contact manifold (X, α).*

Proposition 3.9 ([17]) *There exists a semi-classical symbol $\sigma_k(t)$ so that the unitary group (25) has the form*

$$\hat{U}_k(t) = \hat{\Pi}_k (\hat{g}^{-t})^* \sigma_k(t) \hat{\Pi}_k$$

modulo smooth kernels of order $k^{-\infty}$.

It follows from the above proposition and the Boutet de Monvel–Sjöstand parametrix construction that $\hat{U}_k(t, x, x)$ admits an oscillatory integral representation of the form,

$$\hat{U}_k(t, x, x) \simeq \int_X \int_0^\infty \int_0^\infty \int_{S^1} \int_{S^1} e^{\sigma_1 \hat{\psi}(r_{\theta_1} x, \hat{g}^t y) + \sigma_2 \hat{\psi}(r_{\theta_2} y, x) - ik\theta_1 - ik\theta_2} S_k d\theta_1 d\theta_2 d\sigma_1 d\sigma_2 dy$$

where S_k is a semi-classical symbol, and the asymptotic symbol $\simeq$ means that the difference of the two sides is rapidly decaying in k.

4 Bargmann–Fock Space

In this section, we illustrate the various definition of the background section using the example of Bargmann–Fock (BF) space. We also define the osculating BF space for at the tangent space $T_z M$ for a general Kähler manifold, and show that in the semi-classical limit as $k \to \infty$ the Bergman kernel near the diagonal reduces to the BF model at leading order.

4.1 Set-Up

Let $M = \mathbb{C}^m$ with coordinate $z_i = x_i + \sqrt{-1} y_i$, $L \to M$ be the trivial line bundle. We fix a trivialization and identify $L \cong \mathbb{C}^m \times \mathbb{C}$. We use Kähler form $\omega = i \sum_i dz_i \wedge d\bar{z}_i$ and Kähler potential $\varphi(z) = |z|^2 := \sum_i |z_i|^2$.[2] The Bargmann–Fock space of degree k on $\mathbb{C}^m$ is defined by

$$\mathcal{H}_k = \left\{ f(z) e^{-k|z|^2/2} \mid f(z) \text{ holomorphic function on } \mathbb{C}^m, \quad \int_{\mathbb{C}^m} |f|^2 e^{-k|z|^2} < \infty \right\}.$$

The volume form on $\mathbb{C}^m$ is $d\,\mathrm{Vol}_{\mathbb{C}^m} = \omega^m / m!$.

[2] Our choice of ω may differ from other conventions by factors of 2 or π.

More generally, fix (V, ω) be a real $2m$ dimensional symplectic vector space. Let $J : V \to V$ be a ω compatible linear complex structure, that is $g(v, w) := \omega(v, Jw)$ is a positive-definite bilinear form and $\omega(v, w) = \omega(Jv, Jw)$. There exists a canonical identification of $V \cong \mathbb{C}^m$ up to $U(m)$ action, identifying ω and J. We denote the BF space for (V, ω, J) by $\mathcal{H}_{k,J}$.

The circle bundle $\pi : X \to M$ can be trivialized as $X \cong \mathbb{C}^m \times S^1$. The contact form on X is

$$\alpha = d\theta + (i/2) \sum_j (z_j d\bar{z}_j - \bar{z}_j dz_j).$$

If $s(z)$ is a holomorphic function (section of L^k) on $\mathbb{C}^m$, then its CR-holomorphic lift to X is

$$\hat{s}(z, \theta) = e^{k\left(i\theta - \frac{1}{2}|z|^2\right)} s(z).$$

Indeed, the horizontal lift of $\partial_{\bar{z}_j}$ is $\partial^h_{\bar{z}_j} = \partial_{\bar{z}_j} - \frac{i}{2} z_j \partial_\theta$, and $\partial^h_{\bar{z}_j} \hat{s}(z, \theta) = 0$. The volume form on $X = \mathbb{C}^m \times S^1$ is $d\,\mathrm{Vol}_X = (d\theta/2\pi) \wedge \omega^m/m!$.

4.2 Bergman Kernel on Bargmann–Fock Space

The degree k Bergman kernel downstairs on $\mathbb{C}^m$ is given by

$$\Pi_k(z, w) = \left(\frac{k}{2\pi}\right)^m e^{z\bar{w}}.$$

Given any function $f \in L^2(\mathbb{C}^m, e^{-k|z|^2/2} dVol_{\mathbb{C}^m})$, its orthogonal projection to holomorphic function is given by

$$(\Pi_k f)(z) = \int_{\mathbb{C}^m} \Pi_k(z, w) f(w) e^{-k|w|^2} d\,\mathrm{Vol}_{\mathbb{C}^m}(w).$$

The degree k Bergman (Szegö) kernel $\hat{\Pi}_k(\hat{z}, \hat{w})$ upstairs for $X = \mathbb{C}^m \times S^1$ is given by

$$\hat{\Pi}_k(\hat{z}, \hat{w}) = \left(\frac{k}{2\pi}\right)^m e^{k\hat{\psi}(\hat{z}, \hat{w})},$$

where $\hat{z} = (z, \theta_z)$, $\hat{w} = (w, \theta_w)$ and the phase function is

$$\psi(\hat{z}, \hat{w}) = i(\theta_z - \theta_w) + z\bar{w} - \frac{1}{2}|z|^2 - \frac{1}{2}|w|^2. \tag{26}$$

4.3 Heisenberg Representation

The space $\mathbb{C}^m \times S^1$ can be identified with the reduced Heisenberg group $\mathbb{H}^m_{red}$, where the group multiplication is given by

$$(z,\theta)\circ(z',\theta') = (z+z', \theta+\theta'+\mathrm{Im}(z\bar{z}')).$$

Lemma 4.1 *The contact form $\alpha = d\theta + \frac{i}{2}\sum_j (z_j d\bar{z}_j - \bar{z}_j dz_j)$ on $\mathbf{H}^m_{red}$ is invariant under the left multiplication*

$$L_{(z_0,\theta_0)} : (z,\theta) \mapsto (z_0,\theta_0)\circ(z,\theta) = \left(z+z_0, \theta+\theta_0+\frac{z_0\bar{z}-\bar{z}_0 z}{2i}\right).$$

Proof

$$(L^*_{(z_0,\theta_0)}\alpha)|_{(z,\theta)} = d\left(\theta+\theta_0+\frac{\bar{z}z_0-\bar{z}_0 z}{2i}\right)+\frac{i}{2}\sum_j((z_j+z_{0j})d\bar{z}_j-(\bar{z}_j+\bar{z}_{0j})dz_j) = \alpha|_{(z,\theta)}.$$

□

In particular, $\mathbb{H}^m_{red}$ preserves the volume form $\alpha\wedge(d\alpha)^m/m!$ on X, hence defines a unitary operator acting on the degree k CR functions on X.

The infinitesimal Heisenberg group action on X can be identified with contact vector field generated by a linear Hamiltonian function $H : \mathbb{C}^m \to \mathbb{R}$.

Lemma 4.2 ([19, Section 3.2]) *For any $\beta \in \mathbb{C}^m$, we define a linear Hamiltonian function on $\mathbb{C}^m$ by*

$$H(z) = z\bar{\beta}+\beta\bar{z}.$$

The Hamiltonian vector field on $\mathbb{C}^m$ is

$$\xi_H = -i\beta\partial_z + i\bar{\beta}\partial_{\bar{z}},$$

and its contact lift is

$$\hat{\xi}_H = -i\beta\partial_z + i\bar{\beta}\partial_{\bar{z}} - \frac{1}{2}(z\bar{\beta}+\beta\bar{z})\partial_\theta.$$

The time t flow $\hat{g}^t$ on X is given by left multiplication

$$\hat{g}^t(z,\theta) = (-i\beta t, 0)\circ(z,\theta) = (z-i\beta t, \theta - t\mathrm{Re}(\beta\bar{z})).$$

4.4 Metaplectic Representation

Let $\mathbb{R}^{2m}, \omega = 2\sum_{j=1}^m dx_j \wedge dy_j$ be a symplectic vector space. The space $Sp(m, \mathbb{R})$ consists of linear transformation $S : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$, such that $S^*\omega = \omega$. In coordinates, we write

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = S \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

In complex coordinates $z_i = x_i + iy_i$, we have then

$$\begin{pmatrix} z' \\ \bar{z}' \end{pmatrix} = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} \begin{pmatrix} z \\ \bar{z} \end{pmatrix} =: \mathcal{A} \begin{pmatrix} z \\ \bar{z} \end{pmatrix},$$

where

$$\begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} = \mathcal{W}^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mathcal{W}, \quad \mathcal{W} = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ -iI & iI \end{pmatrix}. \tag{27}$$

The choice of normalization of $\mathcal{W}$ is such that $W^{-1} = W^*$. Thus,

$$P = \frac{1}{2}(A + D + i(C - B)).$$

We say such $\mathcal{A} \in Sp_c(m, \mathbb{R}) \subset M(2n, \mathbb{C})$. The following identities are often useful.

Proposition 4.3 ([7] Prop 4.17) *Let* $\mathcal{A} = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} \in Sp_c$, *then*

(1) $\begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}^{-1} = \begin{pmatrix} P^* & -Q^t \\ -Q^* & P^t \end{pmatrix} = K\mathcal{A}^*K$, *where* $K = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$.
(2) $PP^* - QQ^* = I$ *and* $PQ^t = QP^t$.
(3) $P^*P - Q^t\bar{Q} = I$ *and* $P^t\bar{Q} = Q^*P$.

The (double cover) of $Sp(m, \mathbb{R})$ acts on the (downstairs) BF space $\mathcal{H}_k$ via kernel: given $M = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} \in Sp_c$, we have

$$\mathcal{K}_{k,M}(z, w) = \left(\frac{k}{2\pi}\right)^m (\det P)^{-1/2} \exp\left\{k\frac{1}{2}\left(z\bar{Q}P^{-1}z + 2\bar{w}P^{-1}z - \bar{w}P^{-1}Q\bar{w}\right)\right\}$$

where the ambiguity of the sign the square root $(\det P)^{-1/2}$ is determined by the lift to the double cover. When $\mathcal{A} = Id$, then $\mathcal{K}_{k,\mathcal{A}}(z, \bar{w}) = \Pi_k(z, \bar{w})$. Similarly, we have the kernel upstairs on X as

$$\hat{\mathcal{K}}_{k,\mathcal{A}}(\hat{z}, \hat{w}) = \mathcal{K}_{k,M}(z, \bar{w})e^{k(i\theta_z - |z|^2/2) + k(-i\theta_w - |w|^2/2)}. \tag{28}$$

A quadratic Hamiltonian function $H : \mathbb{C}^m \to \mathbb{R}$ will generates a one-parameter family of symplectic linear transformations $\mathcal{A}_t = g^t : \mathbb{C}^m \to \mathbb{C}^m$. However, $\mathcal{A}_t$ is only $\mathbb{R}$-linear but not $\mathbb{C}$-linear, i.e. M_t does not preserve the complex structure of $\mathbb{C}^m$. Hence, one need to orthogonal project back to holomorphic sections. To compensate for the loss of norm due to the projection, one need to multiply a factor $\eta_{\mathcal{A}_t}$. This is in the spirit of Proposition 3.9.

