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Zero loci of Bernstein–Sato ideals

Nero Budur[1] · Robin van der Veer[1] ·
Lei Wu[2] · Peng Zhou[3]

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Abstract We prove a conjecture of the first author relating the Bernstein–Sato ideal of a finite collection of multivariate polynomials with cohomology support loci of rank one complex local systems. This generalizes a classical theorem of Malgrange and Kashiwara relating the b-function of a multivariate polynomial with the monodromy eigenvalues on the Milnor fibers cohomology.

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✉ Nero Budur
nero.budur@kuleuven.be

Robin van der Veer
robin.vanderveer@kuleuven.be

Lei Wu
lwu@math.utah.edu

Peng Zhou
pzhou.math@gmail.com

1 KU Leuven, Celestijnenlaan 200B, 3001 Leuven, Belgium

2 Department of Mathematics, University of Utah, 155 S. 1400 E, Salt Lake City, UT 84112, USA

3 Institut des Hautes Études Scientifiques, 35 Route de Chartres, Le Bois-Marie, 91440 Bures-sur-Yvette, France

Contents

1 Introduction

1.1.

Let $F = (f_1, \ldots, f_r) : (X, x) \to (\mathbb{C}^r, 0)$ be the germ of a holomorphic map from a complex manifold X. The *(local) Bernstein–Sato ideal* of F is the ideal B_F in $\mathbb{C}[s_1, \ldots, s_r]$ generated by all $b \in \mathbb{C}[s_1, \ldots, s_r]$ such that in a neighborhood of x

$$b \prod_{i=1}^{r} f_i^{s_i} = P \cdot \prod_{i=1}^{r} f_i^{s_i+1} \tag{1.1}$$

for some $P \in \mathscr{D}_X[s_1, \ldots, s_r]$, where $\mathscr{D}_X$ is the ring of holomorphic differential operators. Sabbah [22,23] showed that B_F is not zero.

1.2.

If $F = (f_1, \ldots, f_r) : X \to \mathbb{C}^r$ is a morphism from a smooth complex affine irreducible algebraic variety, the *(global) Bernstein–Sato ideal* B_F is defined as the ideal generated by all $b \in \mathbb{C}[s_1, \ldots, s_r]$ such that (1.1) holds globally with $\mathscr{D}_X$ replaced by the ring of algebraic differential operators. The global Bernstein–Sato ideal is the intersection of all the local ones at points x with some $f_i(x) = 0$, and there are only finitely many distinct local Bernstein–Sato ideals, see [1,8].

1.3.

It was clear from the beginning that B_F contains some topological information about F, e.g. [18,19,22,23]. However, besides the case $r = 1$, it was not clear what precise topological information is provided by B_F. Later, a conjecture based on computer experiments was formulated in [10] addressing this problem. In this article we prove this conjecture.

1.4.

Let us recall what happens in the case $r = 1$. If $f : X \to \mathbb{C}$ is a regular function on a smooth complex affine irreducible algebraic variety, or the germ at $x \in X$ of a holomorphic function on a complex manifold, the monic generator of the Bernstein–Sato ideal of f in $\mathbb{C}[s]$ is called the *Bernstein–Sato polynomial*, or the *b-function*, of f and it is denoted by $b_f(s)$. The non-triviality of $b_f(s)$ is a classical result of Bernstein [5] in the algebraic case, and Björk [6] in the analytic case. One has the following classical theorem, see [16,17,21]:

Theorem 1.4.1 *Let $f : X \to \mathbb{C}$ be a regular function on a smooth complex affine irreducible algebraic variety, or the germ at $x \in X$ of a holomorphic function on a complex manifold, such that f is not invertible. Let $b_f(s) \in \mathbb{C}[s]$ be the Bernstein–Sato polynomial of f. Then:*

(i) (Malgrange, Kashiwara) The set

$$\{exp(2\pi i\alpha) \mid \alpha \text{ is a root of } b_f(s)\}$$

is the set of monodromy eigenvalues on the nearby cycles complex of f.

(ii) (Kashiwara) The roots of $b_f(s)$ are negative rational numbers.

(iii) (Monodromy Theorem) The monodromy eigenvalues on the nearby cycles complex of f are roots of unity.

The definition of the nearby cycles complex is recalled in Sect. 2. In the algebraic case, $b_f(s)$ provides thus an algebraic computation of the monodromy eigenvalues.

1.5.

We complete in this article the extension of this theorem to a finite collection of functions as follows. Let

$$Z(B_F) \subset \mathbb{C}^r$$

be the zero locus of the Bernstein–Sato ideal of F. Let $\psi_F\mathbb{C}_X$ be the specialization complex[1] defined by Sabbah [24]; the definition will be recalled in Sect. 2. This complex is a generalization of the nearby cycles complex to a finite collection of functions, the monodromy action being now given by r simultaneous monodromy actions, one for each function f_i. Let

$$\mathcal{S}(F) \subset (\mathbb{C}^*)^r$$

[1] This is called “le complexe d’Alexander” in [24].

be the support of this monodromy action on $\psi_F \mathbb{C}_X$. In the case $r = 1$, this is the set of eigenvalues of the monodromy on the nearby cycles complex. The support $\mathcal{S}(F)$ has a few other topological interpretations, one being in terms of cohomology support loci of rank one local systems, see Sect. 2. Let $\mathrm{Exp} : \mathbb{C}^r \to (\mathbb{C}^*)^r$ be the map $\mathrm{Exp}(\_) = \exp(2\pi i \_)$.

Theorem 1.5.1 *Let $F = (f_1, \ldots, f_r) : X \to \mathbb{C}^r$ be a morphism of smooth complex affine irreducible algebraic varieties, or the germ at $x \in X$ of a holomorphic map on a complex manifold, such that not all f_i are invertible. Then:*

(i) $\mathrm{Exp}(Z(B_F)) = \mathcal{S}(F)$.
(ii) Every irreducible component of $Z(B_F)$ of codimension 1 is a hyperplane of type $a_1 s_1 + \ldots + a_r s_r + b = 0$ with $a_i \in \mathbb{Q}_{\geq 0}$ and $b \in \mathbb{Q}_{>0}$. Every irreducible component of $Z(B_F)$ of codimension > 1 can be translated by an element of $\mathbb{Z}^r$ inside a component of codimension 1.
(iii) $\mathcal{S}(F)$ is a finite union of torsion-translated complex affine subtori of codimension 1 in $(\mathbb{C}^*)^r$.

Thus in the algebraic case, B_F gives an algebraic computation of $\mathcal{S}(F)$.

Part (i) was conjectured in [10], where one inclusion was also proved, namely that $\mathrm{Exp}(Z(B_F))$ contains $\mathcal{S}(F)$. See also [11, Conjecture 1.4, Remark 2.8].

Regarding part (iii), Sabbah [24] showed that $\mathcal{S}(F)$ is included in a finite union of torsion-translated complex affine subtori of codimension 1. Here a complex affine subtorus of $(\mathbb{C}^*)^r$ means an algebraic subgroup $G \subset (\mathbb{C}^*)^r$ such that $G \cong (\mathbb{C}^*)^p$ as algebraic groups for some $0 \leq p \leq r$. In [12], it was proven that every irreducible component of $\mathcal{S}(F)$ is a torsion-translated subtorus. Finally, part (iii) was proven as stated in [11].

The first assertion of part (ii), about the components of codimension one of $Z(B_F)$, is due to Sabbah [22,23] and Gyoja [14].

In light of the conjectured equality in part (i), it was therefore expected that part (iii) would hold for $\mathrm{Exp}(Z(B_F))$. This is equivalent to the second assertion in part (ii), about the smaller-dimensional components of $Z(B_F)$, and it was confirmed unconditionally by Maisonobe [20, Résultat 3]. This result of Maisonobe will play a crucial role in this article.

