Sheaf Quantization of Legendrian Isotopy

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In memory of Steve

Abstract

Let $\{\Lambda_t^\infty\}$ be an isotopy of Legendrians (possibly singular) in a unit cosphere bundle S^*M, such that they arise as slices of a singular Legendrian $\Lambda_I^\infty \subset S^*M \times T^*I$. Let $\mathcal{C}_t = Sh(M, \Lambda_t^\infty)$ be the differential graded (dg) derived category of constructible sheaves on M with singular support at infinity contained in Λ_t^∞. We prove that if the isotopy of Legendrians embeds into an isotopy of Liouville hypersurfaces, then the family of categories $\{\mathcal{C}_t\}$ is constant in t.

0.1 Motivations and Results

Let M be a smooth manifold, $Sh(M)$ be the cocomplete (dg) derived category of weakly constructible sheaves on M with coefficient in $\mathbb{C}$. In [KS13] and [Tam08, GKS12], it is proved that contact isotopy of the cosphere bundle $T^\infty M = (T^*M - T^*_M M)/\mathbb{R}_+$ acts on $Sh(M)$ as equivalences of categories. In this paper, we consider a (singular) Legendrian $\Lambda^\infty \subset T^\infty M$, and the full subcategory $Sh(M, \Lambda^\infty)$ consisting of sheaves F with singular support at infinity $SS^\infty(F) = (SS(F) - T^*_M M)/\mathbb{R}_+$ contained in Λ^∞. We define a notion of isotopy for the singular Legendrian Λ^∞, and prove that the category $Sh(M, \Lambda^\infty)$ remains invariant under such an isotopy.

Such invariances of constructible sheaf categories are possible because constructible sheaves are closely related to Lagrangians in T^*M [NZ09, GPS18a, NS20], hence enjoy the flexibility of symplectic geometry. More precisely, the full subcategory of compact objects in $Sh(M, \Lambda^\infty)$, denoted as $Sh^w(M, \Lambda^\infty)$ is equivalent to the wrapped Fukaya category of the pair (T^*M, Λ^∞) [GPS18a, NS20],

$$Sh^w(M, \Lambda^\infty) \simeq \mathrm{Fuk}^w(T^*M, \Lambda^\infty)$$

where 'w' stands for 'wrapped'. The traditional constructible sheaves with bounded cohomologies $Sh^{pp}(M, \Lambda^\infty)$ can be recovered as perfect modules [Nad16]

$$Sh^{pp}(M, \Lambda^\infty) = \mathrm{Fun}^{ex}(Sh^w(M, \Lambda^\infty)^{op}, \mathrm{Perf}(\mathbb{C})).$$

There is an analogous result in the wrapped Fukaya category for Liouville sectors [GPS18b, Theorem 1.4]: given a Liouville domain W with a 'stop' $S \subset \partial W$, if the contact complement $\partial W \backslash S$ remains invariant up to contact isotopy as S moves, then the wrapped Fukaya category $\mathrm{Fuk}^w(W, S)$ is invariant. Hence, combined with the comparison results of [GPS18a, NS20], we then get that $Sh(M, \Lambda^\infty)$ is invariant as long as $T^\infty M \backslash \Lambda^\infty$ is invariant up to contact isotopy.

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Our paper gives a similar sufficient condition using 'isotopy' of Λ^∞: we replace Λ^∞ by a tube $U = U(\Lambda^\infty)$ around the Legendrian Λ^∞, and we equip U with a contact flow X that shrinks the tube U back to Λ^∞ (Definition 0.2). Although Λ^∞ maybe singular, the data of (U, X) are smooth, hence we can talk about isotopies of (U, X). The relation with the complement $T^\infty M\backslash\Lambda^\infty$ is that, if (U_t, X_t) varies smoothly, then the complements $\{T^\infty M\backslash\Lambda_t^\infty\}_t$ are contactomorphic, where Λ_t^∞ is the limit of U_t under the shrinking flow X_t.

To state our theorem precisely, we need some definitions.

Let (C, α) be a contact manifold with a contact 1-form α.

Definition 0.1. *A singular Legendrian $\mathcal{L} \subset C$ is a Whitney stratifiable subspace, such that its top dimensional strata are smooth Legendrians, and the closure of the union of the top dimensional strata is $\mathcal{L}$.*

Definition 0.2. *Let $\mathcal{L} \subset C$ be a singular Legendrian. A convex tube (U, X) around $\mathcal{L}$ is an open subset U containing $\mathcal{L}$ with smooth boundary ∂U and a contact vector field X transverse to ∂U and pointing inward to ∂U, such that $\mathcal{L}_X(\alpha) = -\alpha$ and $\cap_{u>0} X^u(U) = \mathcal{L}$, where X^u is the time u flow of X.*

Definition 0.3. *Let $I \subset \mathbb{R}$ be a closed interval and $\{(U_t, X_t, \mathcal{L}_t)\}_{t\in I}$ be a family of singular Legendrians $\mathcal{L}_t$ with convex tubes (U_t, X_t). If ∂U_t and X_t vary smoothly with t, then we say $\{(U_t, X_t, \mathcal{L}_t)\}_{t\in I}$ is an isotopy of convex tubes over I.*

Let M be a smooth manifold with Riemannian metric g. Let $S^*M \subset T^*M$ be the unit cosphere bundle, and $\alpha = \lambda|_{S^*M}$ be a contact 1-form on S^*M where λ is the Liouville 1-form on T^*M (e.g $\lambda = pdx$ on $T^*\mathbb{R}$). We identify S^*M with $T^\infty M$. We equip $S^*M \times T^*I$ with the contact form $\widetilde{\alpha} = \alpha + \tau dt$, where t is the coordinate of I and τ is the coordinate on the cotangent fiber. Then the composition $S^*M \times T^*I \hookrightarrow \dot{T}^*(M \times I) \to T^\infty(M \times I)$ is an open immersion and contactomorphism, with image $(x, t; [p, \tau]) \in T^\infty(M \times I)$ where $p \neq 0$.

Definition 0.4. *Let $I \subset \mathbb{R}$ be a closed interval. A strong isotopy of Legendrians in S^*M over I is a Legendrian $\mathcal{L}_I \subset S^*M \times T^*I$. A strong isotopy of convex tubes is a convex tube (U_I, X_I) of $\mathcal{L}_I$, such that X_I preserves the fibers of $S^*M \times T^*I \to I$.*

Our main theorem is that

Theorem 0.5. *If (U_I, X_I) is a strong isotopy of convex tubes around $\mathcal{L}_I$ in $S^*M \times T^*I$, then for any $t \in I$, we have an equivalence of categories*

$$\iota_t^* : Sh(M \times I, \mathcal{L}_I) \to Sh(M, \mathcal{L}_t)$$

where $\iota_t : M_t = M \times \{t\} \hookrightarrow M_I = M \times I$ is the inclusion of the slice over t.

Given a strong isotopy of Legendrian $\mathcal{L}_I$, we prove that to construct a tube thickening (U_I, X_I), it suffices to construct a Liouville hypersurface thickening of each slice $\mathcal{L}_t$ (See Proposition 1.14).

Although the result is well-expected given the analogous result in Fukaya category, and is superseded by the recent paper [NS20], we hope the purely sheaf theoretic proof and the simpler setting of cotangent bundle make the presentation still worthwhile.

0.2 Previous Works

We first recall the sheaf quantization of a contact isotopy of S^*M.