Proposition 4.4 *Let $\mathcal{A} : \mathbb{C}^m \to \mathbb{C}^m$ be a linear symplectic map, $\mathcal{A} = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}$, and let $\hat{\mathcal{A}} : X \to X$ be the contact lift that fixes the fiber over* 0*, then*

$$\hat{\mathcal{K}}_{k,\mathcal{A}}(\hat{z}, \hat{w}) = (\det P^*)^{1/2} \int_X \hat{\Pi}_k(\hat{z}, \hat{\mathcal{A}}\hat{u})\hat{\Pi}_k(\hat{u}, \hat{w}) d\,\mathrm{Vol}_X(\hat{u})$$

Proof The contact lift $\hat{\mathcal{A}} : \mathbb{C}^m \times S^1 \to \mathbb{C}^m \times S^1$ is given by $\mathcal{A}$ acting on the first factor:

$$\hat{\mathcal{A}} : (z, \theta) \mapsto (Pz + Q\bar{z}, \theta),$$

one can check that $\hat{\mathcal{A}}^*\alpha = \alpha$. The integral over X is a standard complex Gaussian integral, analogous to [7, Prop 4.31], and with determinant Hessian $1/|\det P|$, hence we have $(\det P^*)^{1/2}/|\det P| = (\det P)^{-1/2}$. □

4.5 Toeplitz Construction of the Metaplectic Representation

As in [5], the metaplectic representation $W_J(S)$ of $S \in Mp(n, \mathbb{R})$ on $\mathcal{H}_J$ can also be constructed by the Toeplitz approach. First, let U_S be the unitary translation operator on $L^2(\mathbb{R}^{2n}, dL)$ defined by $U_S F(x, \xi) := F(S^{-1}(x, \xi))$. The metaplectic representation of S on $\mathcal{H}_J$ is given by ([5], (5.5) and (6.3 b))

$$W_J(S) = \eta_{J,S}\Pi_J U_S \Pi_J, \tag{29}$$

where we define (see [5] (6.1) and (6.3a)),

$$\eta_{J,S} = 2^{-n} \det(I - iJ) + S(I + iJ)^{\frac{1}{2}} \tag{30}$$

and Π_J is the Bargmann–Fock Szegö projector (20).

Also define $\beta_{J,SJS^{-1}} = 2^{-n/2}[\det(SJ + JS)]^{1/4}$. Then, $|\eta_{J,S}| = \beta_{J,SJS^{-1}}$. In fact (see [5], above (6.3a), and (B6))

$$|2^{-n} \det(I - iJ) + S(I + iJ)^{\frac{1}{2}}| = [\det(SJ + JS)]^{1/2} = 2^n \beta^2_{J,SJS^{-1}}.$$

We further record the identities,

$$\det(SJ + JS) = \det(I + J^{-1}S^{-1}JS) = \det(I + S^*S).$$

The following identity gives another explanation of the presence of $(\det P_n)^{-\frac{1}{2}}$ in (9).

Lemma 4.5 (see [5], p. 1388)

$$\eta_{JS}\beta^{-2}_{J,SJS^{-1}} = (\eta_{JS})^{*-1} = \eta_{JS}\, 2^n(\det(I + S^*S))^{-\frac{1}{2}}$$

and (cf. [5], p. 1388),

$$(\eta^*_{J,S})^{-1} = \det((I + iJ) + S(I - iJ)) = 2^n \det(A + D + i(B - C)) = \det P^*.$$

Proof The first equality is proved on p. 1388 of [5]. The second asserts that

$$\beta_{J,SJS^{-1}} = 2^{-n/2}(\det(I + S^*S))^{\frac{1}{4}},$$

which follows from (30) and identity (ii) above. □

Corollary 4.6 $\eta_{J,USU^{-1}} = \eta_{J,S}$ *where* $U \in U(m)$.

Proof This follows from replacing S by USU^{-1} and using that $UJ = JU$. □

4.6 Osculating Bargmann–Fock Space

In this subsection, we first define the osculation Bargmann–Fock space for any fixed point $z \in M$, using the triple (T_zM, ω_z, J_z). Then, we define the preferred local coordinates in a neighborhood U of z and a preferred frame section e_L of L over U, which together determines a coordinate system of the circle bundle $X|_U$ over U. In these special coordinate, the Boutet–Sjöstrand phase can be approximated by the Bargmann–Fock–Heisenberg phase function modulo cubic order terms.

Definition 4.7 Given a point $x \in X_h$ (resp. $z \in M$), we define the *osculating Bargmann–Fock space* at x (resp. z) to be the Bargmann–Fock space of (H_xX, J_x, ω_x) resp. (T_zM, J_z, ω_z). We denote it by $\mathcal{H}_{J_x,\omega_x}$ (resp. $\mathcal{H}_{J_z,\omega_z}$).

If z is a periodic point for g^t, let $\gamma = \bigcup_{0\le s\le t} g^s z$ be the corresponding closed geodesic, and we may apply the metaplectic representation to define $W_{J_z}(Dg^t|_z)$ as a unitary operator on $\mathcal{H}_{J_z,\omega_z}$. There is a square root ambiguity which can be resolved as in [5] but for our purposes it is not very important and for brevity we omit it from the discussion.

Definition 4.8 Let $p \in M$. A coordinate system $(z_1, \ldots, z_m)$ on a neighborhood U of p is called *K-coordinates* at p if

$$i \sum_{j=1}^{m} dz_j \wedge d\bar{z}_j = \omega|_p .$$

Let e_L be a local frame and let $\phi(z) = -\log ||e_L(z)||_h^2$, if in a K-coordinates

$$\phi(z) = |z|^2 + \sum_{JK} a_{JK} z^J \bar{z}^K, \quad \text{with } |J| \geq 2, |K| \geq 2. \tag{31}$$

then e_L is called a K-frame.

K-coordinates are defined by Lu–Shiffman in Definition 2.6 of [9]. Existence of K-coordinates and K-frames are proved in [9] (Lemma 2.7). Further, in K-coordinates,

$$\omega = \omega_0 + \sum_{ijk\ell} R_{ijk\ell} z_i \bar{z}_j dz_k \wedge d\bar{z}_\ell + \cdots, \quad \omega_0 = \sum_j dz_j \wedge d\overline{z_j}.$$

The K-frame and K-coordinates together give us 'Heisenberg coordinates':

Definition 4.9 A *Heisenberg coordinate chart* at a point x_0 in the principal bundle X is a coordinate chart $\rho : U \to V$ with $0 \in U \subset \mathbb{C}^m \times S^1$ and $\rho(0) = x_0 \in V \subset X$ of the form

$$\rho(z_1, \ldots, z_m, \theta) = e^{i\theta} \frac{e_L^*(z)}{||e_L^*(z)||_{h^k}},$$

where e_L is a preferred local frame for $L \to M$ at $P_0 = \pi(x_0)$, and $(z_1, \ldots, z_m)$ are K-coordinates centered at P_0. (Note that P_0 has coordinates $(0, \ldots, 0)$ and $e_L^*(P_0) = x_0$.)

In these coordinates, the Boutet–Sjöstrand phase $\psi(x, y)$ may be approximated modulo cubic remainder terms by the Bargmann–Fock–Heisenberg phase (26).

The lifted Szegö kernel is shown in [16] and in Theorem 2.3 of [9] to have the scaling asymptotics,

Theorem 4.10 *Let $P_0 \in M$ and choose a Heisenberg coordinate chart about P_0.*

$$k^{-m} \hat{\Pi}_{h^k} \left(\frac{u}{\sqrt{k}}, \frac{\theta_1}{k}, \frac{v}{\sqrt{k}}, \frac{\theta_2}{k} \right) = \hat{\Pi}_{h_z, J_z}^{T_z M}(u, \theta_1, v, \theta_2) \left(1 + k^{-1} A_1(u, v, \theta_1, \theta_2) + \cdots \right),$$

where $\Pi_{h_z, J_z}^{T_z M}$ is the osculating Bargmann–Fock Szegö kernel for $k = 1$ and for the tangent space $T_z M \simeq \mathbb{C}^m$ equipped with the complex structure J_z and Hermitian metric h_z.

Here we identify the coordinates $(u, \theta_1, v, \theta_2)$ with linear coordinates on $T_z M \times S^1 \times T_z M \times S^1$.

5 Proof of Theorem 2.2

In this section we study the rescaled Weyl sum

$$\Pi^E_{k,f}(z,z) := \sum_j f(k(\mu_{k,j} - E))\Pi_{k,j}(z,z).$$

Our purpose is to prove Theorem 2.2. By comparison with interface asymptotics [19], we now need to consider the Hamiltonian flow for long times.

The main idea of the proof is that aside from the holonomy factor (the value of the phase at the critical point), the data of the principal term in Theorem 2.2 localizes at the periodic point. That is, the data come from the derivative of the first return map and do not involve data along the orbit. Too see this, we use the quadratic Taylor approximation of the phase $\psi(x, \hat{g}^t y) + \psi(y, x)$ in (t, y) around a periodic point (T, x). First, we approximate the phase ψ by its osculating Bargmann–Fock approximation ψ_0 at x. Further we approximate $\hat{g}^t$ by its linear approximation $D\hat{g}^t$. We also need to determine the quadratic approximation to the holonomy term of the phase coming from the θ variable. This part of the calculation is apriori non-local. But we show in Proposition 5.6 that the Hessian of the holonomy term $\hat{\theta}_w(T)$ vanishes at the periodic point. After these Taylor approximations, the calculation is essentially reduced to the linear Bargmann–Fock case of Sect. 4.

5.1 Stationary Phase Integral Expression

Let $z \in M$ and $x \in X$ such that $\pi(x) = z$. Let $f \in \mathcal{S}(\mathbb{R})$ with Fourier transform $\hat{f}(t) = \int f(x)e^{itx}\frac{dx}{2\pi}$ compactly supported. We combine the definition (15) with two compositions of the Boutet de Monvel–Sjoestrand parametrix (24) to get

$$\begin{aligned}\Pi^E_{k,f}(z) &= \int_{\mathbb{R}} \hat{f}(t)e^{-itkE}\hat{U}_k(t,x,x)dt \\ &= \int_{\mathbb{R}}\int_X\int_{S^1}\int_{S^1}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \hat{f}(t)e^{k\Psi(t,x,y,\sigma_1,\sigma_2,\theta_1,\theta_2)}A_k d\sigma_1 d\sigma_2 d\theta_1 d\theta_2 dy dt + O(k^{-\infty}).\end{aligned}$$

where the phase function is given by,

$$\Psi(t,x,y,\sigma_1,\sigma_2,\theta_1,\theta_2) = -itE + \sigma_1\hat{\psi}(r_{\theta_1}x, \hat{g}^t y) + \sigma_2\hat{\psi}(r_{\theta_2}y, x) - i\theta_1 - i\theta_2 \tag{32}$$

and A_k is a semi-classical symbol. We consider the critical points and the determinant of the Hessian matrix of the phase.

We will work with a K-coordinate and K-frame in a neighborhood U of z. In this coordinate, $z = (0,\dots,0) \in \mathbb{C}^m$, $x = (0,\dots,0;0) \in \mathbb{C}^m \times S^1$, and $y = (w;\theta_w) \in \mathbb{C}^m \times S^1$. We denote $\hat{g}^t y = (w(t);\theta_w(t))$. Since $\theta_w(t) - \theta_w$ only depends on w, t but independent of θ_w, then we define the holonomy phase for flow $\hat{g}^t$:

$$\hat{\theta}_w(t) := \theta_w(t) - \theta_w.$$

Similarly, the holonomy phase $\theta^h_w(t)$ for the horizontal flow $\exp(t\xi^h_H)$ is denoted by

$$\exp(t\xi^h_H)(w;\theta_w) = (g^t w; \theta_w + \theta^h_w(t)). \tag{33}$$

Note that $\hat{\theta}_w(t)$ depends on H, where as $\theta_w(t)$ only depend on H modulo constant,or dH.