In this article we complete the proof of Theorem 1.5.1 by proving the other inclusion from part (i):

Theorem 1.5.2 *Let F be as in Theorem 1.5.1. Then* $\mathrm{Exp}(Z(B_F))$ *is contained in $\mathcal{S}(F)$.*

The proof uses Maisonobe's results from [20] and uses an analog of the Cohen-Macaulay property for modules over the noncommutative ring $\mathscr{D}_X[s_1, \dots, s_r]$.

1.6.

Algorithms for computing Bernstein–Sato ideals are now implemented in many computer algebra systems. The availability of examples where the zero loci of Bernstein–Sato ideals contain irreducible components of codimension > 1 suggests that this is not a rare phenomenon, see [1]. The stronger conjecture that Bernstein–Sato ideals are generated by products of linear polynomials remains open, [10, Conjecture 1]. This would imply in particular that all irreducible components of $Z(B_F)$ are linear.

1.7.

In Sect. 2, we recall the definition and some properties of the support of the specialization complex. In Sect. 3 we give the proof of Theorem 1.5.2. Section 4 is an appendix reviewing basic facts from homological algebra for modules over not-necessarily commutative rings.

2 The support of the specialization complex

2.1 Notation

Let $F = (f_1, \dots, f_r) : X \to \mathbb{C}^r$ be a holomorphic map on a complex manifold X of dimension $n > 0$. Let $f = \prod_{i=1}^r f_i$, $D = f^{-1}(0)$, $U = X \setminus D$. Let $i : D \to X$ be the closed embedding and $j : U \to X$ the open embedding. We are assuming that not all f_i are invertible, which is equivalent to $D \neq \emptyset$.

We use the notation $\mathbf{s} = (s_1, \dots, s_r)$ and $\mathbf{f^s} = \prod_{i=1}^r f_i^{s_i}$, and in general tuples of numbers will be in bold, e.g. $\mathbf{1} = (1, \dots, 1)$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r)$, etc.

2.2 Specialization complex

Let $D_c^b(A_D)$ be the derived category of bounded complexes of A_D-modules with constructible cohomology, where A is the affine coordinate ring of $(\mathbb{C}^*)^r$ and A_D is the constant sheaf of rings on D with stalks A. Sabbah [24] defined the *specialization complex* $\psi_F \mathbb{C}_X$ in $D_c^b(A_D)$ by

$$\psi_F \mathbb{C}_X = i^{-1} R j_* R\pi_! (j \circ \pi)^{-1} \mathbb{C}_X,$$

where $\pi : U \times_{(\mathbb{C}^*)^r} \mathbb{C}^r \to U$ is the first projection from the fibered product obtained from $F_{|U} : U \to (\mathbb{C}^*)^r$ and the universal covering map $exp : \mathbb{C}^r \to (\mathbb{C}^*)^r$.

The *support of the specialization complex* $\mathcal{S}(F)$ is defined as the union over all $i \in \mathbb{Z}$ and $x \in D$ of the supports in $(\mathbb{C}^*)^r$ of the cohomology stalks $\mathcal{H}^i(\psi_F\mathbb{C}_X)_x$ viewed as finitely generated A-modules.

If F is only given as the germ at a point $x \in X$ of a holomorphic map, by $\psi_F\mathbb{C}_X$ we mean the restriction of the specialization complex to a very small open neighboorhood of $x \in X$.

When $r = 1$, that is, in the case of only one holomorphic function $f : X \to \mathbb{C}$, the specialization complex equals the shift by $[-1]$ of Deligne's *nearby cycles complex* defined as

$$\psi_f\mathbb{C}_X = i^{-1}R(j \circ \pi)_*(j \circ \pi)^{-1}\mathbb{C}_X.$$

The complex numbers in the support $\mathcal{S}(f) \subset \mathbb{C}^*$ are called the *monodromy eigenvalues* of the nearby cycles complex of f.

2.3 Cohomology support loci

It was proven in [10,11] that $\mathcal{S}(F)$ admits an equivalent definition, without involving derived categories, as the union of cohomology support loci of rank one local systems on small ball complements along the divisor D. More precisely,

$$\mathcal{S}(F) = \{\boldsymbol{\lambda} \in (\mathbb{C}^*)^r \mid H^i(U_x, L_{\boldsymbol{\lambda}}) \neq 0 \text{ for some } x \in D \text{ and } i \in \mathbb{Z}\},$$

where U_x is the intersection of U with a very small open ball in X centered at x, and $L_{\boldsymbol{\lambda}}$ is the rank one $\mathbb{C}$-local system on U obtained as the pullback via $F : U \to (\mathbb{C}^*)^r$ of the rank one local system on $(\mathbb{C}^*)^r$ with monodromy λ_i around the i-th missing coordinate hyperplane.

If F is only given as the germ at (X, x) of a holomorphic map, $\mathcal{S}(F)$ is defined as above by replacing X with a very small open neighboorhood of x.

For one holomorphic function $f : X \to \mathbb{C}$, the support $\mathcal{S}(f)$ is the union of the sets of eigenvalues of the monodromy acting on cohomologies of the Milnor fibers of f along points of the divisor $f = 0$, see [12, Proposition 1.3].

With this description of $\mathcal{S}(F)$, the following involutivity property was proven:

Lemma 2.3.1 ([12, Theorem 1.2]) *Let* $\boldsymbol{\lambda} \in (\mathbb{C}^*)^r$. *Then* $\boldsymbol{\lambda} \in \mathcal{S}(F)$ *if and only if* $\boldsymbol{\lambda}^{-1} \in \mathcal{S}(F)$.

2.4 Non-simple extension loci

An equivalent definition of $\mathcal{S}(F)$ was found by [11, §1.4] as a locus of rank one local systems on U with non-simple higher direct image in the category of perverse sheaves on X:

$$\mathcal{S}(F) = \left\{ \boldsymbol{\lambda} \in (\mathbb{C}^*)^r \mid \frac{Rj_* L_{\boldsymbol{\lambda}}[n]}{j_{!*} L_{\boldsymbol{\lambda}}[n]} \neq 0 \right\},$$

where $L_{\boldsymbol{\lambda}}$ is the rank one local system on U as in 2.3. This description is equivalent to

$$\mathcal{S}(F) = \left\{ \boldsymbol{\lambda} \in (\mathbb{C}^*)^r \mid j_! L_{\boldsymbol{\lambda}}[n] \to Rj_* L_{\boldsymbol{\lambda}}[n] \text{ is not an isomorphism} \right\},$$

the map being the natural one.

2.5 $\mathscr{D}$-module theoretic interpretation

Recall that for $\boldsymbol{\alpha} \in \mathbb{C}^r$,

$$\mathscr{D}_X[\mathbf{s}]\mathbf{f}^{\mathbf{s}}$$

is the natural left $\mathscr{D}_X[\mathbf{s}]$-submodule of the free rank one $\mathscr{O}_X[\mathbf{s}, f^{-1}]$-module $\mathscr{O}_X[\mathbf{s}, f^{-1}] \cdot \mathbf{f}^{\mathbf{s}}$ generated by the symbol $\mathbf{f}^{\mathbf{s}}$. For $r = 1$, see for example Walther [25].