Theorem 0.6 [GKS12] Theorem 3.7, Proposition 3.12. *Let I be an open interval containing* 0, *and $\varphi : I \times T^\infty M \to T^\infty M$ be a smooth map with $\varphi_t = \varphi(t, -)$. Assume φ satisfies (1) $\varphi_0 = id$, and (2) φ_t are contactomorphisms for all $t \in I$. Then for each $t \in I$, we have equivalences of categories*

$$\hat{\varphi}_t : Sh(M) \xrightarrow{\sim} Sh(M), \quad \text{such that } SS^\infty(\hat{\varphi}_t F) = \varphi_t(SS^\infty(F)).$$

Note any isotopy of smooth Legendrian can be extended to a contact isotopy of the ambient manifold. In general, we have the corollary

Corollary 0.7. *If an isotopy of Legendrians $\{\Lambda_t^\infty\}_{t\in I}$ can be embedded into an isotopy $\{\varphi_t\}_{t\in I} : S^*M \to S^*M$ of the contact manifold, that is, $\Lambda_t^\infty = \varphi_t(\Lambda_0^\infty)$, then we have an equivalence of categories*

$$\hat{\varphi}_t : Sh(M, \Lambda_0^\infty) \xrightarrow{\sim} Sh(M, \Lambda_t^\infty).$$

For a deformation of singular Legendrians, there is one necessary condition for the invariance of categories due to Nadler [Nad15].

Definition 0.8 Displaceable Legendrian. *Let (S^*M, α) be the unit cosphere bundle of a Riemannian manifold M with Reeb vector field R and time t Reeb flow R^t. A Legendrian $\mathcal{L} \subset S^*M$ is* **ϵ-displaceable** *for R and for some $\epsilon > 0$, if*

$$\mathcal{L} \cap R^s(\mathcal{L}) = \emptyset, \quad \forall 0 < |s| < \epsilon. \tag{1}$$

We say a family of Legendrian $\{\mathcal{L}_t\}$ is **uniformly ϵ-displaceable** *for R and for some $\epsilon > 0$, if each $\mathcal{L}_t$ is ϵ-displaceable.*

If a family of Legendrians $\{\mathcal{L}_t\}$ can be upgraded to an isotopy of convex tubes $\{U_t, X_t, \mathcal{L}_t\}$, then $\{\mathcal{L}_t\}$ is uniformly displaceable (Proposition 1.9).

Example 0.9. *Consider the following example.*

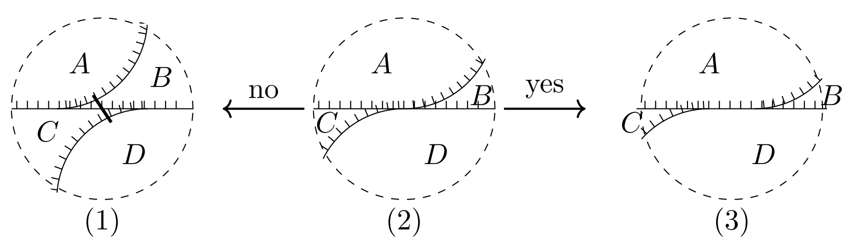
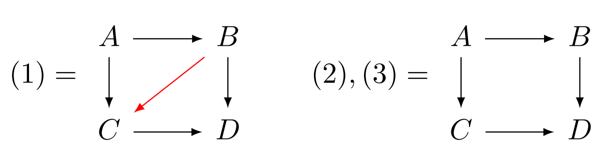


Figure 1. The deformation to the right is uniformly displaceable, and the one to the left is not, due to the appearance of new short Reeb chord (marked by a thick line). (c.f. [Nad15], Example 1.5)

The category of constructible sheaves for the above diagrams are the representation of the following commutative diagrams (each region corresponds to a vertex, and arrow between vertices goes against the direction of the hair).



0.3 A sketch of the proof

Given a convex tube (U, X) of a Legendrian $\mathcal{L} \subset S^*M$, we may define a projection functor as the 'limit' of the flow X

$$\Pi_{\mathcal{L}} : Sh(M, U) \to Sh(M, \mathcal{L}), \quad \Pi(F) := \lim_{T\to\infty} \hat{X}^T(F), \tag{2}$$

where $Sh(M, U)$ means the category of constructible sheaves with $SS^\infty(F) \subset U$, and the limit is defined using the nearby cycle functor in Section 2.6.

Let $(U_I, X_I, \mathcal{L}_I)$ be a strong isotopy of convex tubes in $S^*M \times T^*I$, and let $\{(U_t, X_t, \mathcal{L}_t)\}$ be the slices. Let $F_t \in Sh(M, \mathcal{L}_t)$. We will extend F_t to a sheaf $F_I \subset Sh(M \times I, \mathcal{L}_I)$ such that $F_I|_t = F_t$.

One first shows that such an extension is unique (if it exists), this is equivalent to show that the restriction functor $Sh(M, \mathcal{L}_I) \to Sh(M, \mathcal{L}_t)$ is fully-faithful (Proposition 3.1), i.e.

$$\mathrm{Hom}(F_I, G_I) \xrightarrow{\sim} \mathrm{Hom}(F_t, G_t), \quad \forall F_I, G_I \in Sh(M \times I, \mathcal{L}_I).$$

One needs to show that $\mathcal{H}om(F_I, G_I)(M \times (a, b))$ is independent of the size of the interval, hence one can interpolate from $(a, b) = I$ to an infinitesimal small neighborhood around t. The key technical point is to use the uniform displaceability condition to perturb G_I slice-wise by positive Reeb flow for time s, $G_I \to K_s^! G_I$, to separate $SS^\infty(F_I)$ and $SS^\infty(K_s^! G_I)$.

One then shows that such an extension exists locally, i.e., given a F_t, we may find a small neighborhood $J = (t - \delta, t + \delta)$, such that $\mathcal{L}_t \times T_J^* J \subset U_J = U_I \cap S^*M \times T^*J$ and extend F_t on M_t to F_J on M_J by defining $F_J = \Pi_{\mathcal{L}_J}(F_t \boxtimes \mathbb{C}_J)$.

Finally, we use the uniqueness of extension to patch together the local extensions, and thus we get the global extension result. (c.f. Lemma 1.13 in [GKS12]).

0.4 Acknowledgements

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0.5 Notation

We use $Sh(M)$ to denote the co-complete dg derived category of weakly constructible sheaves. We abuse notation, and use "constructible sheaf" to mean a cohomologically constructible complex of sheaves. All the functors $f_*, f^*, f_!, \mathcal{H}om, \cdots$ are derived.

1. Convex tubes and Isotopy

1.1 Basics of contact geometry

We recall the definition of co-oriented contact manifold as follows. Let C be a $2n+1$ dimensional manifold, $\xi \subset TC$ be a rank $2n$ sub-bundle, such that there exists a one-form (contact one-form) α (up to multiplication of non-negative function) satisfying $\xi = \ker \alpha$ and $\alpha \wedge (d\alpha)^n \neq 0$. If we fix such an α, we have a Reeb vector field R_α given by

$$\iota_{R_\alpha} \alpha = 1, \quad \iota_{R_\alpha} d\alpha = 0.$$

We note that different choices of α will lead to different choices of R_α.

A contact vector field X is a vector field on C that perserves the sub-bundle ξ.

Definition 1.1. *Given a smooth function $H : C \to \mathbb{R}$, the* **contact Hamiltonian vector field** *X_H is uniquely determined by the following conditions*

$$\begin{cases} \langle X_H, \alpha \rangle = H \\ \iota_{X_H} d\alpha = \langle dH, R\rangle \alpha - dH \end{cases} \tag{3}$$

The Reeb vector field is a special case of X_H where $H = 1$.

Proposition 1.2 [Gei08] Theorem 2.3.1. *With a fixed choice of contact form α there is a one-to-one correspondence between contact vector field X and smooth functions $H : C \to \mathbb{R}$. The correspondence is given by*

$$X \mapsto H = \langle \alpha, X \rangle, \quad H \mapsto X_H.$$

Unlike symplectic Hamiltonian vector field, X_H does not preserve the level sets of H.

Lemma 1.3.

$$\langle X_H, dH \rangle = H \langle R, dH \rangle$$

In particular, X_H preserves the zero set of H.