Proposition 5.1 *Fix a K-coordinate and K-frame in a neighborhood U at z. Let $\chi : M \to \mathbb{R}$ be a smooth cut-off function supported in U and constant equals to one near z. Then we have*

$$\Pi^E_{k,f}(z) = \int_{\mathbb{R}}\int_M\int_{S^1}\int_{S^1}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \hat{f}(t)e^{k\Psi'(t,w,\sigma_1,\sigma_2,\theta_1,\theta_2)}\chi(g^t w)\chi(w)S_k d\sigma_1 d\sigma_2 d\theta_1 d\theta_2 dw dt + O(k^{-\infty}).$$

where

$$\Psi'(t,w,\sigma_1,\sigma_2,\theta_1,\theta_2) = -itE + \sigma_1(i\theta_1 - i\hat{\theta}_w(t) - \varphi(w(t)) + \sigma_2(i\theta_2 - \varphi(w)) - i\theta_1 - i\theta_2. \tag{34}$$

Proof Introducing the cut-off function χ in the integral (32) only changes the integral by $O(k^{-\infty})$. Within the support of the cut-off function, we may use the K-coordinates.

Then phase function Ψ can be written as (within the coordinate patch):

$$\begin{aligned}\Psi &= -itE + \sigma_1(i\theta_1 - i\hat{\theta}_w(t) - i\theta_w + \psi(0, w(t)) - \varphi(w(t)) \\ &\quad + \sigma_2(i\theta_2 + i\theta_w + \psi(w, 0) - \varphi(w)) - i\theta_1 - i\theta_2 \\ &= -itE + \sigma_1(i\widetilde{\theta}_1 - i\hat{\theta}_w(t) - \varphi(w(t)) + \sigma_2(i\widetilde{\theta}_2 - \varphi(w)) - i\widetilde{\theta}_1 - i\widetilde{\theta}_2\end{aligned}$$

where $\widetilde{\theta}_1 = \theta_1 - \theta_w$ and $\widetilde{\theta}_2 = \theta_2 + \theta_w$. We note $\psi(0, w) = 0$ due to the choice of K-frame (31). After the change of variables, we see the phase Ψ does not depend on θ_w. Hence we may perform the θ_w integral, and rewrite $\widetilde{\theta}_i$ as θ_i, to get the reduced phase function Ψ'. □

Proposition 5.2 *The critical points for Ψ' (34) are as following:*
(1) If $z \notin H^{-1}(E)$, there is no critical points.
(2) If $z \in H^{-1}(E)$ but $z \notin \mathcal{P}_E$, then the only critical point corresponds to $t = 0$.
(3) If $z \in H^{-1}(E)$ and $z \in \mathcal{P}_E$, then for each $n \in \mathbb{Z}$, there is a critical point with $t = nT_z$, where T_z is the primitive period of g^t at z.

Proof We will prove that the critical points for Ψ' (32) are given by

$$w = 0,\ w(t) = 0,\ \sigma_1 = \sigma_2 = 1,\ \theta_1 = \hat{\theta}_0(t),\ \theta_2 = 0.$$

Taking derivatives of σ_1 and σ_2, we need to have

$$i\theta_1 - i\hat{\theta}_w(t) - \varphi(w(t) = 0, \quad i\theta_2 - \varphi(w) = 0.$$

Hence

$$\theta_1 = \hat{\theta}_0(t), \ \theta_2 = 0,$$

Thus, we may work in a neighborhood of x from now on.

Taking derivatives in θ_1 and θ_2 and setting them to zero, we get

$$\sigma_1 = 1, \quad \sigma_2 = 1.$$

Taking derivative in t and setting it to zero, we have

$$\frac{\partial \Psi'}{\partial t} = -iE + i\sigma_1 \frac{d\hat{\theta}_w(t)}{dt} = -i(E - \sigma_1 H(0)).$$

Thus, using $\sigma_1 = 1$, we have $E = H(0)$.

Finally, taking derivatives in w, we have

$$\frac{\partial \Psi'}{\partial w} = -i\sigma_1 \partial_w \hat{\theta}_w(t) = -i\partial_w \theta_w(T)$$

where T is a period. Since $\hat{g}^T$ preserves horizontal space, and ∂_w is in the horizontal space at $x = (0; 0)$, hence

$$\partial_w \theta_w(T) = \langle \alpha|_x, (\hat{g}^T|_x)_* \partial_w \rangle = \langle \alpha|_x, \partial_w \rangle = \langle d\theta, \partial_w \rangle = 0.$$

$\square$

5.2 Determinant of Hessian of Ψ'

Let T be a period of g^t at z (possibly zero). To compute the contribution at $t = T$, we will do a slight change of variables.

Lemma 5.3 *Define new integration variables*

$$t = T + t', \ w = g^{-t'} w', \ \theta_1 = \theta_1' - \hat{\theta}_{w'}(-t'), \ \theta_2 = \theta_2' + \hat{\theta}_{w'}(-t').$$

Then the Jacobian factor is 1*, and the phase function* Ψ_T *in the new variables is*

$$\begin{aligned}\Psi_T(t', w', \sigma_i, \theta_i') = &-i(T + t')E + \sigma_1(i\theta_1' - i\hat{\theta}_{w'}(T) - \varphi(w'(T)) + \sigma_2(i\theta_2' + \hat{\theta}_{w'}(-t') \\ &- \varphi(w'(-t'))) - i\theta_1' - i\theta_2'.\end{aligned}$$

(We will drop the prime from now on.)

Proof The Jacobian matrix is block-upper-triangular, with the $w - w'$ block having determinant 1, since g^t preserves the volume form.

The holonomy for flow $\hat{g}^t$ can be written as

$$\hat{\theta}_w(t) = \theta_w(t) - \theta_w(0) = \theta_{w'}(T) - \theta_{w'}(-t') = \hat{\theta}_{w'}(T) - \hat{\theta}_{w'}(-t').$$

$\square$

Lemma 5.4 *The Hessian matrix for $\Psi_T(t, w, \sigma_i, \theta_i)$ at $t = 0, w = 0, \sigma_i = 1, \theta_1 = \hat{\theta}_0(T), \theta_2 = 0$ is as*

$$Hess \Psi_T = \begin{array}{c} \\ \sigma_1 \\ \theta_1 \\ \sigma_2 \\ \theta_2 \\ t \\ w \end{array} \begin{array}{c} \begin{array}{cccccc} \sigma_1 & \theta_1 & \sigma_2 & \theta_2 & t & w \end{array} \\ \begin{bmatrix} 0 & i & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial_{tt}\Psi_T & \partial_{tw}\Psi_T \\ 0 & 0 & 0 & 0 & \partial_{wt}\Psi_T & \partial_{ww}\Psi_T \end{bmatrix} \end{array}.$$

In particular, at this critical point, we have

$$\det Hess \Psi_T = \det \begin{pmatrix} \partial_{tt}\Psi_T & \partial_{tw}\Psi_T \\ \partial_{wt}\Psi_T & \partial_{ww}\Psi_T \end{pmatrix}.$$

Proof The calculation is very similar to that in the proof of Proposition 5.2, and is therefore omitted. $\square$

5.3 Quadratic Approximation to the Phase

To compute the Hessian of the phase function Ψ_T in t and w, suffice to set σ_i, θ_i to their critical value, and compute the Taylor expansion of Ψ_T to second order. Thus, we get

$$\Psi'_T(t, w) := -i(T + t)E - i\hat{\theta}_w(T) - \varphi(w(T)) + i\hat{\theta}_w(-t) - \varphi(w(-t)).$$

We will consider second order Taylor expansion in each term. We write $\simeq$ for equal modulo cubic order term.

Suppose H has Taylor expansion

$$H(w) = E + (\alpha\bar{w} + w\bar{\alpha}) + O(|w|^2).$$

We define the corresponding H_{BF} for the osculating BF space $\mathbb{C}^m \cong T_z M$, as the linear term of H:

$$H_{BF}(w) = \alpha\bar{w} + w\bar{\alpha}.$$

We denote the BF model potential as $\varphi_{BF}(z) = |z|^2$. Let $\hat{g}^t_{BF}$ be the flow generated by H_{BF} on $X_{BF} = \mathbb{C}^m \times S^1$, such that

$$\hat{g}^t_{BF}(w; \theta_w) = (w(t)_{BF}; \theta_w + \hat{\theta}_w(t)_{BF}).$$

Then, we have the following comparison result

Proposition 5.5 *(1)* $\hat{\theta}_w(-t) - tE = \hat{\theta}_w(-t)_{BF} + O_3 = \frac{1}{2}(\alpha\bar{z} + z\bar{\alpha})t.$
(2) $\varphi(w(T)) = |Dg^T w|^2 + O_3.$
(3) $\varphi(w(-t)) = |w(-t)_{BF}|^2 + O_3 = |w + i\alpha t|^2 + O_3.$

Proof (1) $\hat{\theta}_w(-t) = \int_0^t \frac{1}{2} d^c\varphi(\xi_H)|_{w(s)} ds + tH(w)$. Since $d^c\varphi|_w = O(|w|)$ and the integral interval is first order in t, hence

$$\int_0^t \frac{1}{2} d^c\varphi(\xi_H)|_{w(s)} ds = t\frac{1}{2} d^c\varphi(\xi_H)|_w + O_3$$
$$= t\langle \frac{1}{2} d^c\varphi|_w, \xi_H|_0\rangle + O_3 = \int_0^t \frac{1}{2} d^c\varphi_{BF}(\xi_{H_{BF}})|_{w(s)} ds + O_3.$$

And $tH(w) = t(E + H_{BF}(w)) + O_3$. Hence

$$\hat{\theta}_w(-t) - tE = \int_0^t \frac{1}{2} d^c\varphi_{BF}(\xi_{H_{BF}})|_{w(s)} ds + t(E + H_{BF}(w)) - tE + O_3 = \hat{\theta}_w(-t)_{BF} + O_3.$$

Finally, we may use Lemma 4.2 to compute the increment in θ.

(2) Since $\varphi(w) = |w|^2 + O(|w|^3)$ and $w(T) = g^T(w) = g^T(0) + Dg^T w + O(|w|^2) = Dg^T w + O(|w|^2)$, hence

$$\varphi(w(T)) = |Dg^T w|^2 + O_3$$

(3) Since $\xi_H = -i\alpha\partial_z + i\bar{\alpha}\partial_{\bar{z}} + O(|z|)$, we have $w(-t) = w + i\alpha t + O_2$, hence

$$\varphi(w(-t)) = |w + i\alpha t|^2 + O_3 = |w(-t)_{BF}|^2 + O_3.$$

□

Proposition 5.6

$$\hat{\theta}_w(T) = \hat{\theta}_0(T) + O(|w|^3).$$

Proof The proof is rather long, so we break it up into the following two Lemmas.