We denote by $D^b_{rh}(\mathscr{D}_X)$ the derived category of bounded complexes of regular holonomic $\mathscr{D}_X$-modules. We denote by $\mathrm{DR}_X : D^b_{rh}(\mathscr{D}_X) \to D^b_c(\mathbb{C}_X)$ the de Rham functor, an equivalence of categories. The following is a particular case of [27, Theorem 1.3 and Corollary 5.5], see also [4]:

Theorem 2.5.1 *Let $F = (f_1, \ldots, f_r) : X \to \mathbb{C}^r$ be a morphism from a smooth complex algebraic variety. Let $\boldsymbol{\alpha} \in \mathbb{C}^r$ and $\boldsymbol{\lambda} = \exp(-2\pi i \boldsymbol{\alpha})$. Let $L_{\boldsymbol{\lambda}}$ be the rank one local system on U defined as in 2.3, and let $\mathcal{M}_{\boldsymbol{\lambda}} = L_{\boldsymbol{\lambda}} \otimes_{\mathbb{C}} \mathscr{O}_U$ the corresponding flat line bundle, so that*

$$\mathrm{DR}_U(\mathcal{M}_{\boldsymbol{\lambda}}) = L_{\boldsymbol{\lambda}}[n]$$

as perverse sheaves on U. For every integer $k \gg \|\alpha\|$ and $\mathbf{k} = (k, \ldots, k) \in \mathbb{Z}^r$, there are natural quasi-isomorphisms in $D^b_{rh}(\mathscr{D}_X)$

$$\mathscr{D}_X[\mathbf{s}]\mathbf{f}^{\mathbf{s}+\mathbf{k}} \otimes_{\mathbb{C}[\mathbf{s}]} \mathbb{C}_{\boldsymbol{\alpha}} = j_! \mathcal{M}_{\boldsymbol{\lambda}},$$
$$\mathscr{D}_X[\mathbf{s}]\mathbf{f}^{\mathbf{s}-\mathbf{k}} \otimes_{\mathbb{C}[\mathbf{s}]} \mathbb{C}_{\boldsymbol{\alpha}} = j_* \mathcal{M}_{\boldsymbol{\lambda}},$$

where $\mathbb{C}_{\boldsymbol{\alpha}}$ is the residue field of $\boldsymbol{\alpha}$ in $\mathbb{C}^r$.

Proposition 2.5.2 *With F as in Theorem 2.5.1,*

$$\mathcal{S}(F) = \mathrm{Exp}\left\{\boldsymbol{\alpha} \in \mathbb{C}^r \mid \frac{\mathscr{D}_X[\mathbf{s}]\mathbf{f}^{\mathbf{s}-\mathbf{k}}}{\mathscr{D}_X[\mathbf{s}]\mathbf{f}^{\mathbf{s}+\mathbf{k}}} \otimes_{\mathbb{C}[\mathbf{s}]} \mathbb{C}_{\boldsymbol{\alpha}} \neq 0 \textit{ for all } k \gg \|\alpha\|\right\}.$$

Proof Applying DR_X directly to Theorem 2.5.1, one obtains that

$$\begin{aligned}&\mathcal{S}(F)\\ &= \mathrm{Exp}\left\{-\boldsymbol{\alpha} \in \mathbb{C}^r \mid \frac{\mathscr{D}_X[\mathbf{s}]\mathbf{f}^{\mathbf{s}-\mathbf{k}}}{\mathscr{D}_X[\mathbf{s}]\mathbf{f}^{\mathbf{s}+\mathbf{k}}} \otimes^L_{\mathbb{C}[\mathbf{s}]} \mathbb{C}_{\boldsymbol{\alpha}} \neq 0 \text{ in } D^b_{rh}(\mathscr{D}_X) \text{ for all } k \gg \|\alpha\|\right\}\end{aligned}$$

by the interpretation of $\mathcal{S}(F)$ from 2.4. Since $j_!\mathcal{M}_{\boldsymbol{\lambda}} \to j_*\mathcal{M}_{\boldsymbol{\lambda}}$ is a morphism of holonomic $\mathscr{D}_X$-modules of same length, the kernel and cokernel must simultaneously vanish or not. Thus, we can replace the derived tensor product with the usual tensor product. We then can replace $-\boldsymbol{\alpha}$ with $\boldsymbol{\alpha}$ by Lemma 2.3.1. □

For related work in a particular case, see [2].

Remark 2.5.3 Note that Theorem 2.5.1 is stated in the algebraic case only. However, the proof from [4,27] extends to the case when X is a complex manifold by replacing $j_!\mathcal{M}_{\boldsymbol{\lambda}}$, $j_*\mathcal{M}_{\boldsymbol{\lambda}}$ with $\mathcal{M}(!D)$, $\mathcal{M}(*D)$, respectively, where $\mathcal{M}$ is the analytic $\mathscr{D}_X$-module $\mathscr{D}_X \cdot \mathbf{f}^{\boldsymbol{\alpha}}$ whose restriction to U is $\mathcal{M}_{\boldsymbol{\lambda}}$. Hence the last proposition also holds in the analytic case.

Since the tensor product is a right exact functor, as a consequence one has the following corollary which also follows from the proof of [10, Proposition 1.7]:

Proposition 2.5.4 *If $\boldsymbol{\alpha}$ is in $\mathbb{C}^r$ and*

$$\frac{\mathscr{D}_X[\mathbf{s}]\mathbf{f}^{\mathbf{s}}}{\mathscr{D}_X[\mathbf{s}]\mathbf{f}^{\mathbf{s}+\mathbf{1}}} \otimes_{\mathbb{C}[\mathbf{s}]} \mathbb{C}_{\boldsymbol{\alpha}} \neq 0,$$

then $\mathrm{Exp}(\boldsymbol{\alpha})$ *is in* $\mathcal{S}(F)$.

This proposition can be interpreted as to say that the difficulty in proving Theorem 1.5.2 is the lack of a Nakayama Lemma for the non-finitely generated $\mathbb{C}[\mathbf{s}]$-module $\mathscr{D}_X[\mathbf{s}]\mathbf{f}^{\mathbf{s}}/\mathscr{D}_X[\mathbf{s}]\mathbf{f}^{\mathbf{s}+\mathbf{1}}$.

3 Relative holonomic modules

In this section we will provide necessary conditions for modules over $\mathscr{D}_X[\mathbf{s}]$ to obey an analog of Nakayama Lemma, and we will see that $\mathscr{D}_X[\mathbf{s}]\mathbf{f}^{\mathbf{s}}/\mathscr{D}_X[\mathbf{s}]\mathbf{f}^{\mathbf{s}+\mathbf{1}}$

satisfies these conditions at least generically. Using Maisonobe's results [20], this will prove Theorem 1.5.2.

3.1.

For simplicity, we will assume from now that we are in the algebraic case, namely, X is a smooth complex affine irreducible algebraic variety. We will treat the analytic case at the end.

We define an increasing filtration on the ring $\mathscr{D}_X$ by setting $F_i\mathscr{D}_X$ to consist of all operators of order at most i, that is, in local coordinates $(x_1, \dots, x_n)$ on X, the order of x_i is zero and the order of $\partial/\partial x_i$ is one.

We let R be a regular commutative finitely generated $\mathbb{C}$-algebra integral domain. We write

$$\mathscr{A}_R = \mathscr{D}_X \otimes_{\mathbb{C}} R,$$

and if $R = \mathbb{C}[\mathbf{s}]$ we write

$$\mathscr{A} = \mathscr{A}_{\mathbb{C}[\mathbf{s}]} = \mathscr{D}_X[\mathbf{s}].$$

The order filtration on $\mathscr{D}_X$ induces the *relative filtration* on $\mathscr{A}_R$ by

$$F_i\mathscr{A}_R = F_i\mathscr{D}_X \otimes_{\mathbb{C}} R.$$

The associated graded ring

$$\operatorname{gr}\mathscr{A}_R = \operatorname{gr}\mathscr{D}_X \otimes_{\mathbb{C}} R$$

is a regular commutative finitely generated $\mathbb{C}$-algebra integral domain, and it corresponds to the structure sheaf of $T^*X \times \operatorname{Spec} R$, where T^*X is the cotangent bundle of X. Thus $\mathscr{A}_R$ is an Auslander regular ring by Theorem 4.3.2. Moreover, the homological dimension is equal to the Krull dimension of $\operatorname{gr}\mathscr{A}_R$,

$$\operatorname{gl.dim}(\mathscr{A}_R) = 2n + \dim(R),$$

by Propositions 4.3.3, 4.4.2, and 4.5.1.