Proof. One may apply ι_{X_H} to the second line of Eq (3), then use the first line. □

We also have the Lie derivative of α along X_H

$$\mathcal{L}_{X_H}(\alpha) = d\iota_{X_H}\alpha + \iota_{X_H} d\alpha = \langle R, H \rangle \alpha.$$

Proposition 1.4. *If $\mathcal{L} \subset C$ is a germ of smooth Legendrian, and H is any locally defined function vanishing on $\mathcal{L}$, then the contact flow X_H is tangential to $\mathcal{L}$.*

Proof. To show that X_H is tangential to $\mathcal{L}$ at $p \in \mathcal{L}$, we only need to show that for any tangent vector $v \in T_p\mathcal{L}$, we have $d\alpha(X_H, v) = 0$ and $\alpha(X_H)|_p = 0$, because these two conditions imply $X_H \in (T_p\mathcal{L})^{\perp_{d\alpha}} \cap \ker(\alpha) = T_p\mathcal{L}$. Indeed, $\alpha(X_H)|_p = H(p) = 0$, and

$$d\alpha(X_H, v) = \iota_{X_H}(d\alpha)(v) = [\langle R, dH \rangle \alpha - dH](v) = \langle R, dH \rangle (\alpha(v)) - H(v) = 0.$$

Hence X_H is tangential to $\mathcal{L}$. □

Example 1.5. *Let M be a smooth manifold, and T^*M the cotangent bundle with canonical Liouville one-form λ and symplectic two-form $\omega = d\lambda$. If we put local Darboux coordinate $(q, p) = (q_1, \cdots, q_m; p_1, \cdots, p_m)$ on T^*M where $m = \dim_{\mathbb{R}} M$, then $\lambda = \sum_{i=1}^m p_i dq_i$ and $\omega = \sum_i dp_i \wedge dq_i$, and we will suppress the indices and summation to write $\lambda = pdq, \omega = dpdq$. Also define $\dot{T}^*M = T^*M \backslash T^*_M M$, $T^\infty M = \dot{T}^*M/\mathbb{R}_{>0}$. The Liouville vector field for λ is defined by $\iota_{V_\lambda}\omega = \lambda$, and here it is given by $V_\lambda = p\partial_p$. On $T(\dot{T}^*M)$, the symplectic orthogonal to the Liouville vector field defines a distribution*

$$\widetilde{\xi} = \{(q, p; v_q, v_p) \in T(\dot{T}^*M) : \omega((v_q, v_p), V_\lambda) = 0\},$$

which projects to a canonical contact distribution ξ on $T^\infty M$. Let g be any Riemannian metric on M, then T^*M has induced norm. Let $S^*M = \{(q, p) \in T^*M \mid |p| = 1\}$ be the unit cosphere bundle with contact form $\alpha = \lambda|_{S^*M}$, then the contact distribution can also be written as $\xi = \ker(\alpha)$.

Define the symplectization of $(C, \xi = \ker \alpha)$ by

$$S := C \times \mathbb{R}_u, \quad \lambda = e^u \alpha, \quad \omega_S = d(e^u \alpha).$$

We have the projection along $\mathbb{R}_u$, and the inclusion of zero section as:

$$\pi_S : S \to C, \quad \iota_C : C \simeq C \times \{0\} \hookrightarrow S.$$

A different choice of α would give rise to the same S up to a fiber preserving symplectomorphism that identifies the 'zero-section' $\mathrm{Im}(\iota_C)$.

A Hamiltonian function $H : C \to \mathbb{R}$ can be extended to a homogeneous degree one function $\widetilde{H} : S \to \mathbb{R}$ by setting $\widetilde{H} = e^u H$. Then the symplectic Hamiltonian vector field $\xi_{\widetilde{H}}$, given by $\omega_S(-, \xi_{\widetilde{H}}) = d\widetilde{H}(-)$, preserves the fiber of π_S and descends to X_H.

1.2 Convex Tubes

Recall the definition of convex tubes in Definition 0.2.

Definition 1.6. *A Liouville hypersurface thickening of a singular Legendrian $\mathcal{L}$ is a hypersurface $\mathcal{H} \supset \mathcal{L}$, such that $(\mathcal{H}, \alpha|_{\mathcal{H}})$ is a Liouville domain with the Liouville skeleton being $\mathcal{L}$.*

First we show that a Liouville hypersurface thickening can be upgraded to a convex tube thickening of $\mathcal{L}$.

Proposition 1.7. *Let $\mathcal{L}$ be a singular Legendrian with a Liouville hypersurface thickening $\mathcal{H}$. Then $\mathcal{L}$ admits a convex tube (U, X), where the contact vector field X preserves $\mathcal{H}$ and X restricts to $\mathcal{H}$ is the downward Liouville flow of $\mathcal{H}$.*

Proof. Let $\epsilon > 0$ be small enough, such that for any $0 < s < \epsilon$ we have $\mathcal{H} \cap R^s \mathcal{H} = \emptyset$. Then define $U = \cup_{|s| \leqslant \epsilon/2} R^s \mathcal{H} \simeq \mathcal{H} \times (-\epsilon/2, +\epsilon/2)$, and let $h : U \to (-\epsilon/2, +\epsilon/2)$ be the projection. Then $X = X_h$ shrinks U to $\mathcal{L}$ and restricts to downward Liouville flow on $\mathcal{H}$. One may smooth the corner of U and achieve transversality of X with ∂U. □

Conversely, we show that each convex tube (U, X) around $\mathcal{L}$ determines a Liouville thickening.

Proposition 1.8. *Let (U, X) be a convex tube around $\mathcal{L}$. Let $h = \alpha(X)$, and $\mathcal{H} = h^{-1}(0) \subset U$. Then $\mathcal{H}$ is a Liouville thickening of $\mathcal{L}$.*

Proof. Since $X = X_h$ preserves $\mathcal{H}$ and shrinks $\mathcal{H}$ to $\mathcal{L}$, we only need to show that $\mathcal{H}$ is transverse to R and X is the downward Liouville flow on $\mathcal{H}$.

Since $\mathcal{L}_X(\alpha) = \langle R, dh \rangle \alpha = -\alpha$, hence $R(h) = -1$. Thus R is transversal to the level sets of h, in particular $\mathcal{H}$. Hence $d\alpha$ is non-degenerate on $\mathcal{H}$, thus $\mathcal{H}$ is exact symplectic. Let $\lambda = \alpha|_{\mathcal{H}}, \omega = d\lambda$. When we restrict to $T\mathcal{H}$, we have

$$\iota_{X_h}(\omega) = \iota_{X_h}(d\alpha) = \langle R, dh \rangle \alpha - dh = -\lambda$$

hence X_h is the downward Liouville flow on $\mathcal{H}$. □

Proposition 1.9. *Let (U, X) be a convex tube of $\mathcal{L}$. Then $\mathcal{L}$ is displaceable (See Definition 0.8). Similarly, let $I = [0, 1]$, let (U_I, X_I) be a strong isotopy of convex tubes for $\mathcal{L}_I$, then the family of Legendrians $\mathcal{L}_t$ are uniformly displaceable.*

Proof. Let $h = \alpha(X)$ be the Hamiltonian function generating X. Then h vanishes on $\mathcal{L}$, and by the normalization condition, we have $R(h) = -1$. If there is a Reeb chord $\gamma : [0, T] \to C$

contained in U and ending on $\mathcal{L}$, then we have

$$\int_0^T \dot\gamma(dh)dt = \int_0^T R(dh)dt = \int_0^T (-1)dt = -T.$$

But on the other hand, we also have

$$\int_0^T \dot\gamma(dh)dt = \int_\gamma dh = h(\gamma(T)) - h(\gamma(0)) = 0,$$

since $\gamma(T) \in \mathcal{L}$, $\gamma(0) \in \mathcal{L}$ and $h|_{\mathcal{L}} = 0$. Thus, there is no Reeb chord ending on $\mathcal{L}$ and contained in U. For any $x \in \mathcal{L}$, let $t(x) = \inf\{t \in \mathbb{R} : R^t(x) \notin U\}$, then $t(x) > 0$ and is continuous on $\mathcal{L}$. Let $\epsilon = \inf\{t(x) : x \in \mathcal{L}\}$, since $\mathcal{L}$ is compact, hence $\epsilon > 0$. Then $\mathcal{L}$ is displaceable.