Lemma 5.7 *There exists a neighborhood $V \subset U$ of z, such that for any $w \in V$, and any path $\gamma : [0, 1] \to V$ from z to w, we have*

$$\hat{\theta}_w(T) = \hat{\theta}_0(T) - \int_\gamma \frac{1}{2} d^c \varphi + \int_{g^T(\gamma)} \frac{1}{2} d^c \varphi.$$

Proof We only give proof for $T = nT_z$, $n > 0$, the $n \le 0$ case is analogous. Let $\{(U_i, e_i, \varphi_i)\}_{i=1}^n$ be a sequence of coordinate patch U_i, such that there exists a partition of $[0, T]$: $0 = t_0 < t_1 < \cdots < t_n = T$, such that U_i covers the ith segment of the orbit $O_i = \{g^s z \mid t_{i-1} \le s \le t_i\}$, and $e_i \in \Gamma(U_i, L)$ are non-vanishing holomorphic sections, and $e^{-\varphi_i} = \|e_i\|^2$. Without loss of generality, we may take $U_1 = U$. We identify index $n + i$ with i.

Since $g^{t_i} z \in U_i \cap U_{i+1}$ for $0 \le i \le n$, hence

$$z \in V := \bigcap_{i=0}^n g^{-t_i}(U_i \cap U_{i+1}).$$

For any $w \in V$, let $\gamma : [0, 1] \to V$ be a path from z to w. Let

$$\gamma_0 = \gamma, \quad \gamma_i = g^{t_i} \gamma.$$

Then

$$Im(\gamma_i) \subset U_i \cap U_{i+1}, \forall 0 \le i \le n.$$

Over $U_i \cap U_{i+1}$, define transition function $g_i = \log(e_{i+1}/e_i)$, such that $g_i = a_i + \sqrt{-1} b_i$, with $b_i(g^{t_i} z) \in [0, 2\pi)$. Then we have

$$\|e_{i+1}\| = |g_i| \|e_i\| \Rightarrow e^{-\frac{1}{2}\varphi_{i+1}} = e^{a_i} e^{-\frac{1}{2}\varphi_i} \Rightarrow \varphi_{i+1} - \varphi_i = -2a_i.$$

Over U_i, let $\theta_i = e_i^* / \|e_i^*\|$ be the section in the co-circle bundle X. Then over $U_i \cap U_{i+1}$, we have

$$\log(e_{i+1}^*/e_i^*) = 1/g_i = e^{-a_i - \sqrt{-1} b_i} \Rightarrow \theta_{i+1} - \theta_i \equiv -b_i \mod 2\pi.$$

where we used additive notation for section valued in S^1.

Then, the holonomy can be expressed using Lemma 3.5 in each coordinate patch U_i

$$\hat{\theta}_w(T) = \theta_w(T) - \theta_w = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{1}{2} \langle d^c \varphi_i, \xi_H \rangle|_{g^s w} ds - (t_{i+1} - t_i) H(w) + b_i(g^{t_i} w).$$

Thus, we may take the difference

$$\begin{aligned}\hat{\theta}_w(T) - \hat{\theta}_0(T) = &\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{1}{2} \langle d^c \varphi_i, \xi_H \rangle|_{g^s w} ds - \int_{t_{i-1}}^{t_i} \frac{1}{2} \langle d^c \varphi_i, \xi_H \rangle|_{g^s z} ds - (t_{i+1} - t_i)(H(w) - H(z)) \\ &+ \sum_{i=1}^n b_i(g^{t_i} w) - b_i(g^{t_i} z)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \int_0^1 \int_{t_{i-1}}^{t_i} \omega(\partial_t, \partial_s) dt ds - (t_{i+1} - t_i)(H(w) - H(z)) \\
&\quad + \sum_{i=1}^{n} - \int_{\gamma_{i-1}} \frac{1}{2} d^c \varphi_i + \int_{\gamma_i} \frac{1}{2} d^c \varphi_i + \sum_{i=1}^{n} \int_{\gamma_i} db_i \\
&= \sum_{i=1}^{n} \int_0^1 \int_{t_{i-1}}^{t_i} dH(\partial_s) dt ds - (t_{i+1} - t_i)(H(w) - H(z)) \\
&\quad - \int_{\gamma_0} \frac{1}{2} d^c \varphi_1 + \sum_{i=1}^{n} \int_{\gamma_i} \frac{1}{2} d^c (\varphi_i - \varphi_{i+1}) + \int_{\gamma_n} \frac{1}{2} d^c \varphi_{n+1} + \sum_{i=1}^{n} \int_{\gamma_i} db_i \\
&= - \int_{\gamma_0} \frac{1}{2} d^c \varphi_1 + \int_{\gamma_n} \frac{1}{2} d^c \varphi_{n+1} + \sum_{i=1}^{n} \int_{\gamma_i} (d^c a_i + db_i) \\
&= - \int_{\gamma_0} \frac{1}{2} d^c \varphi_1 + \int_{\gamma_n} \frac{1}{2} d^c \varphi_1
\end{aligned}
$$

where in the last step, we used

$$
d^c(a_i + \sqrt{-1} b_i) = d(\sqrt{-1} a_i - b_i) \Rightarrow d^c a_i = -db_i.
$$

□

Lemma 5.8 *For any fixed path $\gamma : [0, 1] \to U$ starting from 0, and for any $1 \gg \epsilon > 0$, we have*

$$
\int_{\gamma([0,\epsilon])} d^c \varphi = \int_0^\epsilon \langle d^c \varphi, \dot{\gamma}(s) \rangle ds = O(\epsilon^3)
$$

Proof If a path $\gamma : [0, 1] \to U$ with $\gamma(0) = 0$ and $\gamma(1) = w$ is a straight-line, then

$$
\int_\gamma d^c \varphi = O(|w|^3).
$$

Indeed, consider the Taylor expansion of $\varphi(z)$ at $z = 0$,

$$
\varphi(z) = |z|^2 + O(|z|^3)
$$

then

$$
d^c \varphi = -2 \sum_i |z_i|^2 d\theta_i + \sum_i (O(|z|^2) dz_i + O(|z|^2) d\bar{z}_i).
$$

However, along a straight line path from 0 to w, θ_i is constant, hence the leading term of $d^c \varphi$ vanishes in the integral. For the remainder term, we have $|\int_\gamma dz_i| = O(|w|)$, hence proving the claim.

Next, we consider a general path as in the statement of the lemma. For each ϵ, we may consider the straight-line path $\beta : [0, \epsilon] \to U$ from 0 to $\gamma(\epsilon)$. From the previous claim, we know $\int_{\beta(\epsilon)} d^c \varphi = O(\epsilon^3)$. Let

$$
\Sigma_\epsilon : [0, \epsilon] \times [0, 1] \to U, (t, u) \mapsto u\gamma(t) + (1 - u)\beta(t).
$$

Then, we may verify that

$$\int_{\gamma([0,\epsilon])} d^c\varphi < C\left|\int_{\Sigma_\epsilon}\omega\right| + O(\epsilon^3) < O(\epsilon^3).$$

where the estimate of $\int_\Sigma \omega = 2\sum_i \int_\Sigma dx_i \wedge dy_i$ can be done by noting for any smooth function f,

$$\int_0^\epsilon f(x)dx - \epsilon\frac{1}{2}(f(0)+f(\epsilon)) = O(\epsilon^3).$$

$\square$

Using above two lemma, we have

$$\hat\theta_w(T) = \hat\theta_0(T) = -\int_\gamma \frac{1}{2}d^c\varphi + \int_{g^T(\gamma)}\frac{1}{2}d^c\varphi = O(|w|^3) + O(|g^T w|^3) = O(|w|^3).$$

This finishes the proof for Proposition 5.6. $\square$

5.4 Reduction to Osculating BF Model

We continue the calculation of the contribution to the stationary phase integral for period T orbit. The reduced phase function $\Psi'_T(t,w)$ has the following expansion:

$$\begin{aligned}\Psi'_T(t,w) &= -iTE - i\hat\theta_0(T) + it\mathrm{Re}(\alpha\bar w) - |w+i\alpha t|^2/2 - |Dg^T w|^2/2 + O_3.\\ &= -iTE - i\hat\theta_0(T) + iw\bar\alpha t - |w|^2/2 - |\alpha t|^2/2 - |Dg^T w|^2/2 + O_3.\end{aligned}$$

We may write the critical value as

$$\Psi'_T(0,0) = \Psi_T|_{crit} = -iTE - i\hat\theta_0(T) = -i\theta_0^h(T)$$

using holonomy phase of the horizontal flow (33).

The leading term of the stationary integral can be obtained by the following model result on BF space.

Proposition 5.9 *Let $H = \alpha\bar z + z\bar\alpha$. Let $\mathcal{A}:\mathbb{C}^m \to \mathbb{C}^m$ be a symplectic linear map, $\mathcal{A}w = Pw + Q\bar w$. Suppose ξ_H is invariant under $\mathcal{A}$. Then*
(1)

$$\begin{aligned}&(\det P^*)^{1/2}\left(\frac{k}{2\pi}\right)^{2m}\int_{\mathbb{C}^m} e^{k(itw\overline{\alpha}-|w|^2/2-t^2|\alpha|^2/2-|\mathcal{A}w|^2/2)}d\,\mathrm{Vol}_{\mathbb{C}^m}(w)\\ &= \hat{\mathcal{K}}_{k,\mathcal{A}}((0;0),\hat g^t(0;0))\end{aligned}$$

$$= (\det P)^{-1/2} \left(\frac{k}{2\pi}\right)^m e^{-kt^2(|\alpha|^2 - \bar{\alpha} P^{-1} Q \bar{\alpha})/2}$$

where the metaplectic representation kernel $\hat{\mathcal{K}}_{k,\mathcal{A}}(\hat{z}, \hat{w})$ *is defined in* (28).

(2)

$$\int_{\mathbb{R}} \hat{\mathcal{K}}_{k,\mathcal{A}}((0;0), \hat{g}^t(0;0)) dt = \left(\frac{k}{2\pi}\right)^{m-1/2} (\det P)^{-1/2} (\bar{\alpha} P^{-1} \alpha).$$

Proof (1) We note that

$$\left(\frac{k}{2\pi}\right)^m e^{k(-|\mathcal{A}w|^2/2)} = \hat{\Pi}_k(0, (\mathcal{A}w; 0)),$$

and

$$\left(\frac{k}{2\pi}\right)^m e^{k(itw\bar{\alpha} - |w|^2/2 - t^2|\alpha|^2/2)} = \hat{\Pi}_k(\hat{g}^{-t}(w;0), 0) = \hat{\Pi}_k((w;0), \hat{g}^t(0;0)).$$

Hence by Proposition 4.4, we have

$$\begin{aligned}
&(\det P^*)^{1/2} \left(\frac{k}{2\pi}\right)^{2m} \int_{\mathbb{C}^m} e^{k(itw\bar{\alpha} - |w|^2/2 - t^2|\alpha|^2/2 - |\mathcal{A}w|^2/2)} d\,\mathrm{Vol}_{\mathbb{C}^m}(w) \\
&= (\det P^*)^{1/2} \int_{\mathbb{C}^m} \hat{\Pi}_k(0, (\mathcal{A}w; 0)) \hat{\Pi}_k((w;0), \hat{g}^t(0;0)) dw \\
&= \hat{\mathcal{K}}_{k,\mathcal{A}}((0;0), \hat{g}^t(0;0)).
\end{aligned}$$

And the last line follows by $\hat{g}^t(0;0) = (-i\alpha t; 0)$ and definition for $\hat{\mathcal{K}}_{k,\mathcal{A}}$.