3.2.

Let N be a left (or right) $\mathscr{A}_R$-module. A *good filtration F on N over R* is an exhaustive filtration compatible with the relative filtration on $\mathscr{A}_R$ such that the associated graded module $\operatorname{gr} N$ is finitely generated over $\operatorname{gr}\mathscr{A}_R$, cf. 4.2. If N is finitely generated over $\mathscr{A}_R$, then good filtrations over R exist on N. We define the *relative characteristic variety of N over R* to be the support of gr N inside $T^*X \times \operatorname{Spec} R$, denoted by

$$\mathrm{Ch}^{\mathrm{rel}}(N).$$

Equivalently, $\mathrm{Ch}^{\mathrm{rel}}(N)$ is defined by the radical of the annihilator ideal of gr N in gr $\mathscr{A}_R$. The relative characteristic variety $\mathrm{Ch}^{\mathrm{rel}}(N)$ and the multiplicities $m_{\mathfrak{p}}(N)$ of gr N at generic points $\mathfrak{p}$ of the irreducible components of $\mathrm{Ch}^{\mathrm{rel}}(N)$ do not depend on the choice of a good filtration for N, by 4.2.1.

Remark 3.2.1 The good filtration F on N localizes, that is, if S is a multiplicatively closed subset of R, then

$$F_i(S^{-1}N) = S^{-1}F_iN$$

form a good filtration of $S^{-1}N$ over $S^{-1}R$, and hence

$$\mathrm{gr}\,(S^{-1}N) \simeq S^{-1}\mathrm{gr}\,N.$$

For a finitely generated $\mathscr{A}_R$-module N, we will denote by $j_{\mathscr{A}_R}(N)$, or simply $j(N)$, the grade number of N defined as in 4.3.

Lemma 3.2.2 *Suppose that N is a finitely generated $\mathscr{A}_R$-module. Then:*

(1) $j(N) + \dim(\mathrm{Ch}^{\mathrm{rel}}(N)) = 2n + \dim(R)$;
(2) if

$$0 \to N' \to N \to N'' \to 0$$

is a short exact sequence of finitely generated $\mathscr{A}_R$-modules, then

$$\mathrm{Ch}^{\mathrm{rel}}(N) = \mathrm{Ch}^{\mathrm{rel}}(N') \cup \mathrm{Ch}^{\mathrm{rel}}(N'')$$

and if $\mathfrak{p}$ is the generic point of an irreducible component of $\mathrm{Ch}^{\mathrm{rel}}(N)$ *then*

$$m_{\mathfrak{p}}(N) = m_{\mathfrak{p}}(N') + m_{\mathfrak{p}}(N'').$$

Proof Propositions 4.4.2 and 4.5.1 give (1). Proposition 4.2.1 gives (2). □

Note that the lemma does not require, nor does it imply, that $\mathrm{Ch}^{\mathrm{rel}}(N)$ is equidimensional.

Definition 3.2.3 We say that a finitely generated $\mathscr{A}_R$-module N is *relative holonomic over R* if its relative characteristic variety over R is a finite union

$$\mathrm{Ch}^{\mathrm{rel}}(N) = \bigcup_w \Lambda_w \times S_w$$

where Λ_w are irreducible conic Lagrangian subvarieties in T^*X and S_w are algebraic irreducible subvarieties of Spec R.

Lemma 3.2.4 *Suppose that N is relative holonomic over R. Then:*

(1) every nonzero subquotient of N is relative holonomic over R;
(2) if $\operatorname{Ext}^j_{\mathscr{A}_R}(N, \mathscr{A}_R) \neq 0$ for some integer j, then $\operatorname{Ext}^j_{\mathscr{A}_R}(N, \mathscr{A}_R)$ is relative holonomic (as a right $\mathscr{A}_R$-module if N is a left $\mathscr{A}_R$-module and vice versa), and

$$\mathrm{Ch}^{\mathrm{rel}}(\operatorname{Ext}^j_{\mathscr{A}_R}(N, \mathscr{A}_R)) \subset \mathrm{Ch}^{\mathrm{rel}}(N).$$

Proof By Proposition 4.2.2, there exist good filtrations on N and $\operatorname{Ext}^j_{\mathscr{A}_R}(N, \mathscr{A}_R)$ such that gr $(\operatorname{Ext}^j_{\mathscr{A}_R}(N, \mathscr{A}_R))$ is a subquotient of $\operatorname{Ext}^j_{\mathrm{gr}\,\mathscr{A}_R}(\mathrm{gr}\, N, \mathrm{gr}\, \mathscr{A}_R)$. It follows that

$$\mathrm{Ch}^{\mathrm{rel}}(\operatorname{Ext}^j_{\mathscr{A}_R}(N, \mathscr{A}_R)) \subset \mathrm{Ch}^{\mathrm{rel}}(N).$$

Then part (2) follows from Proposition 3.2.5. Part (1) is proved similarly, using Lemma 3.2.2 (2). □

The following is a straight-forward generalization of the algebraic case of [20, Proposition 8] where one replaces $\mathbb{C}[\mathbf{s}]$ by R:

Proposition 3.2.5 *If N is a finitely generated module over $\mathscr{A}_R$ such that $\mathrm{Ch}^{\mathrm{rel}}(N)$ is contained in $\Lambda \times \mathrm{Spec}\, R$ for some conic Lagrangian, not necessarily irreducible, subvariety Λ of T^*X, then N is relative holonomic over R.*

Proof The Poisson bracket on $\mathrm{gr}\,\mathscr{A}_R$ is the R-linear extension of the Poisson bracket on $\mathrm{gr}\,\mathscr{D}_X$. Let J be the radical ideal of the annihilator in $\mathrm{gr}\,\mathscr{A}_R$ of $\mathrm{gr}\, N$. By Gabber's Theorem [7, A.III 3.25], J is involutive with respect to the Poisson bracket on $\mathrm{gr}\,\mathscr{A}_R$, that is, $\{J, J\} \subset J$. Let $\mathfrak{m}$ be a maximal ideal in R corresponding to a point q in the image of $\mathrm{Ch}^{\mathrm{rel}}(N)$ under the second projection

$$p_2 : T^*X \times \mathrm{Spec}\, R \to \mathrm{Spec}\, R.$$

By R-linearity of the Poisson bracket, it follows that $J + \mathfrak{m} \cdot \mathscr{A}_R$ is involutive. Therefore the image $\bar{J}$ of J in the ring $\mathrm{gr}\,\mathscr{A}_R \otimes_R R/\mathfrak{m} \simeq \mathrm{gr}\,\mathscr{D}_X$ is involutive under the Poisson bracket on $\mathrm{gr}\,\mathscr{D}_X$. If this ideal would be radical, we could conclude that all the irreducible components of the fiber $\mathrm{Ch}^{\mathrm{rel}}(N) \cap p_2^{-1}(q)$ have dimension at least $\dim X$. Note however that the same assertions on involutivity are true for the associated sheaves since the Poisson bracket on a $\mathbb{C}$-algebra induces a canonical Poisson bracket on the localization of the algebra with respect to any multiplicatively closed subset, cf. [15, Lemma 1.3]. Thus, restricting to an open subset of $\mathrm{Ch}^{\mathrm{rel}}(N)$ where the second projection p_2 has smooth reduced fibers, and assuming $q = p_2(y)$ for a point y in this open subset, the involutivity implies that $\dim_y(\mathrm{Ch}^{\mathrm{rel}}(N) \cap p_2^{-1}(q)) \geq \dim X$. By the upper-semicontinuity on $\mathrm{Ch}^{\mathrm{rel}}(N)$ of the function $y \mapsto \dim_y(\mathrm{Ch}^{\mathrm{rel}}(N) \cap p_2^{-1}(p_2(y)))$, every irreducible component of a non-empty fiber $\mathrm{Ch}^{\mathrm{rel}}(N) \cap p_2^{-1}(q)$ has dimension $\geq \dim X$. (So far, this is an elaborate adaptation of proof of the algebraic case of [20, Proposition 5] to the case when $\mathbb{C}[\mathbf{s}]$ is replaced by R.)