For the uniform displaceable statement, note that $I = [0,1]$ is compact, and $\epsilon(t)$ for (U_t, X_t) is continuous in t, hence $\epsilon = \inf\{\epsilon(t)\} > 0$. □

1.3 The Construction of Strong Isotopies of Convex Tubes

Consider the unit cosphere bundle (S^*M, α) and a closed interval $I \subset \mathbb{R}$. Let a point in S^*M be denoted as $(x,p) \in T^*M$ with $|p| = 1$. Let a point in T^*I be denoted as $(t,\tau) \in I \times \mathbb{R}$. Let $S^*M \times T^*I$ be equipped with the contact 1-form

$$\alpha_I = \alpha + \tau dt.$$

Let $\pi_t : S^*M \times T^*I \to I$.

Proposition 1.10. *The Reeb flow R_I on $S^*M \times T^*I$ for α_I is the pullback of the Reeb flow R on S^*M.*

Proof. Let R denote the pullback to $S^*M \times T^*I$. We may verify that $\iota_R(\alpha_I) = 1, \iota_R(d\alpha_I) = 0$. □

Let $\mathcal{L}_I$ be a strong isotopy of Legendrian. Let

$$\mathcal{L}_t = \{(x,p) \in S^*M \mid \exists (x,p,t,\tau) \in \mathcal{L}_I\}.$$

Lemma 1.11. *$\mathcal{L}_t$ is a singular Legendrian in S^*M.*

Proof. Take any $p \in \mathcal{L}_t$ that is the image of a point $\widetilde{p}$ in the smooth loci $\mathcal{L}_I^{sm}$, and for any tangent vector $v \in T_p\mathcal{L}_t$, it can be lifted to $\widetilde{v} \in T_{\widetilde{p}}\mathcal{L}_I$. Concretely, $\widetilde{v} = v + c\partial_\tau$. Since $0 = (\alpha + \tau dt)(\widetilde{v}) = \alpha(v)$, we see $T_p\mathcal{L}_t$ is in the $\ker(\alpha)$. Hence a dense open part of $\mathcal{L}_t$ is Legendrian, thus $\mathcal{L}_t$ is a singular Legendrian. □

Let $(U_I, X_I, \mathcal{L}_I)$ be a strong isotopy of convex tubes. First we define restriction to $S^*M \times T^*_tI$. Since X_I preserves the t coordinate, hence for each t, we have the vector field

$$\hat{X}_t := X_I|_t \in \mathrm{Vect}(S^*M \times T^*_tI).$$

Also denote the restriction

$$\hat{U}_t = U_I \cap S^*M \times T^*_tI, \quad \hat{\mathcal{L}}_t = \mathcal{L}_I \cap S^*M \times T^*_tI.$$

Next, we define (U_t, X_t). Define the projection map $\hat\pi_t : S^*M \times T^*_tI \to S^*M$, and denote

$$U_t = \hat\pi_t(\hat{U}_t), \quad \mathcal{L}_t = \hat\pi_t(\hat{\mathcal{L}}_t).$$

Let $h_I = \alpha_I(X_I)$, since X_I has no ∂_t component, then $\partial_\tau h_I = 0$, hence h_I is independent of τ. For each $t \in I$, we define

$$h_t(x,p) := h_I(x,p,t) \quad \forall (x,p) \in U_t$$

and we let X_t be the contact vector field generated by h_t.

PROPOSITION 1.12. *With the above setup, we have*

$$\hat{X}_t = X_t + (-\tau - \partial_t h_t)\partial_\tau$$

Proof. We split a tangent vector v on $S^*M \times T^*I$ as two components $v = v_1 + v_2$, where v_1, v_2 are along the S^*M and T^*I factors respectively. Similarly, we decompose $X_I = X_{I,1} + X_{I,2}$, where $X_{I,2} = a\partial_\tau$.

By the definition of X_I, we have

$$\iota_{X_I}(\alpha + \tau dt) = h_I$$

and

$$\iota_{X_I} d(\alpha + \tau dt) = \langle R_I, h_I\rangle(\alpha + \tau dt) - dh_I,$$

which we will refer to as the first and second equations below.

Since $\tau dt(X_{I,2}) = 0$, the first equation becomes

$$\alpha(X_{I,1}) = h_t(x,p).$$

For the second equation, if we restrict to the tangent space on S^*M, we have

$$\iota_{X_{I,1}} d\alpha = \langle R, h_t\rangle\alpha - dh_t$$

Thus $X_{I,1} = X_t$ is the contact vector field on S^*M generated by $h_t(x,p)$.

Finally, if we restrict the second equation to the tangent space of T^*I, we get

$$\iota_{X_{I,2}}(d\tau \wedge dt) = \langle R, h_I\rangle(\tau dt) - \partial_t h_t dt$$

If we plug in $X_{I,2} = a\partial_t$ and $\langle R, h_I\rangle = -1$, we get the desired result. □

PROPOSITION 1.13. *For any $t \in I$, the above defined (U_t, X_t) is a convex tube for $\mathcal{L}_t$. Furthermore, the family $\{(U_t, X_t)\}_t$ varies smoothly with t hence is an isotopy of convex tubes for $\{\mathcal{L}_t\}$.*

Proof. From Proposition 1.12, we know that the flow of $\hat{X}_t$ preserves the fibers of $S^*M \times T^*_t I \to S^*M$ and the induced flow on S^*M is generated by X_t. Since the flow of $\hat{X}_t$ shrinks $\hat{U}_t$ to $\hat{\mathcal{L}}_t$, i.e, $\hat{\mathcal{L}}_t = \cap_{u>0}\hat{X}_t^u\hat{U}_t$, the sequence of open sets $\hat{X}_t^u\hat{U}_t$ is monotonously decreasing in u, and furthermore $\hat{\pi}_t(\hat{X}_t^u\hat{U}_t) = X_t^u(U_t)$, thus we have

$$\mathcal{L}_t = \cap_{u>0} X_t^u(U_t).$$

□

Finally, the last proposition allows us to upgrade from an isotopy of Liouville hypersurfaces to a strong isotopy of convex tubes.

PROPOSITION 1.14. *If $\mathcal{L}_I$ is a Legendrian in $S^*M \times T^*I$, and if $\{\mathcal{H}_t\}$ is a smooth family of Liouville hypersurfaces in S^*M such that $\mathcal{L}_t$ is the skeleton of $\mathcal{H}_t$, then we have a strong isotopy of convex tubes (U_I, X_I) around $\mathcal{L}_I$.*

Proof. First we use $\mathcal{H}_t$ to get a family of convex tubes (U_t, X_t) and the associated Hamiltonian functions h_t, where $h_t|_{\mathcal{H}_t} = 0$ and $R(h_t) = -1$. The family of functions h_t determines the lifted function $h_I(x,p,t,\tau) = h_t(x,p)$ which is defined when $(x,p) \in U_t$. In turn, h_I determines the contact vector field X_I, which restricts to the fiber $S^*M \times T^*_t I$ as given by Proposition 1.12. Thus, we only need to specify the subset $\hat{U}_t \subset U_t \times T^*_t I$, such that its boundary $\partial\hat{U}_t$ is transverse to the vector field $\hat{X}_t$, and it is compressed by the flow of $\hat{X}_t$ to $\hat{\mathcal{L}}_t = \mathcal{L}_I \cap S^*M \times T^*_t I$.

Let
$$C = 1 + \sup\{|\partial_t h_t(x,p)| \mid (x,p) \in \bar{U}_t, t \in I\}$$
and let
$$\hat{U}_t = U_t \times (-C, C) \subset S^*M \times T^*I.$$
Then the flow $\hat{X}_t$ is tranverse to the boundary ∂U_t. We only need to show that $\cap_{u>0} \hat{X}_t^u(\hat{U}_t) = \hat{\mathcal{L}}_t$. Since $\hat{U}_t \to U_t$ with fiber $(-C, C)$, and $\hat{X}_t$ restricted to the fiber gives the equation for τ as
$$(d/du)\tau(u) = -\tau - \partial_t h_t(x(u), p(u)).$$
This is a contracting flow with a unit contraction rate in the sense that, for any initial condition τ_1, τ_2 at $u = 0$, we have $\tau_1(u) - \tau_2(u) = (\tau_1 - \tau_2)e^{-u}$ for $u > 0$.