(2) Next, we use the fact that ξ_H is preserved by $\mathcal{A}$, i.e.

$$\begin{pmatrix} -i\alpha \\ i\bar{\alpha} \end{pmatrix} = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} \begin{pmatrix} -i\alpha \\ i\bar{\alpha} \end{pmatrix}$$

Thus

$$\alpha = P\alpha - Q\bar{\alpha} \tag{35}$$

hence

$$|\alpha|^2 - \bar{\alpha} P^{-1} Q \bar{\alpha} = |\alpha|^2 - \bar{\alpha} P^{-1}(P\alpha - \alpha) = \bar{\alpha} P^{-1} \alpha$$

Then, we have

$$\left(\frac{k}{2\pi}\right)^m (\det P)^{-1/2} \int_{\mathbb{R}} e^{-k\frac{1}{2}t^2(\bar{\alpha}P^{-1}\alpha)} dt = \left(\frac{k}{2\pi}\right)^{m-1/2} (\bar{\alpha} P^{-1} \alpha)^{-1/2} (\det P)^{-1/2}$$

□

Combining all the steps before, we have proven the following proposition.

Proposition 5.10 *Let $z \in M$ be a periodic point for the flow ξ_H and $H(z) = E$, then*

$$\Pi^E_{k,f}(z,z) = \sum_{n\in\mathbb{Z}} \hat{f}(nT_z)e^{-ikn\theta^h_z}\mathcal{G}_n\left(\frac{k}{2\pi}\right)^{m-1/2}(1+O(k^{-1}))$$

where if $Dg^{nT_z}|_z$ in K-coordinate at z can be written as $\begin{pmatrix} P_n & Q_n \\ \bar{Q}_n & \bar{P}_n \end{pmatrix}$, then

$$\mathcal{G}_n = (\det P_n)^{-1/2}(\bar{\alpha}P_n^{-1}\alpha)^{-1/2}.$$

6 Proof of Proposition 1.6

The issue at hand is the regularity of the measures $\mu_k^{z,1,E}$ defined on test functions $f \in \mathcal{S}(\mathbb{R})$ with $\hat{f} \in C_0^\infty(\mathbb{R})$ in Theorem 2.2. It is only an interesting question when $z \in \mathcal{P}_E$. In this case,

$$\int_{\mathbb{R}} f\mu_k^{z,1,E} = \left(\frac{k}{2\pi}\right)^{m-1/2}\sum_{n\in\mathbb{Z}} \hat{f}(nT_z)\;\mathcal{G}_n(z)e^{-ink\theta^h_z} + O(k^{m-3/2}).$$

Unravelling the Fourier transform gives that, in the sense of distributions,

$$d\mu_k^{z,1,E}(x) = \left(\frac{k}{2\pi}\right)^{m-1/2}\sum_{n\in\mathbb{Z}} e^{inT_zx}\;\mathcal{G}_n(z)e^{-ikn\theta^h_z}dx + O(k^{m-3/2}).$$

The proposition asserts first that this series converges absolutely and uniformly when the orbit through z is real hyperbolic. To prove this we need to consider the behavior of the matrix element $\bar{\alpha}P_n^{-1}\alpha$ and the determinant $\det P_n$ as $n \to \infty$, where as in (7)

$$P_n := P_J S^n P_J : T_z^{(1,0)}M \to T_z^{(1,0)}M.$$

We first develop the symplectic linear algebra introduced in Sect. 3.1.

6.1 *Matrix Elements and Determinants of Positive Definite Symplectic Matrices*

We are interested in $P_J S P_J$ with $P_J = \frac{1}{2}(I - iJ)$. We also use the notation $\langle\alpha,\beta\rangle = \bar{\beta}^t \cdot \alpha$ for the sesquilinear inner product.

First we prove

Proposition 6.1 *If S is positive definite symmetric symplectic, with invariant vector ξ and $\alpha = P_J\xi$, and if the spectrum of S is $\{e^{\lambda_j}, e^{-\lambda_j}\}_{j=1}^n$ with $\lambda_j \geq 0$ then*

$$\begin{cases} (i) \quad [P_J S P_J]^{-1}\alpha = \alpha, \\ (ii) \ \det P_J S P_J|_{T_0^{1,0}\mathbb{R}^{2n}} = \prod_{j=1}^n [\cosh \lambda_j]. \end{cases}$$

Proof The proof is through a series of lemmas:

Lemma 6.2 *If S is positive definite symplectic, then*

$$P_J S P_J = \frac{1}{2} P_J (S + S^{-1}) = \frac{1}{2}(S + S^{-1}) P_J$$

Proof

$$\begin{aligned} P_J S P_J &= \tfrac{1}{4}(I - iJ)S(I - iJ) = \tfrac{1}{4}[S - iJS - iSJ - JSJ] \\ &= \tfrac{1}{4}[S + S^{-1}] - \tfrac{i}{4}[J[S + S^{-1}] = \tfrac{1}{4}\left((S + S^{-1}) - iJ(S + S^{-1})\right) = \tfrac{1}{2} P_J (S + S^{-1}). \end{aligned}$$

since $JSJ = -S^{-1}$ if S is symmetric. Also,

$$J(S + S^{-1}) = JS + SJ = (S^{-1} + S)J$$

so that $P_J(S + S^{-1}) = (S + S^{-1})P_J$. □

Lemma 6.3 *Let S be positive definite symmetric symplectic and e_j be eigenvectors of S for eigenvalues $\lambda_1, \ldots, \lambda_n$. Consider the basis $P_J e_k$ of $H_J^{1,0}$. Then*

$$[P_J S P_J] P_J e_k = \cosh(\lambda_j) P_J e_k,$$

and $[P_J S P_J]^{-1} = P_J[S + S^{-1}]^{-1} P_J$.

Proof Follows from the previous lemma and the fact that $(S + S^{-1})$ commutes with P_J:

$$[P_J S P_J] P_J e_k = \frac{1}{2} P_J (S + S^{-1}) e_k = \frac{1}{2}(e^{\lambda_j} + e^{-\lambda_j}) P_J e_k = \cosh(\lambda_j) P_J e_k.$$

□

Statement (i) of the Proposition follows from the fact that

$$[P_J S P_J]\alpha = \frac{1}{2}(1 + 1)\alpha = \alpha.$$

Statement (ii) follows from the fact that the eigenvalues of $P_J S P_J$ are $\cosh \lambda_j$ by Lemma 6.3. □

6.2 Strong Hyperbolicity Hypothesis

Let z be a periodic point of the Hamiltonian flow g^t. Under this hypothesis, we have the following result.

Proposition 6.4 *If* $\dim_{\mathbb{C}} M = m > 1$, *and z be a periodic point with primitive period T, satisfying the strong hyperbolic hypothesis. Then*

$$\sum_{n\in\mathbb{Z}} |\mathcal{G}_n(z)| < \infty.$$

Proof Let the spectrum of $S := Dg^T$ be $\{e^{\lambda_j}, e^{-\lambda_j}\}_{j=1}^m$, with $\lambda_1 = 0$ and $\lambda_j > 0$ for $j = 2, \ldots, n$. Then, recall that

$$\mathcal{G}_n(z) = [\det(P_J S^n P_J)\langle(P_J S^n P_J)^{-1}\alpha, \alpha\rangle]^{-1/2}.$$

Then, from previous section, we have $\det(P_J S^n P_J) = \prod_{j=1}^n \cosh(n\lambda_j)$, and $\langle(P_J S^n P_J)\alpha, \alpha\rangle = \langle\alpha, \alpha\rangle$ independent of n. Since $\lambda_j > 0$ for $j = 2, \ldots, m$, hence

$$|\mathcal{G}_n| = |\det(P_J S^n P_J)\langle\alpha, \alpha\rangle|^{-1/2} < Ce^{-|n|\sum_j \lambda_j}$$

for some positive constant C. Thus the sum $\sum_{n\in\mathbb{Z}} |\mathcal{G}_n(z)|$ converges exponentially fast. □

6.3 Proof of Proposition 1.6

By Proposition 6.4, the family of measures

$$d\nu_T(\lambda) := \sum_{|n|\leq T} \rho_T(nT(z))e^{-i\lambda nT_z}e^{-ikn\theta_z^h(T_z)}\,\mathcal{G}_n(z)d\lambda, \quad (T \in \mathbb{R}_+)$$

converges in the weak* sense of distributions on the space $\mathcal{S}(\mathbb{R})$ of Schwartz functions to the limit distribution,

$$d\nu(\lambda) := \sum_{n\in\mathbb{Z}} e^{-i\lambda nT_z}e^{-ikn\theta_z^h(T_z)}\,\mathcal{G}_n(z)d\lambda,$$

since the coefficients $\mathcal{G}_n(z)$ are bounded in n and by dominated convergence,

$$\int_{\mathbb{R}} f(\lambda)d\nu_T(\lambda) = \sum_{|n|\leq T} \rho_T(nT(z))\hat{f}(nT(z))\,\mathcal{G}_n(z) \to \sum_{n\in\mathbb{Z}} \hat{f}(nT(z))\,\mathcal{G}_n(z),$$

where the sum on the right side converges absolutely.

7 Proof of Theorem 1.7

In this section we apply Theorem 2.2 and a Tauberian theorem to prove Theorem 1.7. We are concerned with the Weyl sums,

$$\Pi_{k,[E_1,E_2]}(z) = \int_{E_1}^{E_2} d\mu_k^{z,1,E} = \sum_{j:k(\mu_{kj}-H(z))\in[E_1,E_2]} \Pi_{k,j}(z).$$

The basic idea is to convolve $\mathbf{1}_{[E_1,E_2]}$ with a well-chosen Schwartz test function depending on (h, T), apply Theorem 2.2 and then estimate the remainder.

We consider both families of measures of (3), μ_k^z and $\mu_k^{z,1,E}$. The main difference is the range of eigenvalues involved. The measures μ_k^z have a fixed compact support, the range $H(M) = [H_{\min}, H_{\max}]$ of H, and the mean level spacing between the k^m point masses μ_{kj} is k^{-m}. The measures $\mu_k^{z,1,E}$ are scaled versions,

$$\mu_k^{z,1,E}[-M, M] = \sum_{j:|\mu_{jk}-E|<\frac{M}{k}} \Pi_{kj}(z),$$

and the mean level spacing between the point masses is k^{-m+1}. Of course,

$$\sum_{j:|\mu_{jk}-E|<\frac{M}{k}} \Pi_{kj}(z) = \mu_k^{z,1,E}[-M, M] = \mu_k^z\left[\frac{-M}{k}, \frac{M}{k}\right], \tag{36}$$

As a preliminary, we quote a result from [19, Theorem 3]:

Theorem 7.1 *Let E be a regular value of H and $z \in H^{-1}(E)$. If ϵ is small enough, such that the Hamiltonian flow trajectory starting at z does not return to z for time $|t| < 2\pi\epsilon$, then for any Schwarz function $f \in \mathcal{S}(\mathbb{R})$ with $\hat{f}$ supported in $(-\epsilon, \epsilon)$ and $\hat{f}(0) = \int f(x)dx = 1$, and for any $\alpha \in \mathbb{R}$ we have*

$$\int_{\mathbb{R}} f(x)d\mu_k^{z,1,\alpha}(x) = \left(\frac{k}{2\pi}\right)^{m-1/2} e^{-\frac{\alpha^2}{\|\xi_H(z)\|^2}} \frac{\sqrt{2}}{2\pi\|\xi_H(z)\|}(1 + O(k^{-1/2})).$$

There is a further integrated version of the Weyl law with remainder,

$$\#\left\{j : |\mu_{kj} - E| \leq \frac{M}{k}\right\} = \frac{2M}{(2\pi)^n}\mathrm{Vol}(h^{-1}(E))k^{m-1} + o(k^{m-1}). \tag{37}$$

The constraint in the sum (36) is a 'codimension one' condition localizing around $H^{-1}(E)$. The extra integration in (37) gives an extra factor of $k^{-\frac{1}{2}}$ in the stationary phase expansion. Note that $\int_M \Pi_{kj}(z)dV(z) = \mathrm{Mult}(\bar{}_{\mathrm{kj}})$ (the multiplicity of the eigenvalue, generically equal to 1), so the integrated Weyl law does not deal with non-uniform weights $\Pi_{kj}(z)$. The integrated Weyl law (essentially contained in [3]).