Since Λ is equidimensional with $\dim \Lambda = \dim X$, and Λ contains every non-empty fiber $\mathrm{Ch}^{\mathrm{rel}}(N) \cap p_2^{-1}(q)$, it follows that $\mathrm{Ch}^{\mathrm{rel}}(N) \cap p_2^{-1}(q)$ is a finite union of some of the irreducible conic Lagrangian subvarieties Λ_w of T^*X which are irreducible components of Λ. Define S_w to be the subset of closed points q in $\operatorname{Spec} R$ such that Λ_w is an irreducible component of $\mathrm{Ch}^{\mathrm{rel}}(N) \cap p_2^{-1}(q)$. Then $\mathrm{Ch}^{\mathrm{rel}}(N) = \cup_w(\Lambda_w \times S_w)$. Moreover, setting λ_w to be a general point of Λ_w,

$$\{\lambda_w\} \times S_w = \mathrm{Ch}^{\mathrm{rel}}(N) \cap p_1^{-1}(\lambda_w),$$

where $p_1 : T^*X \times \operatorname{Spec} R \to T^*X$ is the first projection. Since the right-hand side is defined in $\operatorname{Spec} R$ by finitely many algebraic regular functions, S_w is Zariski closed in $\operatorname{Spec} R$. It follows that $\mathrm{Ch}^{\mathrm{rel}}(N)$ is relative holonomic over R. □

3.3.

Recall from 4.3 the definition of pure modules over $\mathscr{A}_R$. Examples of pure modules are given by the following.

Definition 3.3.1 We say that a nonzero finitely generated $\mathscr{A}_R$-module N is *Cohen-Macaulay*, or more precisely j-*Cohen-Macaulay*, if for some $j \geq 0$

$$\mathrm{Ext}^k_{\mathscr{A}_R}(N, \mathscr{A}_R) = 0 \quad \text{if } k \neq j.$$

Remark 3.3.2 If N is a Cohen-Macaulay $\mathscr{A}_R$-module, then:

(1) N is j-pure (see Definition 4.3.4), by Lemma 4.3.5 (2);
(2) $\mathrm{Ch}^{\mathrm{rel}}(N)$ is equidimensional of codimension j, by Propositions 4.4.1, 4.4.2, and 4.5.1.

Lemma 3.3.3 *If N is relative holonomic over R and $j(N) = n + \dim(R)$, then it is $(n + \dim(R))$-Cohen-Macaulay.*

Proof The condition on $j(N)$ implies that $N \neq 0$ by Lemma 3.2.2 (1). If $\operatorname{Ext}^k_{\mathscr{A}_R}(N, \mathscr{A}_R) \neq 0$ for some $k > n + \dim(\operatorname{Spec} R)$, then by Lemma 3.2.4 (2), $\operatorname{Ext}^k_{\mathscr{A}_R}(N, \mathscr{A}_R)$ is relative holonomic. Hence $\dim(\operatorname{Ch}^{\mathrm{rel}}(\operatorname{Ext}^k_{\mathscr{A}_R}(N, \mathscr{A}_R))) \geq n$. Since $\mathscr{A}_R$ is an Auslander regular ring, $j(\operatorname{Ext}^k_{\mathscr{A}_R}(N, \mathscr{A}_R)) \geq k$. This contradicts Lemma 3.2.2 (1). □

3.4.

For a finitely generated $\mathscr{A}_R$-module N, since N is also an R-module, we write

$$B_N = \operatorname{Ann}_R(N)$$

and denote by $Z(B_N)$ the reduced subvariety in Spec R defined by the radical ideal of B_N. Since in general N is not finitely generated over R, it is a priori not clear that $Z(B_N)$ is the R-module support of N, $\operatorname{supp}_R(N)$, consisting of closed points with maximal ideal $\mathfrak{m} \subset R$ such that the localization $N_{\mathfrak{m}} \neq 0$.

Lemma 3.4.1 *If N is relative holonomic over R, then*

$$Z(B_N) = p_2(\operatorname{Ch}^{\mathrm{rel}}(N)),$$

where $p_2 \colon T^*X \times \operatorname{Spec} R \to \operatorname{Spec} R$ the natural projection. In particular,

$$Z(B_N) = \operatorname{supp}_R(N).$$

Proof For $R = \mathbb{C}[\mathbf{s}]$ and in the analytic setting, this is [20, Proposition 9], whose proof can be easily adapted to our case. Since N is relative holonomic, $p_2(\operatorname{Ch}^{\mathrm{rel}}(N))$ is closed. Since the contraction of a radical ideal is a radical ideal, the ideal defining $p_2(\operatorname{Ch}^{\mathrm{rel}}(N))$ is $R \cap \sqrt{\operatorname{Ann}_{\operatorname{gr}\mathscr{A}_R}(\operatorname{gr} N)}$. Hence the first assertion is equivalent to

$$R \cap \sqrt{\operatorname{Ann}_{\operatorname{gr}\mathscr{A}_R}(\operatorname{gr} N)} = \sqrt{\operatorname{Ann}_R(N)},$$

where R is viewed as a $\mathbb{C}$-subalgebra of $\operatorname{gr}\mathscr{A}_R = \operatorname{gr}\mathscr{D}_X \otimes_{\mathbb{C}} R$ via the map $a \mapsto 1 \otimes a$ for a in R. Let b be in R. If $b^k N = 0$ for some $k \geq 1$, then $b^k(\operatorname{gr} N) = 0$ as well. Conversely, if $b^k(\operatorname{gr} N) = 0$ for some $k \geq 1$, then $b^k(F_i N) \subset F_{i-1}N$ for all i. Since $\operatorname{gr} N$ is finitely generated over $\operatorname{gr}\mathscr{A}_R$, the filtration F on N is bounded from below. Then by induction applied to the short exact sequence

$$0 \to F_{i-1}N \to F_i N \to \operatorname{gr}^F_i N \to 0,$$

it follows that for each i there exist a multiple k_i of k such that $b^{k_i}(F_i N) = 0$, and k_i form an increasing sequence. Fix a finite set of generators of N over

$\mathscr{A}_R$. Since F is exhaustive, there exists an index j such that all the generators are contained in $F_j N$. Then $b^{k_j} N = 0$.