Under the projection map $\hat{\pi}_t : S^*M \times T_t^*I \to S^*M$, we have the surjection
$$\hat{\pi}_t : \hat{\mathcal{L}}_t' := \cap_{u>0} \hat{X}_t^u \hat{U}_t \to \cap_{u>0} X_t^u U_t = \mathcal{L}_t$$
and by the contracting property of the flow $\hat{X}_t$, the fiber can only consist of one point, thus $\hat{\pi}_t : \hat{\mathcal{L}}_t' \to \mathcal{L}_t$ is a bijection.

Let $U_I = \cup_{t\in I} \hat{U}_t \subset S^*M \times T^*I$, and put the slices $\hat{\mathcal{L}}_t'$ together into $\mathcal{L}_I' = \cap_{u>0} X_I^u U_I$. Recall $\hat{\pi} : S^*M \times T^*I \to S^*M \times I$, then $\hat{\pi}(\mathcal{L}_I) = \hat{\pi}(\mathcal{L}_I')$, and $\mathcal{L}_I'$ is homeomorphic to its image. Since a smooth family of smooth Legendrians in $S^*M \times I$ has a unique lift to $S^*M \times T^*I$, thus $\mathcal{L}_I$ and $\mathcal{L}_I'$ agree over the smooth loci of $\hat{\pi}(\mathcal{L}_I)$. Since $\mathcal{L}_I$ is the closure of its smooth part, we have $\mathcal{L}_I = \mathcal{L}_I'$, finishing the proof of the proposition. □

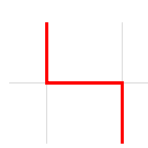
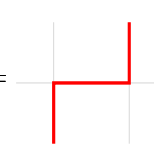
2. Non-characteristic isotopy of sheaves

2.1 Constructible sheaves

We give a quick working definition for constructible sheaves used in this paper, and point to [KS13] for a proper treatment. A constructible sheaf F on M is a sheaf valued in chain complex of $\mathbb{C}$-vector spaces, such that its cohomology is locally constant with finite rank with respect to some Whitney stratification [1] $\mathcal{S} = \{\mathcal{S}_\alpha\}_{\alpha\in A}$ on M, where $\mathcal{S}_\alpha$ are disjoint locally closed smooth submanifolds with nice adjacency condition and $M = \sqcup_{\alpha\in A} \mathcal{S}_\alpha$. The singular support $SS(F)$ of F is a closed conical Lagrangian in T^*M, contained in $\cup_{\alpha\in A} T^*_{\mathcal{S}_\alpha} M$, such that $SS(F) \cap T^*_M M$ equals the support of F, and $(p, q) \in SS(F) \backslash T^*_M M$ if there exists a locally defined function f with $f(q) = 0, df(q) = p$, such that the restriction map $F(B_\epsilon(q) \cap \{f < \delta\}) \to F(B_\epsilon(q) \cap \{f < -\delta\})$ fails to be a quasi-isomorphism for $0 < \delta \ll \epsilon \ll 1$. We denote by $SS^\infty(F) = SS(F) \cap S^*M$ the singular support of F at infinity.

If $\Lambda \subset T^*M$ is a conical Lagrangian containing zero section (as always assumed in this paper), we write $Sh(M, \Lambda^\infty)$ for the dg derived category of constructible sheaves with object F satisfying $SS^\infty(F) \subset \Lambda^\infty$.

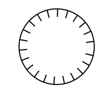
Example 2.1. *For example, on $\mathbb{R}$, if $\mathbb{C}_{[0,1]}$ (resp. $\mathbb{C}_{(0,1)}$) denote the constant sheaf with stalk $\mathbb{C}$ on $[0, 1]$ (resp. on $(0, 1)$) and zero stalk elsewhere, then their singular supports in $T^*\mathbb{R}$ are*

$SS(\mathbb{C}_{[0,1]}) =$  , $SS(\mathbb{C}_{(0,1)}) =$  .

[1] More precisely, μ-stratification, see [KS13, Section 8.3].

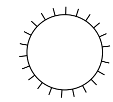
EXAMPLE 2.2. *Let $j : U = B(0,1) \hookrightarrow \mathbb{R}^2$ be the inclusion of an open unit ball in $\mathbb{R}^2$. Then $j_*\mathbb{C}_U$ is supported on the closed set $\overline{U}$, with singular support at infinity as*

$$SS^\infty(j_*\mathbb{C}_U) = \{(x,\eta) \in S^*\mathbb{R}^2 \mid x \in \partial U, \eta = -d|x|\} =$$



And $j_!\mathbb{C}_U$ is supported on the open set U, with singular support at infinity as

$$SS^\infty(j_!\mathbb{C}_U) = \{(x,\eta) \in S^*\mathbb{R}^2 \mid x \in \partial U, \eta = d|x|\} =$$



Here the Legendrians are represented by co-oriented hypersurfaces in $\mathbb{R}^2$ with hairs indicating the co-orientation.

2.2 Operation on sheaves

In this subsection, we deviate from our running convention and use $Sh(X)$ to denote the co-complete dg derived category of sheaves on X without any constructibility condition. Let $f : Y \to X$ be a map of real analytic manifolds. Then we have the following pairs of adjoint functors

$$\begin{aligned} - \otimes F :& Sh(X) \leftrightarrow Sh(X) \colon \mathcal{H}om(F,-) \\ f^* :& Sh(X) \leftrightarrow Sh(Y) \colon f_* \\ f_! :& Sh(Y) \leftrightarrow Sh(X) \colon f^!. \end{aligned}$$

Given an open subset U of X and its closed complement Z,

$$\text{open inclusion:} \quad U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z, \quad \text{closed inclusion,}$$

we have $j^* = j^!$ and $i_* = i_!$. Furthermore, there are exact triangles

$$i_!i^! \to id \to j_*j^* \xrightarrow{[1]}, \quad j_!j^! \to id \to i_*i^* \xrightarrow{[1]} .$$

These are sheaf-theoretic incarnations of excisions: applied to the constant sheaf on X and taking global sections, we get

$$H^*(Z, i^!\mathbb{C}) \to H^*(X,\mathbb{C}) \to H^*(U,\mathbb{C}) \xrightarrow{[1]}, \quad H^*_c(U,\mathbb{C}) \to H^*_c(X,\mathbb{C}) \to H^*_c(Z,\mathbb{C}) \xrightarrow{[1]} .$$

Let X_i, $i = 1, 2$, be spaces, and $K \in Sh(X_1 \times X_2)$. We define the following pair of adjoint functors

$$K_! : Sh(X_1) \leftrightarrow Sh(X_2) : K^! \tag{4}$$

$$K_! : F \mapsto \pi_{2!}(K \otimes \pi_1^* F), \qquad K^! : G \mapsto \pi_{1*}(\mathcal{H}om(K, \pi_2^! G)) \tag{5}$$

In [KS13],$K_! = \Phi_K$ and $K^! = \Psi_K$ and with X_1, X_2 switched. The notation here is suggestive for them to be adjoint functors.

2.3 Isotopy of Constructible Sheaves

Let $I = (a,b) \subset \mathbb{R}$. For any $t \in I$, let

$$j_t : M_t := M \times \{t\} \hookrightarrow M_I := M \times I$$

be the inclusion of the t-slice M_t into the total space M_I, and let $\pi_I : M_I \to I$ be the projection. Let $\mathbb{C}_{M_t}$ be the constant sheaf on M_t with stalk $\mathbb{C}$. We have then

$$SS(\mathbb{C}_{M_t}) = \{(x,t;0,\tau) \in T^*M_I\}, \quad SS^\infty(\mathbb{C}_{M_t}) = \{(x,t;0,\pm 1) \in S^*M_I \simeq T^\infty M\}.$$

Definition 2.3. *Let M be a smooth manifold, I a closed interval of $\mathbb{R}$.*

(i) *An* **isotopy of (constructible) sheaves** *is a constructible sheaf $F_I \in Sh(M \times I)$, such that $SS^\infty(F_I)$ is a strong isotopy of Legendrians in $S^*M \times T^*I$ (Definition 0.4). Or equivalently for any $t \in I$, we have*

$$SS^\infty(F_I) \cap SS^\infty(\mathbb{C}_{M_t}) = \emptyset.$$

If F_I is an isotopy of sheaves, then for any $t \in I$, we denote the **restriction of F_I at t** *as*

$$F_t := F_I|_{M_t} \in Sh(M).$$

(ii) *Two isotopies of sheaves $F_I, G_I \in Sh(M \times I)$ are said to be* **non-characteristic** *if*

$$SS^\infty(F_I)|_t \cap SS^\infty(G_I)|_t = \emptyset, \text{ for all } t \in I.$$

Some easy to check properties are in order.