The remainder estimate requires the use of a semi-classical Tauberian theorem for a sequence $\mu_k^{z,1,E}$ of measures. Before getting started, let us note some basic facts about this sequence. First, $\mu_k^{z,1,E}$ is not normalized to be a probability measure, but it is finite and could be normalized by dividing by its mass $\Pi_{h^k}(z) \simeq k^m + O(k^{m-1})$). In the following discussion, we divide by the mass. Second, note that $\Pi_{h^k}(z)^{-1} d\mu_k^{z,1,E}$ is a centered re-scaling of $\Pi_{h^k}(z)^{-1} d\mu_k^z$ (3). That is $D_k \tau_E d\mu_k^{z,1E} = d\mu_k^z$ where the dilation operator is defined by $D_k \nu(I) = \nu(kI)$ for any interval I and measure ν. Also $\tau_E f(x) = f(x-E)$. Now, μ_k^z is supported in $H(M)$ (the range of $H : M \to \mathbb{R}$), hence $\mu_k^{z,1,E}$ is supported in $k(H-E)(M)$. In [19] we studied $\Pi_{h^k}(z)^{-1}\mu_k^{z,\frac{1}{2},E} := D_{\sqrt{k}} \Pi_{h^k}(z)^{-1} \mu_k^{z,1,E}$, whose support is $\sqrt{k}(H-E)(M)$ and proved that it tends to a Gaussian. In particular, its Fourier transform is continuous at 0, and by Levy's continuity theorem (or by direct analysis), the sequence $\Pi_{h^k}(z)^{-1}\mu_k^{z,1/2,E}$ is tight. By comparison, $\Pi_{h^k}(z)^{-1}\mu_k^{z,1,E}$ is not tight, and indeed the $\Pi_{h^k}(z)^{-1}\mu_k^{z,1,E}([a,b]) \simeq k^{-\frac{1}{2}}$, so that the mass is spreading out to infinity and it does not weak* converge on $C_b(\mathbb{R})$.

Theorem 1.7 not only gives the leading order term but also the order of the remainder. As is well-known from work of Duistermaat–Guillemin, Ivrii, Safarov and others, obtaining a sharp remainder term requires the use of something similar to Fourier transform methods and in particular Fourier Tauberian theorems. As mentioned before, Theorem 1.7 is analogous to Safarov's non-classical pointwise Weyl asymptotics for the spectral function of a Laplace operator Δ, or more precisely, asymptotics on intervals $[\lambda, \lambda+1]$ for $\sqrt{-\Delta}$. The Q-notation is adopted from [14, 15]. Since we are working on phase space, Q involves closed orbits rather than loops in configuration space. However, we need to use a semi-classical Tauberian theorem rather than the homogeneous Tauberian theorem of [15], i.e. we are considering a sequence of measures $\mu_k^{z,1,E}$ on a fixed interval rather than a fixed measure on expanding intervals $[0, \lambda]$.

Semi-classical Tauberian theorems have been known for a long time. It is a classical fact that to obtain sharp remainder estimates, one must make use of the Fourier transform of the measures on long time intervals $[-T, T]$. A Tauberian theorem of the needed type is proved in [12], adapting the statement of Safarov's non-classical Weyl asymptotics to a semi-classical problem. This theorem does not quite apply to our setting for various reasons: (i) It assumes the sequence of measures have fixed compact support; (ii) it assumes the 'weights' or masses of the point masses are uniform. On the contrary, the 'weights' $\Pi_{k,j}(z)$ of $\mu_k^{z,1,E}$ are highly non-uniform in a way that is inconsistent with the hypotheses of the Tauberian Theorem of [12]. Consider the graph of the weights $\Pi_{k,j}(z)$ as a function of μ_{kj}, i.e. the coefficients of the point masses of μ_k^z (3). On average the weights are of order 1 since there are k^m terms and the total sum is $\Pi_k(z) \simeq \mathrm{Vol(M,\omega)k^m}$. But the weights are highly non-uniform:

(1) they peak when $\mu_{kj} \simeq H(z)$; indeed, it is shown inf [19, Theorem 1] that μ_k^z tends weakly to $\delta_{H(z)}$.
(2) By [19, Theorem 2], $\sum_{j:|\mu_{kj}-H(z)|<Mk^{-\frac{1}{2}}} \Pi_{k,j}(z) \sim Mk^m$ while the number of terms is of order $k^{m-\frac{1}{2}}$. Thus, on average, $\Pi_{k,j}(z)$ is of size $k^{\frac{1}{2}}$ in this eigenvalue range.

(3) Further, $\Pi_{k,j}(z) \lesssim k^{-C}$ when $|H(z) - \mu_{kj}| \geq Ck^{-\frac{1}{2}} \log k$. Hence, the weights decay rapidly when μ_{kj} lies outside of the range $|H(z) - \mu_{kj}| \leq Ck^{-\frac{1}{2}} \log k$. Consequently, the sequence of dilated measures $\mu_k^{z,1,E}$ concentrates in the sets $[-k^{\frac{1}{2}} \log k, k^{\frac{1}{2}} \log k]$.

Since we need to modify the Tauberian Theorem of [12] to accommodate the strong peaking of the weights around $H(z)$, we go through the modified proof in detail.

7.1 Mollifiers and Convolution

We use the following notation: Let $\rho_1 \in C_0^\infty(-1,1)$ satisfy $\rho_1(t) = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$, $\rho_1(-t) = \rho_1(t)$. We may also assume $\mathcal{F}\rho_1(\tau) \geq 0$ and $\mathcal{F}\rho_1(\tau) \geq \delta_0 > 0$ for $|\tau| \leq \epsilon_0$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the standard Fourier transform and its inverse,

$$\hat{f}(x) := (\mathcal{F}f)(x) = (2\pi)^{-1} \int f(t) e^{-itx} dt, \quad \check{f}(x) = (\mathcal{F}^{-1}f)(x) = \int f(t) e^{itx} dx$$

Then set,

$$\rho_T(\tau) = \rho_1\left(\frac{\tau}{T}\right), \quad \theta_T(x) := \hat{\rho}_T(x) = T\hat{\rho}_1(xT). \tag{38}$$

In particular, $\int \theta_T(x) dx = 1$ and $\theta_T(x) > T\delta_0$ for $|x| < \epsilon_0/T$. Let

$$\sigma_k^{z,1,E}(x) = \mu_k^{z,1,E}(-\infty, x].$$

7.2 Tauberian Theorem for $\mu_k^{z,1,E}$

In this section we determine the asymptotics of

$$\sigma_k^{z,1,E}(E_2) - \sigma_k^{z,1,E}(E_1) = \int_{E_1}^{E_2} d\mu_k^{z,1,E}(x) = \sum_{j : \frac{E_1}{k} \leq \mu_{jk} - E \leq \frac{E_2}{k}} \Pi_{k,j}(z).$$

We recall that the mean level spacings of $k(\mu_{k,j} - E)$ is k^{-m+1} so that the number of terms in the sum is of order k^{m-1}. The plan is to mollify the measures by convolution with θ_T (38), so that it suffices to determine the asymptotics of

$$\begin{aligned} &\sigma_k^{z,1,E} * \theta_T(E_2) - \sigma_k^{z,1,E} * \theta_T(E_1) \\ &+ \left(\sigma_k^{z,1,E}(E_2) - \sigma_k^{z,1,E}(E_1)\right) - \left(\sigma_k^{z,1,E} * \theta_T(E_2) - \sigma_k^{z,1,E} * \theta_T(E_1)\right) \end{aligned} \tag{39}$$

Since

$$\sigma_k^{z,1,E} * \theta_T(E_2) - \sigma_k^{z,1,E} * \theta_T(E_1) = \int_{E_1}^{E_2} \theta_{h,T} * d\mu_k^{z,1,E}(\lambda),$$

we have

$$\begin{aligned}&\left(\sigma_k^{z,1,E}(E_2) - \sigma_k^{z,1,E}(E_1)\right) - \left(\sigma_k^{z,1,E} * \theta_T(E_2) - \sigma_k^{z,1,E} * \theta_T(E_1)\right)\\ &= \int_{E_1}^{E_2} (\theta_T * d\mu_k^{z,1,E} - d\mu_k^{z,1,E}).\end{aligned} \tag{40}$$

First we consider the top terms of (39).

Proposition 7.2 *Assume that $H(z) = E$, $z \in \mathcal{P}_E$. Then*

$$\frac{d}{dx}(\sigma_k^{z,1,E} * \theta_T)(x) = \left(\frac{k}{2\pi}\right)^{m-1/2} \sum_{n\in\mathbb{Z}} \rho_T(nT_z) e^{-ixnT_z} e^{-ikn\theta_z^h(T_z)} \mathcal{G}_n(z) + O_T(k^{m-3/2}) \tag{41}$$

and

$$\begin{aligned}&\sigma_k^{z,1,E} * \theta_T(E_2) - \sigma_k^{z,1,E} * \theta_T(E_1)\\ &= k^{m-\frac{1}{2}} \int_{E_1}^{E_2} \sum_{|nT_z|\le T} \rho_T(nT_z) e^{-i\lambda nT_z} e^{-ikn\theta_z^h(T_z)}\, \mathcal{G}_n(z) d\lambda + O(k^{m-1})\ .\end{aligned}$$

Proof

$$\begin{aligned}\frac{d}{dx}(\sigma_k^{z,1,E} * \theta_T)(x) &= \int \theta_T(x-y) d\mu_k^{z,1,E}(y)\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \rho_T(-t) e^{-it(x-y)} \sum_j \delta_{k(\mu_{k,j}-E)}(y) \Pi_{k,j}(z) dy dt\\ &= \int_{\mathbb{R}} \rho_T(t) e^{-itx} \sum_j e^{itk(\mu_{k,j}-E)} \Pi_{k,j}(z) dt\\ &= \int_{\mathbb{R}} \rho_T(t) e^{-itx-itkE} U_k(t,z,z) dt\\ &= \left(\frac{k}{2\pi}\right)^{m-1/2} \sum_{n\in\mathbb{Z}} \rho_T(nT_z) e^{-ixnT_z} e^{-ikn\theta_z^h(T_z)} \mathcal{G}_n(z)(1+O(k^{-1})).\end{aligned}$$

where the last line follows from Theorem 2.2 to $f(y) = \theta_T(x-y)$. □

Corollary 7.3 *Under the strong hyperbolicity hypothesis (Definition 1.5), there exists constants $\gamma_0(z), C_1(T,z)$, such that*

$$\frac{d}{dx}(\sigma_k^{z,1,E} * \theta_T)(x) \le \left(\frac{k}{2\pi}\right)^{m-1/2} \gamma_0(z) + C_1(T,z) k^{m-3/2}.$$

Proof We start from (41), and let $T \to \infty$. By Proposition 6.4, the sum in (41) with ρ_T replaced by 1 converges absolutely. □

We now employ a semi-classical Fourier Tauberian theorem to estimate (40). In fact, since we already semi-classically scaled $d\mu_k^z$ by k, we do not need to scale again. We only refer to the Tauberian as semi-classical because it applies to a sequence $\mu_k^{z,1,E}$ of measures on a fixed interval rather than to a fixed measure on a dilated family of intervals as in the homogeneous Tauberian theorem.