We proved thus the first claim, or equivalently, that $Z(B_N) = \operatorname{supp}_R(\operatorname{gr} N)$. Hence the second assertion follows from the equality

$$\operatorname{supp}_R(\operatorname{gr} N) = \operatorname{supp}_R(N)$$

which is proved as follows. If $\mathfrak{m}$ is a maximal ideal in R such that $(\operatorname{gr}_i^F N)_{\mathfrak{m}} \neq 0$ for some i, then $(F_i N)_{\mathfrak{m}} \neq 0$ since localization is an exact functor. Then, again by exactness, $N_{\mathfrak{m}} \neq 0$ since $F_i N$ injects into N. Thus $\operatorname{supp}_R(\operatorname{gr} N)$ is a subset of $\operatorname{supp}_R(N)$. Conversely, if $N_{\mathfrak{m}} \neq 0$, take i to be the minimum integer with the property that $(F_i N)_{\mathfrak{m}} \neq 0$ but $(F_{i-1} N)_{\mathfrak{m}} = 0$. Then $(\operatorname{gr}_i^F N)_{\mathfrak{m}} \neq 0$. □

Lemma 3.4.2 *Suppose that N is relative holonomic over R and $(n+l)$-pure for some $0 \leq l \leq \dim(R)$. If b is an element of R not contained in any minimal prime ideal containing B_N, then the morphisms given by multiplication by b*

$$N \xrightarrow{b} N$$

and

$$\operatorname{Ext}^{n+l}_{\mathscr{A}_R}(N, \mathscr{A}_R) \xrightarrow{b} \operatorname{Ext}^{n+l}_{\mathscr{A}_R}(N, \mathscr{A}_R)$$

are injective. Furthermore, there exists a good filtration of N over R so that

$$\operatorname{gr} N \xrightarrow{b} \operatorname{gr} N$$

is also injective.

Proof We first prove that $N \xrightarrow{b} N$ is injective. If on the contrary its kernel $K \neq 0$, then by Lemma 3.2.2 (2)

$$\operatorname{Ch}^{\mathrm{rel}}(K) \subset \operatorname{Ch}^{\mathrm{rel}}(N).$$

By purity, we know that $j(K) = j(N) = n + l$. Thanks to Lemma 3.2.2 (1),

$$\dim(\operatorname{Ch}^{\mathrm{rel}}(K)) = \dim(\operatorname{Ch}^{\mathrm{rel}}(N)).$$

By Proposition 4.4.1, we can choose good filtrations on K and N so that both $\operatorname{gr} K$ and $\operatorname{gr} N$ are $(n+l)$-pure over $\operatorname{gr} \mathscr{A}_R$. Hence $\operatorname{Ch}^{\mathrm{rel}}(K)$ and $\operatorname{Ch}^{\mathrm{rel}}(N)$ are equidimensional of dimension $n + \dim(R) - l$, by Propositions 4.4.2 and 4.5.1. In particular, $\operatorname{Ch}^{\mathrm{rel}}(K)$ is a union of some irreducible components of $\operatorname{Ch}^{\mathrm{rel}}(N)$.

By the relative holonomicity of N, the irreducible components of $\mathrm{Ch}^{\mathrm{rel}}(N)$ are $\Lambda_i \times Z_i$ with i in some finite index set I, for some conic irreducible Lagrangian subvarieties $\Lambda_i \subset T^*X$ and some irreducible closed subsets $Z_i \subset \operatorname{Spec} R$. The equidimensionality of $\mathrm{Ch}^{\mathrm{rel}}(N)$ implies that $\dim Z_i = \dim(R) - l$.

By Lemma 3.4.1, $Z(B_N) = \cup_{i\in I} Z_i$, and the assumption on b is that $(b = 0)$ does not contain any irreducible component of $Z(B_N)$, where by $(b = 0)$ we mean the reduced closed subset of Spec R defined by the radical ideal of b. We hence have

$$\mathrm{Ch}^{\mathrm{rel}}(K) \not\subset T^*X \times (b = 0.$$

However, since b annihilates K, $\mathrm{Ch}^{\mathrm{rel}}(K) \subset T^*X \times (b = 0)$, which is a contradiction.

Similarly, since $\mathrm{gr}\, N$ is $(n + l)$-pure over $\mathrm{gr}\, \mathscr{A}_R$, we can run the above argument by replacing $\mathrm{Ch}^{\mathrm{rel}}(K)$ with the support of the kernel of the map

$$\mathrm{gr}\, N \xrightarrow{b} \mathrm{gr}\, N$$

to obtain the injectivity of the latter.

By Lemma 3.2.4 (2), $\mathrm{Ext}^{n+l}_{\mathscr{A}_R}(N, \mathscr{A}_R)$ is relative holonomic and

$$\mathrm{Ch}^{\mathrm{rel}}(\mathrm{Ext}^{n+l}_{\mathscr{A}_R}(N, \mathscr{A}_R)) \subset \mathrm{Ch}^{\mathrm{rel}}(N).$$

Since $\mathrm{Ext}^{n+l}_{\mathscr{A}_R}(N, \mathscr{A}_R)$ is always $(n+l)$-pure, cf. Lemma 4.3.5 (1), by a similar argument we conclude that

$$\mathrm{Ext}^{n+l}_{\mathscr{A}_R}(N, \mathscr{A}_R) \xrightarrow{b} \mathrm{Ext}^{n+l}_{\mathscr{A}_R}(N, \mathscr{A}_R)$$

is also injective. □

The following is the key technical result of the article. For simplicity, we take Spec R to be an open set of $\mathbb{C}^r$, the only case we need for the proof of the main result.

Proposition 3.4.3 *Let* Spec R *be a nonempty open subset of* $\mathbb{C}^r$. *Let* N *be an* $\mathscr{A}_R$*-module that is relative holonomic over* R *and* $(n + l)$*-Cohen-Macaulay over* $\mathscr{A}_R$ *for some* $0 \le l \le r$. *Then*

$$\boldsymbol{\alpha} \in Z(B_N) \ \textit{ if and only if } N \otimes_R \mathbb{C}_{\boldsymbol{\alpha}} \neq 0,$$

where $\mathbb{C}_{\boldsymbol{\alpha}}$ *is the residue field of the closed point* $\boldsymbol{\alpha} \in$ Spec R.

Proof We first assume $N \otimes_R \mathbb{C}_{\boldsymbol{\alpha}} \neq 0$. Then $N \otimes_R R_{\mathfrak{m}} \neq 0$, where $\mathfrak{m} \subset R$ is the maximal ideal of $\boldsymbol{\alpha}$ and $R_{\mathfrak{m}}$ is the localization of R at $\mathfrak{m}$. Then $\boldsymbol{\alpha}$ belongs to $\operatorname{supp}_R(N) = Z(B_N)$, by Lemma 3.4.1.

Conversely, we fix a point $\boldsymbol{\alpha}$ in $Z(B_N)$. Since N is $(n+l)$-Cohen-Macaulay, it is in particular $(n+l)$-pure as a module over $\mathscr{A}_R$. By Proposition 4.4.1, we then can choose a good filtration F on N so that $\operatorname{gr} N$ is also pure over $\operatorname{gr} \mathscr{A}_R$. Hence $\operatorname{Ch}^{\mathrm{rel}}(N)$ is purely of dimension $n + r - l$. By relative holonomicity and Lemma 3.4.1, $Z(B_N)$ is also purely of dimension $r - l$.