Proposition 2.4. *Let M be a compact real analytic manifold.*
(1) If F_I is an isotopy of sheaves, and $\Lambda_I^\infty = SS^\infty(F_I)$, then

$$SS^\infty(F_t) \subset \Lambda_t^\infty.$$

(2) If F_I is an isotopy of sheaves, $\pi_I : M_I \to I$, then $(\pi_I)_* F_I$ is a local system on I.

2.4 Invariance of morphisms under non-characteristic isotopies

We use the same notations for $M_I = M \times I, M_t, \mathbb{C}_{M_t}, \cdots$ as in the previous subsection.

Lemma 2.5. *Let $F \in Sh(M)$. Let $\varphi : M \to \mathbb{R}$ be a C^1 function, such that $d\varphi(x) \neq 0$ for $x \in \varphi^{-1}([0,1])$.*
(1) For $s \in (0,1)$, let $U_s = \{x : \varphi(x) < s\}$, and let $U_1 = \cup_s U_s$. If

$$SS^\infty(\mathbb{C}_{U_s}) \cap SS^\infty(F) = \emptyset, \ \forall\, 0 < s < 1,$$

then

$$\mathrm{Hom}(\mathbb{C}_{U_1}, F) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{C}_{U_s}, F), \ \forall\, 0 < s < 1.$$

(2) For $s \in (0,1)$, let $Z_s = \{x : \varphi(x) \leqslant s\}$, and let $Z_0 = \cap_s Z_s$. If

$$SS^\infty(\mathbb{C}_{Z_s}) \cap SS^\infty(F) = \emptyset, \ \forall\, 0 < s < 1,$$

then

$$\mathrm{Hom}(\mathbb{C}_{Z_s}, F) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{C}_{Z_0}, F), \ \forall\, 0 < s < 1.$$

Proof. (1) is a special case in [GKS12, Prop 1.8]. (2) follows from (1) and

$$0 \to \mathbb{C}_{M \setminus \mathbb{Z}_s} \to \mathbb{C}_M \to \mathbb{C}_{Z_s} \to 0.$$

□

The following lemma is also often used.

Lemma 2.6 Petrowsky theorem for sheaves [KS13]. *Let $F, G \in Sh(M)$ be (cohomologically) constructible complexes of sheaves. If $SS^\infty(F) \cap SS^\infty(G) = \emptyset$, then the natural morphism*

$$\mathcal{H}om(F, \mathbb{C}_M) \otimes G \to \mathcal{H}om(F, G)$$

is an isomorphism.

Corollary 2.7. *If F_I be an isotopy of sheaves, then*

$$\mathcal{H}om(\mathbb{C}_{M_t}, F_I) \simeq \mathbb{C}_{M_t}[-1] \otimes F_I$$

Proposition 2.8. *Let G_I and F_I be non-characteristic isotopy of sheaves, then $\mathcal{H}om(F_I, G_I)$ is an isotopy of sheaves. In particular,*

$$\mathrm{Hom}(F_t, G_t) \simeq \mathrm{Hom}(F_s, G_s) \quad \textit{for all } t, s \in I$$

Proof. G_I and F_I being non-characteristic implies $SS^\infty(G_I) \cap SS^\infty(F_I) = \emptyset$, hence we can bound the singular support of the hom sheaf as [KS13]

$$SS(\mathcal{H}om(F_I, G_I)) \subset SS(G_I) + SS(F_I)^a.$$

Again, using G_I and F_I being non-characteristic, we have

$$SS^\infty(\mathcal{H}om(F_I, G_I)) \cap SS^\infty(\mathbb{C}_{M_t}) = \emptyset \quad \text{for all } t, s \in I.$$

Hence $\mathcal{H}om(F_I, G_I))$ is an isotopy of sheaves. For the second statement, we have

$$\begin{aligned} &\mathrm{Hom}(F_t, G_t) \\ =\ &\mathrm{Hom}(j_t{}^* F_I, j_t{}^* G_I) \simeq \mathrm{Hom}(F_I, j_{t*} j_t^* G_I) \simeq \mathrm{Hom}(F_I, \mathbb{C}_{M_t} \otimes G_I) \\ \simeq\ &\mathrm{Hom}(F_I, \mathcal{H}om(\mathbb{C}_{M_t}, G_I)[1]) \simeq \mathrm{Hom}(\mathbb{C}_{M_t}, \mathcal{H}om(F_I, G_I))[1] \\ \simeq\ &\mathrm{Hom}(\mathbb{C}_t, \pi_{I*}\mathcal{H}om(F_I, G_I))[1] \simeq [\pi_{I*}\mathcal{H}om(F_I, G_I)]_t \end{aligned} \tag{6}$$

then the result follows since $\pi_{I*}(\mathcal{H}om(F_I, G_I))$ is a local system. □

2.5 Invariance of morphisms under Reeb perturbations

Sometimes we want to vary G, F while preserving $\mathrm{Hom}(F, G)$, but $SS^\infty(G) \cap SS^\infty(F) \neq \emptyset$, e.g. $F = G$. Here we borrow an idea from infinitesimally wrapped Fukaya-category [NZ09], that to compute $\mathrm{Hom}_{Fuk}(L_1, L_2)$ one needs to do perturbation to separate L_1, L_2 at infinity, one can perturb $L_2 \rightsquigarrow R^t L_2$ or $L_1 \rightsquigarrow R^{-t} L_1$ where R^t is the unit speed geodesic flow on T^*M (smoothed near zero section) for positive small time t, small enough so that no new intersections are created between L_1, L_2 at infinity.

Fix a Riemannian metric g on M, and identify S^*M with $T^\infty M$, so that the Reeb flow R^t is the unit speed geodesic flow. Let $r_{inj}(M, g)$ be the injective radius of (M, g). Let $\hat{R}^t$ be the GKS quantization of R^t. The remaining part of this subsection will be devoted to prove the following Proposition.

Proposition 2.9. *Let $\Lambda^\infty \subset T^\infty M$ be a Legendrian, and $0 < \epsilon < r_{inj}(M, g)$ be small enough such that*

$$\Lambda^\infty \cap R^t \Lambda^\infty = \emptyset, \quad \forall\, 0 < |t| < \epsilon.$$

(1) For any $F \in Sh(M, \Lambda), 0 \leqslant t < \epsilon$, we have a canonical morphism

$$F \to \hat{R}^t F.$$

(2) For any $F, G \in Sh(M, \Lambda), 0 \leqslant t < \epsilon$, we have canonical quasi-isomorphisms

$$\mathrm{Hom}(F, G) \xrightarrow{\sim} \mathrm{Hom}(F, \hat{R}^t G), \quad \mathrm{Hom}(F, G) \xrightarrow{\sim} \mathrm{Hom}(\hat{R}^{-t} F, G)$$

Proof. For any $0 \leqslant t < \epsilon$, define

$$K_t = \mathbb{C}_{\{(x,y) | d_g(x,y) \leqslant t\}} \in Sh(M \times M).$$

Then from [GKS12], we have

$$\hat{R}^t F = \pi_{1*}\mathcal{H}om(K_t, \pi_2^! F) = K_t^! F,$$

and

$$\hat{R}^{-t} F = \pi_{2!}(K_t \otimes \pi_1^* F) = (K_t)_! F,$$

where π_1 and π_2 are the projections from $M \times M$ to the first and second factor, and $\mathcal{H}om$ is the (dg derived) sheaf-hom. From the canonical restriction morphism $K_t \to K_0 = \mathbb{C}_\Delta$, where $\Delta \subset M \times M$ is the diagonal subset, we have

$$F = \pi_{1*}\mathcal{H}om(K_0, \pi_2^! F) \to \pi_{1*}\mathcal{H}om(K_t, \pi_2^! F) = \hat{R}^t F.$$

For the second statement, we first prove the following lemma.