The Tauberian theorem states:

Proposition 7.4 *There exist constant $\gamma(z), C(T, z)$ such that, for any $T > 0$,*

$$\int_{E_1}^{E_2} (\theta_T * d\mu_k^{z,1,E} - d\mu_k^{z,1,E}) \leq \frac{\gamma(z)}{T} k^{m-\frac{1}{2}} + C(T, z) k^{m-3/2}.$$

Together with Proposition 7.2 this gives

Corollary 7.5 *For any $T > 0$, there exist $\gamma_0(z, \tau), \gamma, C_1(T, z, \tau) > 0$ so that*

$$\sigma_k^{z,1,E}(E_2) - \sigma_k^{z,1,E}(E_1)$$
$$= \left(\frac{k}{2\pi}\right)^{m-\frac{1}{2}} \int_{E_1}^{E_2} \sum_{|nT_z| \leq T} \rho_T(nT_z) e^{-i\lambda n T_z} e^{-ikn\theta_z^h(T_z)} \mathcal{G}_n(z) d\lambda + \frac{1}{T} O(k^{m-\frac{1}{2}}) + O_T(k^{m-3/2}).$$

7.3 Proof of Proposition 7.4

As mentioned above, the hypotheses of [12, Theorem 3.1] do not hold in our setting. Hence we must extract from [12, Theorem 3.1] the key elements that pertain to our setting.

We have,

$$\begin{aligned}
\int_{E_1}^{E_2} (\theta_T * d\mu_k^{z,1,E} - d\mu_k^{z,1,E}) &= \int_{\mathbb{R}} (\mu_k([E_1, E_2] - \tau) - \mu_k[E_1, E_2]) \, \theta_T(\tau) d\tau \\
&= T \int_{\mathbb{R}} (\mu_k([E_1, E_2] - \tau) - \mu_k[E_1, E_2])) \, \hat{\rho}_1(\tau T) d\tau \\
&= T \int_{|\tau| \leq \frac{1}{T}} (\mu_k([E_1, E_2] - \tau) - \mu_k[E_1, E_2])) \, \hat{\rho}_1(\tau T) d\tau \\
&+ T \int_{|\tau| > \frac{1}{T}} (\mu_k([E_1, E_2] - \tau) - \mu_k[E_1, E_2])) \, \hat{\rho}_1(\tau T) d\tau \\
&=: I_1 + I_2.
\end{aligned}$$

Evidently, the key objects to estimate are the increments

$$\mu_k([E_1, E_2] - \tau) - \mu_k([E_1, E_2])$$

The key point is to prove the analogue of [12, Proposition 3.2]:

Proposition 7.6 *There exist constants $\gamma_1(z)$ and $C_1(T, z)$ such that, for any $T > 0$,*

$$|(\mu_k([E_1, E_2] - \tau) - \mu_k[E_1, E_2]))| \leq \gamma_1(z)\left(\frac{1}{T} + |\tau|\right) k^{m-\frac{1}{2}} + C_1(T, z) O(k^{m-3/2})$$

We now show that Proposition 7.6 implies Proposition 7.4.

Proof First, observe that Proposition 7.6 implies,

$$I_1 \leq \sup_{|\tau| \leq \frac{1}{T}} |\mu_k([E_1, E_2] - \tau) - \mu_k([E_1, E_2])|,$$

and Proposition 7.6 immediately implies the desired bound of Proposition 7.4 for $|\tau| \leq \frac{1}{T}$. For I_2 one uses that $\hat{\rho}_1 \in \mathcal{S}(\mathbb{R})$. Since $T \int_{|\tau| \geq \frac{1}{T}} \hat{\rho}_1(\tau T) d\tau \leq 1$, Proposition 7.6 implies,

$$I_2 \lesssim k^{m-\frac{1}{2}} \gamma_1(z) T \int \left(\frac{1}{T} + |\tau|\right) \hat{\rho}_1(T\tau) d\tau + C_1(T, z) O(k^{m-3/2}) T \int_{|\tau| > \frac{1}{T}} \hat{\rho}_1(\tau T) d\tau$$

If one changes variables to $r = T\tau$ one also gets the estimate of Proposition 7.4. □

We now prove Proposition 7.6.

Proof We need to estimate $(\mu_k[E_1, E_2] - \tau) - \mu_k[E_1, E_2]))$. The estimate depends both on the position of $[E_1, E_2]$ relative to the center of mass at 0 and on the position of τ. We recall the the total mass of $\mu_k = \mu_k^{z,1,E}$ on the complement of $[-\sqrt{k} \log k, \sqrt{k} \log k]$ is rapidly decaying in k. Hence we may assume that at least one of the following occurs:

- $[E_1, E_2] \cap [-\sqrt{k} \log k, \sqrt{k} \log k] \neq \emptyset$, i.e. $E_1 \geq -\sqrt{k} \log k$, $E_2 \leq \sqrt{k} \log k$.
- $[E_1, E_2] - \tau \cap [-\sqrt{k} \log k, \sqrt{k} \log k] \neq \emptyset$, i.e. $E_1 - \tau - \sqrt{k} \log k$, $E_2 - \tau \leq \sqrt{k} \log k$.

The proof is broken up into 3 cases: (1) $|\tau| \leq \frac{\epsilon_0}{T}$, (2) $\tau = \frac{\ell}{T}\epsilon_0$, (3) $\frac{\ell}{T}\epsilon_0 \leq \tau \leq \frac{\ell+1}{T}\epsilon_0$, for some $\ell \in \mathbb{Z}$.

(1) Assume $|\tau| \leq \frac{\epsilon_0}{T}$. Assume $\tau > 0$ since the case $\tau < 0$ is similar. Write

$$\mu_k([E_1, E_2] - \tau) - \mu_k[E_1, E_2]) = \int_{\mathbb{R}} [\mathbf{1}_{[E_1 - \tau, E_2 - \tau]} - \mathbf{1}_{[E_1, E_2]}](x) d\mu_k(x).$$

For T sufficiently large so that $\tau \ll E_2 - E_1$,

$$[\mathbf{1}_{[E_1-\tau, E_2-\tau]} - \mathbf{1}_{[E_1, E_2]}](x) = \mathbf{1}_{[E_1-\tau, E_1]} - \mathbf{1}_{[E_2-\tau, E_2]}.$$

We do not expect cancellation between the terms for arbitrary E_1, E_2, τ and therefore must show that each term satisfies the desired estimate. Since they are similar we only consider the $[E_1 - \tau, E_1]$ interval. Since for $|\tau| < \epsilon_0/T$, we have $\theta_T(\tau) > T\delta_0$, thus

$$\begin{aligned} \mu_k([E_1 - \tau, E_1]) &\leq \frac{1}{T\delta_0} \int_{\mathbb{R}} \theta_T(E_1 - x) d\mu_k(x) \\ &\sim \frac{1}{T\delta_0} \frac{d}{dx} (\sigma_k^{z,1,E} * \theta_T)(E_1) \\ &< \frac{\gamma_0(z)}{T\delta_0} k^{m-1/2} \end{aligned}$$

It follows that

$$|\mu_k([E_1, E_2] - \tau) - \mu_k[E_1, E_2])| \leq \frac{2\gamma_0(z)}{T\delta_0} k^{m-1/2}.$$

(2) Assume $\tau = \ell \frac{\epsilon_0}{T}$, $\ell \in \mathbb{Z}$. With no loss of generality, we may assume $\ell \geq 1$. Write

$$\begin{aligned} &\mu_k([E_1, E_2]) - \mu_k \left([E_1, E_2] - \frac{\ell}{T} \epsilon_0 \right) \\ &= \sum_{j=1}^{\ell} \mu_k \left([E_1, E_2] - \frac{j-1}{T} \epsilon_0 \right) - \mu_k \left([E_1, E_2] - \frac{j}{T} \epsilon_0 \right) \end{aligned}$$

and apply the estimate of (1) to upper bound the sum by

$$\frac{2\ell\gamma_0(z)}{T\delta_0} k^{m-1/2} = \frac{2\gamma_0}{\epsilon_0\delta_0} \tau k^{m-1/2}$$

(3) Assume $\frac{\ell}{T}\epsilon_0 \leq \tau \leq \frac{\ell+1}{T}\epsilon_0$ and $|\tau h| \leq \epsilon_1$ with $\ell \in \mathbb{Z}$. Write

$$\begin{aligned} \mu_k([E_1, E_2] + \tau) - \mu_k([E_1, E_2]) &= \mu_k([E_1, E_2] + \tau) - \mu_k([E_1, E_2] + \tfrac{\ell}{T}\epsilon_0) \\ &\quad + \mu_k([E_1, E_2] + \tfrac{\ell}{T}\epsilon_0) - \mu_k([E_1, E_2]). \end{aligned}$$

Apply (1) and (2) , it follows that

$$|\mu_k([E_1, E_2] + \tau) - \mu_k([E_1, E_2])| \leq \frac{2\gamma_0(z)}{\delta_0} \left(\frac{\tau}{\epsilon_0} + \frac{1}{T} \right) \gamma_0(\sigma, \lambda) k^{m-\frac{1}{2}}.$$

$\square$

8 Comparison with BPU

In this section we compare our formula for the leading coefficient in Theorem 2.2 with that in [2]. To do so, we need to introduce the notation and terminology of that article.

Let ϕ_τ^h be the horizontal lift of the Hamiltonian flow to X_h (denoted P in [2]). At each point $p \in P$, define $T_p^h P$ to be the horizontal subspace and Λ_p to be the positive definite Lagrangian subspace of $T_p^h P \otimes \mathbb{C}$ (i.e. the type $(1, 0)$ subspace). By the analysis of [3, p. 98] there exists a one-dimensional kernel $\mathcal{W}_p$ of this action, the line of ground states $\mathcal{W}_p \subset H_\infty(T_p^h P)$. A normalized section of the bundle $\mathcal{W} \to P$ defined by $\mathcal{W}_p$ is denoted by e_p. Further denote by $M_\tau : H_\infty(T_p^h P) \to H_\infty(T_p^h P)$ the metaplectic representation of the symplectic group of the horizontal space $H(T_p^h P)$.

Let Ξ denote the Hamilton vector field ξ_H. It is written in [2] that "Ξ acts on $H(T_p^h P)$ and hence on $H_\infty(T_p^h P)$ by via the Heisenberg representation. The action is by translations. The projection from $H_\infty(T_p^h(P))$ to generalized invariant vectors under Ξ is defined by

$$P_\Xi v := \int_{-\infty}^{\infty} e^{it\Xi} v dt$$

the projection from $H_\infty(T_p^h P)$ to the invariant vectors for the flow of Ξ p above z.