Let us consider the case when $l < r$. We then can choose a linear polynomial $b \in \mathbb{C}[\mathbf{s}]$ so that $(b = 0)$ contains $\boldsymbol{\alpha}$, but does not contain any of the irreducible components of $Z(B_N)$. By Lemma 3.4.2, the morphisms given by multiplication by b

$$N \xrightarrow{b} N \text{ and } \operatorname{gr} N \xrightarrow{b} \operatorname{gr} N$$

are both injective, the good filtration from Lemma 3.4.2 being constructed in the same way. Thus for every i the vertical maps are injective in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_{i-1}N & \longrightarrow & F_iN & \longrightarrow & F_iN/F_{i-1}N & \longrightarrow & 0 \\ & & \downarrow b & & \downarrow b & & \downarrow b & & \\ 0 & \longrightarrow & F_{i-1}N & \longrightarrow & F_iN & \longrightarrow & F_iN/F_{i-1}N & \longrightarrow & 0 \end{array}$$

and hence by the snake lemma we get an exact sequence

$$0 \to F_{i-1}N \otimes_R R/(b) \to F_iN \otimes_R R/(b) \to \operatorname{gr}_i^F N \otimes_R R/(b) \to 0. \tag{3.1}$$

Note that b is also injective on N/F_iN. Indeed, if not, then there exists some $v \in F_jN$ with $j > i$, $v \notin F_{j-1}N$, and $bv \in F_iN$. But then b must annihilate the class of v in $\operatorname{gr}_j^F N$, which contradicts the injectivity of b on $\operatorname{gr} N$. Running a similar snake lemma as above after applying the multiplication by b on the short exact sequence

$$0 \to F_iN \to N \to N/F_iN \to 0,$$

we obtain a short exact sequence

$$0 \to F_iN \otimes_R R/(b) \to N \otimes_R R/(b) \to (N/F_iN) \otimes_R R/(b) \to 0 \tag{3.2}$$

The injectivity from (3.1) and (3.2) implies that the induced filtration on $N \otimes_R R/(b)$,

$$F_i(N \otimes_R R/(b)) = \mathrm{im}(F_i N \to N \otimes_R R/(b)) \simeq F_i N/(F_i N \cap bN),$$

is the filtration by

$$F_i N \otimes_R R/(b) \simeq F_i N/bF_i N,$$

and the surjectivity from (3.1) then implies

$$\mathrm{gr}\,(N \otimes_R R/(b)) \simeq \mathrm{gr}\,N \otimes_R R/(b). \tag{3.3}$$

By Lemma 3.4.1, $p_2^{-1}(\boldsymbol{\alpha})$ intersects non-trivially the support of $\mathrm{gr}\,N$, hence the same is true for $p_2^{-1}(b = 0)$. By Nakayama's Lemma for the finitely generated module $\mathrm{gr}\,N$ over $\mathrm{gr}\,\mathscr{A}_R$, we hence have

$$0 \neq \frac{\mathrm{gr}\,N}{b \cdot \mathrm{gr}\,N} \simeq \mathrm{gr}\,N \otimes_R R/(b).$$

Together with the isomorphism (3.3), this implies that $N \otimes_R R/(b) \neq 0$. Since $N \otimes_R R/(b)$ is also a finitely generated $\mathscr{A}_{R/(b)}$-module and $\mathrm{gr}\,(N \otimes_R R/(b))$ is a finitely generated $\mathrm{gr}\,\mathscr{A}_{R/(b)}$-module, we further conclude from (3.3) that the relative characteristic variety over $R/(b)$

$$\mathrm{Ch}^{\mathrm{rel}}(N \otimes_R R/(b)) = (\mathrm{id}_{T^*X} \times \Delta)^{-1}(\mathrm{Ch}^{\mathrm{rel}}(N)), \tag{3.4}$$

where $\Delta\colon \mathrm{Spec}\,R/(b) \hookrightarrow \mathrm{Spec}\,R$ is the closed embedding. Hence $N \otimes_R R/(b)$ is relative holonomic over $R/(b)$. By Lemma 3.4.1, we further have

$$Z(B_{N\otimes R/(b)}) = \Delta^{-1}(Z(B_N)).$$

In particular,

$$\Delta^{-1}(\boldsymbol{\alpha}) \in Z(B_{N\otimes R/(b)}).$$

Since

$$N \otimes_R \mathbb{C}_{\boldsymbol{\alpha}} \simeq N \otimes_R R/(b) \otimes_{R/(b)} \mathbb{C}_{\Delta^{-1}(\boldsymbol{\alpha})},$$

where $\mathbb{C}_{\Delta^{-1}(\boldsymbol{\alpha})}$ is the residue field of $\Delta^{-1}(\boldsymbol{\alpha}) \in \mathrm{Spec}\,R/(b)$, our strategy will be to prove

$$N \otimes_R \mathbb{C}_{\boldsymbol{\alpha}} \neq 0$$

by repeatedly replacing N by $N \otimes_R R/(b)$ and R by $R/(b)$.

To make this work, we need first to prove that $N \otimes_R R/(b)$ remains Cohen-Macaulay over $\mathscr{A}_{R/(b)}$. By taking a free resolution of N, one can see that

$$\mathrm{RHom}_{\mathscr{A}_R}(N, \mathscr{A}_R) \otimes^L_{\mathscr{A}_R} \mathscr{A}_{R/(b)} \simeq \mathrm{RHom}_{\mathscr{A}_{R/(b)}}(N \otimes^L_R R/(b), \mathscr{A}_{R/(b)}) \tag{3.5}$$

in the derived category of right $\mathscr{A}_{R/(b)}$-modules. Since the multiplication by b is injective on N, we further have

$$\mathrm{RHom}_{\mathscr{A}_{R/(b)}}(N \otimes^L_R R/(b), \mathscr{A}_{R/(b)}) \simeq \mathrm{RHom}_{\mathscr{A}_{R/(b)}}(N \otimes_R R/(b), \mathscr{A}_{R/(b)}). \tag{3.6}$$

We will use the Grothendieck spectral sequence associated with the left-hand side of (3.5) to compute the Ext modules from the right-hand side of (3.6). Let us assume without harm that N is a left $\mathscr{A}_R$-module. Then viewing $\mathrm{Hom}_{\mathscr{A}_R}(\_, \mathscr{A}_R)$ as a covariant right-exact functor on the opposite category of the category of left $\mathscr{A}_R$-modules, the composition of the two derived functors $\mathrm{RHom}_{\mathscr{A}_R}(\_, \mathscr{A}_R)$ and $(\_) \otimes^L_{\mathscr{A}_R} \mathscr{A}_{R/(b)}$ gives us a convergent first quadrant homology spectral sequence

$$E^2_{p,q} = \mathrm{Tor}^{\mathscr{A}_R}_p(\mathrm{Ext}^q_{\mathscr{A}_R}(N, \mathscr{A}_R), \mathscr{A}_{R/(b)}) \Rightarrow \mathrm{Ext}^{-p+q}_{\mathscr{A}_{R/(b)}}(N \otimes_R R/(b), \mathscr{A}_{R/(b)}),$$

by [26, Corollary 5.8.4]. Note that the conditions from *loc. cit.* are satisfied in our case, since a projective object in the opposite category of the category of left $\mathscr{A}_R$-modules is an injective left $\mathscr{A}_R$-module I, and thus $\mathrm{Hom}_{\mathscr{A}_R}(I, \mathscr{A}_R)$ is a projective right $\mathscr{A}_R$-module, and so acyclic for the left exact functor $(\_) \otimes_{\mathscr{A}_R} \mathscr{A}_{R/(b)}$.

Since N is $(n+l)$-Cohen-Macaulay over $\mathscr{A}_R$,

$$\mathrm{Ext}^q_{\mathscr{A}_R}(N, \mathscr{A}_R) = 0 \quad \text{for } q \neq n + l.$$

Then

$$\mathrm{Tor}^{\mathscr{A}_R}_p(\mathrm{Ext}^{n+l}_{\mathscr{A}_R}(N, \mathscr{A}_R), \mathscr{A}_{R/(b)}) = 0 \quad \text{for } p \neq 0$$

thanks to Lemma 3.4.2, since the complex $\mathscr{A}_R \xrightarrow{b} \mathscr{A}_R$ is a resolution of $\mathscr{A}_{R/b}$. Therefore the above spectral sequence degenerates at E^2,

$$\mathrm{Ext}^q_{\mathscr{A}_{R/(b)}}(N \otimes_R R/(b), \mathscr{A}_{R/(b)}) = 0 \quad \text{for } q \neq n + l,$$

and

$$\begin{aligned}\operatorname{Ext}^{n+l}_{\mathscr{A}_{R/(b)}}(N \otimes_R R/(b), \mathscr{A}_{R/(b)}) &\simeq \operatorname{Ext}^{n+l}_{\mathscr{A}_R}(N, \mathscr{A}_R) \otimes_{\mathscr{A}_R} \mathscr{A}_{R/(b)} \\ &\simeq \operatorname{Ext}^{n+l}_{\mathscr{A}_R}(N, \mathscr{A}_R) \otimes_R R/(b).\end{aligned}$$

As a consequence, $N \otimes_R R/(b)$ is $(n+l)$-Cohen-Macaulay over $\mathscr{A}_{R/(b)}$.