LEMMA 2.10.

$$SS^\infty(K_t) \cap SS^\infty(\mathcal{H}om(\pi_1^* F, \pi_2^! G)) = \emptyset, \quad \forall 0 < t < \epsilon. \tag{7}$$

Proof. We identify the contact infinity $T^\infty M$ with the unit cosphere bundle S^*M. Assume that the intersection is non-empty and contains the point (x_1, x_2, p_1, p_2). Since $(x_1, x_2; p_1, p_2) \in SS^\infty(K_t)$, we have

$$d_g(x_1, x_2) = t.$$

Since $t < \epsilon < r_{inj}(M, g)$, there is a unique length t geodesic γ connecting x_1, x_2, and p_i is the unit tangent vector along γ at x_i pointing to the interior of the geodesic.

$$p_i = -\partial_{x_i} d_g(x_1, x_2).$$

Hence the geodesic flow on S^*M relates (x_i, p_i),

$$R^t(x_1, p_1) = (x_2, -p_2), \quad R^t(x_2, p_2) = (x_1, -p_1). \tag{8}$$

On the other hand, since $(x_1, x_2; p_1, p_2) \in SS^\infty(\mathcal{H}om(\pi_1^* F, \pi_2^! G))$, we have

$$(x_1, -p_1) \in SS^\infty(F), \quad (x_2, p_2) \in SS^\infty(G). \tag{9}$$

Hence, combining (8) and (9), we have

$$(x_1, -p_1) \in R^t(SS^\infty(G)) \cap SS^\infty(F) \subset R^t \Lambda^\infty \cap \Lambda^\infty$$

This contradicts with the displaceability of Λ^∞ for $t < \epsilon$. □

Now we come back to the proof of the main proposition. We have

$$\begin{aligned}
\mathrm{Hom}(F, G) &\simeq \Gamma(M, \mathcal{H}om(F, G)) \\
&\simeq \Gamma(M \times M, \mathcal{H}om(\mathbb{C}_\Delta, \mathcal{H}om(\pi_1^* F, \pi_2^! G)) \\
&\xrightarrow{\sim} \Gamma(M \times M, \mathcal{H}om(K_t, \mathcal{H}om(\pi_1^* F, \pi_2^! G)) \\
&\simeq \Gamma(M \times M, \mathcal{H}om(\pi_1^* F, \mathcal{H}om(K_t, \pi_2^! G)) \\
&\simeq \Gamma(M, \mathcal{H}om(F, \pi_{1*}\mathcal{H}om(K_t, \pi_2^! G)) \\
&\simeq \mathrm{Hom}(F, \hat{R}^t G).
\end{aligned}$$

where in the third step when we replace $\mathbb{C}_\Delta$ by K_t, we used the canonical morphism $K_t \to \mathbb{C}_\Delta$, and used Lemma 2.10 and Lemma 2.5(2) to show it is a quasi-isomorphism. □

We will use the following purely sheaf-theoretical statement later to study family of GKS quantization.

Proposition 2.11. *Let $I = (0,1)$, and $K_I \in Sh(M \times M \times I)$ be an isotopy of sheaves, such that $K_t = \mathbb{C}_{\Delta_t}$ for some closed subsets $\{\Delta_t\}_{0<t<1}$ satisfying*

$$\Delta_t \subset \Delta_s, \quad \forall 0 < t < s < 1, \text{ and } \bigcap_{t\in I} \Delta_t = \Delta_M = \{(x,x) : x \in M\}$$

Let $F, G \in Sh(M, \Lambda)$, and $\mathcal{H}om(\pi_1^* F, \pi_2^! G) \in Sh(M \times M)$ be the hom-sheaf. Assume

$$SS^\infty(K_t) \cap SS^\infty(\mathcal{H}om(\pi_1^* F, \pi_2^! G)) = \emptyset, \quad \forall t \in I$$

then

$$\mathrm{Hom}(F,G) \simeq \mathrm{Hom}(F, K_t^! G) \simeq \mathrm{Hom}(K_{t!} F, G), \quad \forall t \in I$$

where $K_t^!, K_{t!}$ are defined in (5).

Its proof is exactly as in Proposition 2.9 (2), where the condition provided in Lemma 2.10 is put into the hypothesis, hence we do not repeat here.

2.6 Limit of contact isotopy

Let $I = (0,1)$ and denote the following inclusions as

$$(0,1) \overset{j_I}{\hookrightarrow} \mathbb{R} \overset{j_0}{\hookleftarrow} \{0\}.$$

Proposition 2.12. *[TWZ19, Lemma 7.1] Let $F_I \in Sh(M_I)$ be an isotopy of constructible sheaves, and let $\Lambda_I^\infty = SS^\infty(F_I)$. Suppose the family $(\Lambda_t^\infty, t) \subset T^\infty M \times (0,1)$ has a closure in $T^\infty M \times [0,1)$ whose intersection with $T^\infty M \times \{0\}$ is a Legendrian Λ_0^∞, then the sheaf*

$$F_0 := (j_0)^* (j_I)_* F_I. \tag{10}$$

has $SS^\infty(F_0) \subset \Lambda_0^\infty$.

Proof. These are corollaries of results in [KS13]. By [KS13, Thm 6.3.1], a point $(x,p;0,-1) \in \dot{T}^*M \times T^*\mathbb{R}$ belongs to $SS((j_I)_* F_I)$ only if (x,p) is the limit of a sequence of point $(x_n, p_n) \in \Lambda_{t_n}$ where $t_n \to 0$, i.e., $(x,p) \in \Lambda_0$. By [KS13, Prop 5.4.5], $SS(F_0) \subset SS((j_I)_* F_I)|_0 = \Lambda_0$, hence $SS^\infty(F_0) \subset \Lambda_0^\infty$. □

Let (U, X) be a convex tube for a Legendrian $\mathcal{L} \subset S^*M$. Let X be extended from a neighborhood of $\bar{U}$ to all of S^*M. Let

$$\hat{X}^{[0,\infty)} : Sh(M) \to Sh(M \times [0,\infty))$$

be the sheaf quantization of the flow X. And let

$$j_{[0,\infty)} : \ [0,\infty) \hookrightarrow \ [0,\infty] \ \hookleftarrow \{\infty\} : \ j_\infty.$$

Then we define the functor $\Pi_X := (id_M \times j_\infty)^* \circ (id_M \times j_{[0,\infty)})_* \circ \hat{X}^{[0,\infty)} : Sh(M) \to Sh(M)$.

Let $Sh(M,U)$ denote the subcategory of $Sh(M)$ consisting of sheaves F with $SS^\infty(F) \subset U$.

Proposition 2.13. *When restricted to $Sh(M,U)$, we have $\Pi_{U,X} = \Pi_X|_{Sh(M,U)} : Sh(M,U) \to Sh(M,\mathcal{L})$*

Proof. This follows from the definition of convex tube and Proposition 2.12. □

3. Existence and uniqueness of the extension

In this section, we prove our main result, Theorem 0.5. In the remaining part of this section, we will sometimes identify $\Lambda_t^\infty \subset T^\infty M$ with $\mathcal{L}_t \subset S^*M$, and identify Reeb flow with geodesic flow.

3.1 Uniqueness of extension

Recall from Proposition 1.9, existence of strong isotopy of convex tubes implies uniform displaceablity of the family $\{\mathcal{L}_t\}$.