Further let Q be a first order pseudo-differential operator on $L^2(P)$ so that $\Pi Q \Pi = D \Pi M_H \Pi$ and so that $[Q, \Pi] = 0$. Let q be the symbol of Q, which generates a contact flow ϕ_t on P. Then the flow maps $\Lambda_p \to \Lambda_{\phi_t(p)}$ and M_τ maps e_p to a multiple of $e_{\phi_t(p))}$. Define $c(t)$ by $\Xi_q e_{\phi_t(p)} = ic(t) e_{\phi_t(p)}$.

Then the formula of [2] for the leading coefficient at a periodic orbit of period τ is

$$C_{\tau,0} = \frac{1}{2\pi^{n+1}} \langle M_\tau^{-1} e_{p_1}, P_\Xi(e_{p_1}) \rangle e^{-i \int_0^\tau (\sigma_{sub}(Q) + c(t)) dt}.$$

The approach of this paper is to replace $H_\infty(T_p^h)$ by the osculating Bargmann–Fock space, i.e. the Bargmann–Fock space on $H_z^{1,0} M$ which carries a complex structure and Hermitian metric and hence a Gaussian inner product. In effect, the quadratic part of the scaled phase of $U_k(t, z, z)$ replaces the symbol calculus. We do not use Q but the related operator in our setting is $\hat{H}_k$. The P_Ξ operator there corresponds to the dt integral near a period in our approach. We now verify that our formula agrees with theirs to the extent possible.

We would like to compare the expression (9) with the one in [2],

$$\langle M_T^{-1} e_0, P_\Xi e_0 \rangle = \langle \eta_{J, Dg^T} \Pi_J U_{Dg^T}^{-1} e_0, \int_{\mathbb{R}} g_*^{BF,\tau} e_0 d\tau \rangle = \eta_{J, Dg^T} \int_{\mathbb{R}} \langle U_{Dg^T}^{-1} e_0, g_*^{BF,\tau} e_0 \rangle d\tau$$

where g^τ is the BF translation (Heisenberg representation) of the constant vector field $\xi_H(0)$ by time τ. Here, we dropped the projection operator Π_J, since it is acting on $g_*^{BF,\tau} e_0$, which is holomorphic already.

Let

$$v = e^{-k|z|^2/2}$$

be the (unnormalized) coherent state centered at 0. We first review how Heisenberg group and Metaplectic group acts on it.
(i) Let $w \in \mathbb{C}^m$. Let $\beta(w)$ be translation by w. Then

$$[\beta(w)v](z) = e^{k[z\bar{w}-|z|^2/2-|w|^2/2]} = e^{k[i\mathrm{Im}(z\bar{w})-|z-w|^2/2]}$$

Indeed, it is centered at w, with a non-trivial phase factor $i\mathrm{Im}(z\bar{w})$.

(ii) Let $M = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} \in Sp_c$, with $M^{-1} = \begin{pmatrix} P^* & -Q^t \\ -Q^* & P^t \end{pmatrix}$. Then

$$(Mv)(z) = \frac{1}{(\det P)^{1/2}} e^{k[\frac{1}{2}z\bar{Q}P^{-1}z - \frac{1}{2}|z|^2]}.$$

And for our purpose, we also need

$$(M^{-1}v)(z) = \frac{1}{(\det P^*)^{1/2}} e^{k[-\frac{1}{2}zQ^*(P^*)^{-1}z - \frac{1}{2}|z|^2]}$$

Let $\Xi = -i\alpha\partial_z + i\bar{\alpha}\partial_{\bar{z}}$, the Hamiltonian vector field for $H = \alpha\bar{z} + \bar{\alpha}z$. Then, we can write $P_\Xi v$ as

$$(P_\Xi v)(z) = \int_{\mathbb{R}} \beta(-i\alpha t)v dt = \int_{\mathbb{R}} e^{k[itz\bar{\alpha}-|z|^2/2-|\alpha t|^2/2]}dt$$

It is possible to perform the Gaussian integral, then we get

$$(P_\Xi v)(z) = \sqrt{\frac{2\pi}{k|\alpha|^2}} e^{k[-|z|^2/2-(z\bar{\alpha})^2/2|\alpha|^2)]}$$

We will see, it is better not to evaluate the dt integral first.

Proposition 8.1

$$\langle M^{-1}v, P_\Xi v\rangle = \left(\frac{k}{2\pi}\right)^{-m-1/2} (\bar{\alpha}(P^*)^{-1}\alpha)^{-1/2}(\det P^*)^{-1/2}$$

The power of $\left(\frac{k}{2\pi}\right)$ does not matter, since we did not choose a normalized coherent state. The difference between P and P^* with previous result may be due to the difference of time $+T$ or $-T$ trajectories. Since we will sum time $\{nT \mid n \in \mathbb{Z}\}$ trajectories, the difference does not matter in the end.

Proof

$$\begin{aligned}\langle M^{-1}v, P_\Xi v\rangle &:= \int_{\mathbb{C}^m}\int_{\mathbb{R}} \frac{1}{(\det P^*)^{1/2}} e^{k[-zQ^*(P^*)^{-1}z-|z|^2/2]}\overline{e^{k[itz\bar\alpha-|z|^2/2-|\alpha t|^2/2]}}dtd\,\mathrm{Vol}(z)\\ &= \int_{\mathbb{R}}\int_{\mathbb{C}^m} e^{k[-it\bar z\alpha-zQ^*(P^*)^{-1}z/2-|z|^2-|\alpha t|^2/2]}d\,\mathrm{Vol}(z)dt\\ &= \int_{\mathbb{R}}\int_{\mathbb{C}^m} e^{-\frac{1}{2}k\Psi(t,z)}d\,\mathrm{Vol}(z)dt\end{aligned}$$

Let us do the complex Gaussian integral. The phase function is quadratic

$$\Psi = \begin{pmatrix} t & z^t & \bar z^t\end{pmatrix}\begin{pmatrix} |\alpha|^2 & 0 & -i\alpha^t\\ 0 & Q^*(P^*)^{-1} & I\\ -i\alpha & I & 0\end{pmatrix}\begin{pmatrix} t\\ z\\ \bar z\end{pmatrix}$$

We have

$$\det\begin{pmatrix} |\alpha|^2 & 0 & -i\alpha^t\\ 0 & Q^*(P^*)^{-1} & I\\ -i\alpha & I & 0\end{pmatrix} = \det\begin{pmatrix} |\alpha|^2 & i\alpha^t Q^*(P^*)^{-1} & -i\alpha^t\\ 0 & 0 & I\\ -i\alpha & I & 0\end{pmatrix}$$

$$= \det\begin{pmatrix} |\alpha|^2-\alpha^t Q^*(P^*)^{-1}\alpha & i\alpha^t Q^*(P^*)^{-1} & -i\alpha^t\\ 0 & 0 & I\\ 0 & I & 0\end{pmatrix} = (-1)^n(|\alpha|^2-\alpha^t Q^*(P^*)^{-1}\alpha)$$

Again, we use ξ_H is invariant under M, to get (35), taking conjugate we have

$$\bar\alpha^t = \bar\alpha^t P^* - \alpha^t Q^*$$

Hence

$$|\alpha|^2-\alpha^t Q^*(P^*)^{-1}\alpha = |\alpha|^2-(\bar\alpha^t P^*-\bar\alpha^t)(P^*)^{-1}\alpha = \bar\alpha^t(P^*)^{-1}\alpha$$

Thus, doing the complex Gaussian integral, and note that $(-1)^{n/2}$ from determinant Hessian, should cancels with i^n coming from the volume form, we get

$$\langle M^{-1}v, P_\Xi v\rangle = \left(\frac{k}{2\pi}\right)^{-m-1/2}(\bar\alpha(P^*)^{-1}\alpha)^{-1/2}(\det P^*)^{-1/2}.$$

□

References

1. Abraham, R., Marsden, J.E.: Foundations of Mechanics. Advanced Book Program. Benjamin/Cummings Publishing Co., Inc., Reading (1978)
2. Borthwick, D., Paul, T., Uribe, A.: Semiclassical spectral estimates for Toeplitz operators. Ann. Inst. Fourier (Grenoble) **48**(4), 1189–1229 (1998)
3. Boutet de Monvel, L., Guillemin, V.: The Spectral Theory of Toeplitz Operators. Annals of Mathematics Studies, vol. 99. Princeton University Press, Princeton (1981)
4. Boutet de Monvel, L., Sjöstrand, J.: Sur la singularité des noyaux de Bergman et de Szegö. Asterisque **34–35**, 123–164 (1976)
5. Daubechies, I.: Coherent states and projective representation of the linear canonical transformations. J. Math. Phys. **21**(6), 1377–1389 (1980)
6. de Gosson, M.: Symplectic Geometry and Quantum Mechanics. Operator Theory: Advances and Applications, vol. 166. Advances in Partial Differential Equations. Birkhäuser Verlag, Basel (2006)
7. Folland, G.: Harmonic Analysis in Phase Space. Annals of Mathematics Studies, vol. 122. Princeton University Press, Princeton (1989)
8. Long, Y.: Index Theory for Symplectic Paths with Applications. Progress in Mathematics, vol. 207. Birkhäuser Verlag, Basel (2002)
9. Lu, Z., Shiffman, B.: Asymptotic expansion of the off-diagonal Bergman kernel on compact Kähler manifolds. J. Geom. Anal. **25**(2), 761–782 (2015)
10. Paoletti, R.: Scaling asymptotics for quantized Hamiltonian flows. Int. J. Math. **23**(10), 1250102 (2012)
11. Paoletti, R.: Local scaling asymptotics in phase space and time in Berezin-Toeplitz quantization. Int. J. Math. **25**(6), 1450060 (2014), 40 pp
12. Petkov, V., Robert, D.: Asymptotique semi-classique du spectre d'hamiltoniens quantiques et trajectoires classiques periodiques. Commun. Partial Differ. Equ. **10**(4), 365–390 (1985)
13. Rubinstein, Y.A., Zelditch, S.: The Cauchy problem for the homogeneous Monge-Ampre equation, I. Toeplitz quantization. J. Differ. Geom. **90**(2), 303–327 (2012)
14. Safarov, Y.G.: Asymptotics of a spectral function of a positive elliptic operator without a nontrapping condition, (Russian) Funktsional. Anal. i Prilozhen. **22**, 53–65 (1988), 96; translation in Funct. Anal. Appl. **22**(3) (1988), 213–223 (1989)
15. Safarov, Y.G., Vassiliev, D.: The asymptotic distribution of eigenvalues of partial differential operators, Translated from the Russian manuscript by the authors. Translations of Mathematical Monographs, vol. 155. American Mathematical Society, Providence (1997)
16. Shiffman, B., Zelditch, S.: Asymptotics of almost holomorphic sections of ample line bundles on symplectic manifolds. J. Reine Angew. Math. **544**, 181–222 (2002)
17. Zelditch, S.: Index and dynamics of quantized contact transformations. Ann. Inst. Fourier **47**, 305–363 (1997), MR1437187, Zbl 0865.47018
18. Zelditch, S., Zhou, P.: Interface asymptotics of partial Bergman kernels on S^1-symmetric Kähler manifolds, to appear in J. Symp. Geom. arXiv:1604.06655
19. Zelditch, S., Zhou, P.: Central limit theorem for spectral partial Bergman kernels. arXiv:1708.09267