Proposition 3.1. *Let Λ_t^∞ be a family of Legendrian in $T^\infty M$ that are uniformly displaceable with parameter ϵ. Then, the restriction functor ι_t^* is fully-faithful for all t.*

Proof. For $0 \leqslant s < \epsilon$, we define a family of kernels in $Sh((M_1 \times I_1) \times (M_2 \times I_2))$:

$$K_s := \mathbb{C}_{d(x_1,x_2) \leqslant s} \boxtimes \mathbb{C}_{t_1 = t_2}. \tag{11}$$

One can check that K_s generates the slice-wise geodesic flow, i.e., if $F_I \in Sh(M_I)$, and

$$K_s^! F_I := \pi_{1*} \mathcal{H}om(K_s, \pi_2^! F_I)$$

then we have

$$SS^\infty((K_s^! F_I)|_{M_t}) = R^s SS^\infty(F_I|_{M_t})$$

where π_i is the projection from $(M_1 \times I_1) \times (M_2 \times I_2)$ to $M_i \times I_i$, and R^s is the Reeb (geodesic) flow for time s.

We first prove the following claim: for any $F_I, G_I \in Sh(M_I, \Lambda_I^\infty)$, we have

$$\mathrm{Hom}(\mathbb{C}_{M \times (a,b)}, \mathcal{H}om(F_I, G_I)) \text{ is independent of } (a,b) \subset I.$$

It suffices to prove the case for the right end-point b. To use the estimate of the singular support of the hom-sheaf, we would like to perturb G_I by the fiberwise Reeb flow.

Lemma 3.2. *For any $0 < s < \epsilon$, we have*

$$\mathrm{Hom}(\mathbb{C}_{M \times \{t\}}, \mathcal{H}om(F_I, G_I)) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{C}_{M \times \{t\}}, \mathcal{H}om(F_I, K_s^! G_I)).$$

Furthermore, $\mathrm{Hom}(\mathbb{C}_{M \times \{t\}}, \mathcal{H}om(F_I, K_s^! G_I))$ is independent of t. The same is true if we replace $\{t\}$ by any sub-interval, eg. $[a,b]$, (a,b) of I.

Proof. Unwinding the definition of $K_s^!$, we have

$$\begin{aligned}
&\mathrm{Hom}(\mathbb{C}_{M \times \{t\}}, \mathcal{H}om(F_I, K_s^! G_I)) \\
&= \mathrm{Hom}(\mathbb{C}_{M \times \{t\}}, \mathcal{H}om(F_I, \pi_{1*} \mathcal{H}om(K_s, \pi_2^! G_I))) \\
&= \mathrm{Hom}(\mathbb{C}_{M \times \{t\}}, \pi_{1*} \mathcal{H}om(\pi_1^* F_I, \mathcal{H}om(K_s, \pi_2^! G_I))) \\
&= \mathrm{Hom}(\pi_1^* \mathbb{C}_{M \times \{t\}}, \mathcal{H}om(K_s, \mathcal{H}om(\pi_1^* F_I, \pi_2^! G_I)))
\end{aligned}$$

We claim that

$$SS^\infty(\pi_1^* \mathbb{C}_{M \times \{t\}}) \cap SS^\infty \mathcal{H}om(K_s, \mathcal{H}om(\pi_1^* F_I, \pi_2^! G_I)) = \emptyset, \ \forall\, 0 < s < \epsilon. \tag{12}$$

The verification is straightforward though a bit tedious, thus we leave it to the readers.

From this claim, and

$$\begin{aligned}
&\mathrm{Hom}(\pi_1^* \mathbb{C}_{M \times \{t\}}, \mathcal{H}om(K_s, \mathcal{H}om(\pi_1^* F_I, \pi_2^! G_I))) \\
&\simeq \mathrm{Hom}(\pi_1^* \mathbb{C}_{M \times \{t\}} \otimes K_s, \mathcal{H}om(\pi_1^* F_I, \pi_2^! G_I)))
\end{aligned}$$

we may apply Lemma 2.5 (2) on shrinking closed set, to get

$$\mathrm{Hom}(\pi_1^* \mathbb{C}_{M \times \{t\}} \otimes K_s, \mathcal{H}om(\pi_1^* F_I, \pi_2^! G_I))) \simeq \mathrm{Hom}(\pi_1^* \mathbb{C}_{M \times \{t\}} \otimes K_0, \mathcal{H}om(\pi_1^* F_I, \pi_2^! G_I)))$$

for all $0 < s < \epsilon$. This proves the first statement of the Lemma.

The statement about independence of t follows from (12) and Proposition 2.8.

The case for sub-interval can be proved similarly, and we omit the details. □

Now, we finish to prove the proposition. By Lemma 3.2,

$$\mathrm{Hom}(\mathbb{C}_{M\times(a,b)}, \mathcal{H}om(F_I, G_I))$$

is independent of (a,b), hence we may shrink from $(0,1)$ to an arbitrary small neighborhood of t. Then we have

$$\begin{aligned}\mathrm{Hom}(F_I, G_I) &\simeq [\pi_{I*}(\mathcal{H}om(F_I, G_I))]_t \simeq [\pi_{I*}(\mathcal{H}om(F_I, K_s^! G_I)]_t \\ &\simeq \mathrm{Hom}(\iota_t^* F_I, \iota_t^* K_s^! G_I) \simeq \mathrm{Hom}(F_t, R^s G_t) \simeq \mathrm{Hom}(F_t, G_t)\end{aligned}$$

where $0 < s < \epsilon$, and we used small Reeb perturbation to make $F_I, K_s^! G_I$ non-charactersitic isotopy of sheaves, then apply Eq.(6) in Proposition 2.8. □

Proposition 3.3. *Let $\{\Lambda_t^\infty\}$ be a family of Legendrian in $T^\infty M$ that are uniformly displaceable with parameter ϵ. For a given t, let $F_t \in Sh(M, \Lambda_t^\infty)$. Suppose we have F_I' and F_I'' in $Sh(M_I, \Lambda_I^\infty)$ and isomorphisms*

$$f : F_I'|_t \xrightarrow{\sim} F_t, \quad g : F_I''|_t \xrightarrow{\sim} F_t,$$

then there exists a canonical isomorphism

$$\Phi : F_I' \to F_I''$$

such that $\Phi|_t = g^{-1} \circ f : F_I'|_t \to F_I''|_t$.

Proof. The proof follows from Proposition 3.1 by standard argument. □

3.2 Existence of local extension

Proposition 3.4. *Let $I = [0,1]$. Let $\mathcal{L}_I$ be a strong isotopy of Legendrians in $S^*M \times T^*I$ with slice over t denoted as $\mathcal{L}_t$. Let (U_I, X_I) be a strong isotopy of convex tubes for $\mathcal{L}_I$. Then for any $t \in I$ and $F_t \in Sh(M, \mathcal{L}_t)$, there exists an interval $J \supset t$ and $F_J \in Sh(M_J, \mathcal{L}_J)$, such that $F_J|_t = F_t$, where $M_J = M \times J, \mathcal{L}_J = \mathcal{L}_I \cap S^*M \times T^*J$.*

Proof. For any interval $J \subset I$, let $U_J = U_I \cap S^*M \times T^*J$. Then, for J small enough containing t, we have $\mathcal{L}_t \times T_J^* J \subset U_J$. Let X_J denote the restriction of X_I to X_J, then if we define (see Proposition 2.13 for definition of $\Pi_{U,X}$)

$$F_J := \Pi_{(U_J, X_J)}(F_t \boxtimes \mathbb{C}_J),$$

we have $F_J|_t = F_t$ and $SS^\infty(F_J) \in \mathcal{L}_J$. □

3.3 Proof of theorem 0.5

By the local extension result (Proposition 3.4) and uniqueness of extension result, for any $t \in I = [0,1]$ and $F_t \in Sh(M, \mathcal{L}_t)$, we can extend F_t to $F_I \in Sh(M_I, \mathcal{L}_I)$, such that $F_I|_t = F_t$. Hence the functor ι_t^* is fully-faithful (Proposition 3.1) and essentially surjective, thus is an equivalence.

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