Since b is linear, $\mathbb{C}^{r-1} \simeq \operatorname{Spec} \mathbb{C}[\mathbf{s}]/(b)$, and the latter contains $\operatorname{Spec} R/(b)$ an open subset. We then repeatedly replace R by $R/(b)$, N by $N \otimes_R R/(b)$, and $\boldsymbol{\alpha}$ by $\Delta^{-1}(\boldsymbol{\alpha})$. Each time r drops by 1, l stays unchanged, and N remains nonzero, relative holonomic, and $(n+l)$-Cohen-Macaulay. This reduces us to the case $l = r$.

If $0 = l = r$, the claim is trivially true.

We now assume $0 < l = r$. Since N is now relative holonomic and $(n+r)$-Cohen-Macaulay, hence $(n+r)$-pure, we have

$$\operatorname{Ch}^{\mathrm{rel}}(N) = \sum_w \Lambda_w \times \{p_w\},$$

where p_w are points in $\mathbb{C}^r$. Hence $Z(B_N)$ is a finite union of points in $\operatorname{Spec} R$. Counting multiplicities, by Lemma 3.2.2 (2) we see that N is of finite length.

We now fix a linear polynomial $b \in \mathbb{C}[\mathbf{s}]$ with $b(\boldsymbol{\alpha}) = 0$ but not vanishing at the other points of $Z(B_N)$. We then have an exact sequence

$$0 \to K \to N \xrightarrow{b} N \to N \otimes_R R/(b) \to 0,$$

where K is the kernel. We claim that $K \neq 0$. To see this, chose a polynomial $c \in \mathbb{C}[\mathbf{s}]$ not vanishing at $\boldsymbol{\alpha}$ but vanishing at all other points of $Z(B_N)$. Then by Nullstellensatz, there is $m > 0$ the smallest power such that $(bc)^m$ is in B_N. On the other hand, c^m is not in B_N. Taking $p \geq 1$ to be the smallest with $b^p c^m \in B_N$, we see that there exists v in N, such that $b^{p-1} c^m v$ is a nonzero element of K.

Since $K \neq 0$ and since endomorphisms of modules of finite length are isomorphisms if and only if they are surjective, we have $N \otimes_R R/(b) \neq 0$. By Lemma 3.2.4 (1), $N \otimes_R R/(b)$ is relative holonomic over R, and by Lemma 3.2.2 (2), every irreducible component of its relative characteristic variety over R is one of the components $\Lambda_w \times \{p_w\}$ of $\operatorname{Ch}^{\mathrm{rel}}(N)$. Since b annihilates $N \otimes_R R/(b)$, only the components with $b(p_w) = 0$, and hence with $p_w = \boldsymbol{\alpha}$, appear. We conclude that $N \otimes_R R/(b)$ is also relative holonomic over $R/(b)$. By Lemma 3.2.2 (1), we have $j_{\mathscr{A}_{R/(b)}}(N \otimes_R R/(b)) = n + r - 1$. Then by Lemma 3.3.3, $N \otimes_R R/(b)$ is $(n+r-1)$-Cohen-Macaulay over $\mathscr{A}_{R/(b)}$.

We therefore can replace N by $N \otimes_R R/(b)$, R by $R/(b)$, and assume that $\operatorname{Ch}^{\mathrm{rel}}(N) = \cup_w \Lambda_w \times \{\boldsymbol{\alpha}\}$ for some irreducible conic Lagrangian subvarieties

Λ_w of T^*X. Repeating this process, each time r drops by 1, N remains nonzero, relative holonomic, and $(n+r)$-Cohen-Macaulay. The process finishes at the case $r=0$, in which case there is nothing to prove anymore. □

Remark 3.4.4 A result similar to Proposition 3.4.3 is proved by a different method in [3, Appendix B] for $\mathscr{D}_X[\mathbf{s}]\mathbf{f}^{\mathbf{s}}/\mathscr{D}_X[\mathbf{s}]\mathbf{f}^{\mathbf{s+1}}$ when $\mathbf{f}$ is a reduced free hyperplane arrangement.

3.5.

We consider now the left $\mathscr{A}$-module

$$M = \mathscr{D}_X[\mathbf{s}]\mathbf{f}^{\mathbf{s}}/\mathscr{D}_X[\mathbf{s}]\mathbf{f}^{\mathbf{s+1}}.$$

In this case, the annihilator B_M is the Bernstein–Sato ideal B_F, since M is a cyclic $\mathscr{A}$-module generated by the class of $\mathbf{f}^{\mathbf{s}}$ in M.

It is well-known that the zero locus $Z(B_F)$ in $\mathbb{C}^r$ has dimension $r-1$. Indeed, since B_F is the intersection of the local Bernstein–Sato ideals, by restricting attention to the neighborhood of a smooth point of the zero locus of $\prod_{i=1}^r f_i$, one reduces the assertion to the case when $f_i = x_1^{a_i}$ for some $a_i \in \mathbb{N}$ for all $i=1,\dots r$ with $\mathbf{a} = (a_1,\dots,a_r) \neq (0,\dots,0)$. In this case, the Bernstein–Sato ideal is principal, generated by $\prod_{j=1}^{|\mathbf{a}|}(\mathbf{a}\cdot\mathbf{s}+j)$ with $|\mathbf{a}| = a_1+\ldots+a_r$.

In addition, it is known that every top-dimensional irreducible component of $Z(B_F)$ is a hyperplane in $\mathbb{C}^r$ defined over $\mathbb{Q}$ by [22,23].

We will use the following result of Maisonobe, which also holds in the local analytic case, cf. 3.6:

Theorem 3.5.1 (Maisonobe) *The $\mathscr{A}$-module M is relative holonomic over* $\mathbb{C}[\mathbf{s}]$, *has grade number $j(M)=n+1$ over $\mathscr{A}$, and* $\dim \mathrm{Ch}^{\mathrm{rel}}(M) = n+r-1$. *Every irreducible component of $Z(B_F)$ of codimension >1 can be translated by an element of $\mathbb{Z}^r$ into a component of codimension one.*

Proof In [20, Résultat 3] it is shown that $\mathrm{Ch}^{\mathrm{rel}}(M) = \cup_{i\in I}\Lambda_i \times Z_i$ for some finite set I with $\Lambda_i \subset T^*X$ conic Lagrangian, $Z_i \subset \mathbb{C}^r$ algebraic closed subset of dimension $\leq r-1$. Thus M is relative holonomic over $\mathbb{C}[\mathbf{s}]$. Lemma 3.4.1 shows that $Z(B_F) = \cup_{i\in I} Z_i$, cf. also the remark after [20, Résultat 2]. Since $\dim Z(B_F) = r-1$, it follows that $\dim \mathrm{Ch}^{\mathrm{rel}}(M) = n+r-1$, and hence $j(M)=n+1$ by Lemma 3.2.2 (1). The last claim is contained in the statement of [20, Résultat 3]. □

We next observe that over an open subset of $\mathbb{C}^r$, M behaves particularly nice:

Lemma 3.5.2 *There exists an open affine subset $V = \mathrm{Spec}\, R \subset \mathbb{C}^r$ such that the intersection of V with each irreducible component of codimension one of*

$Z(B_F)$ is not empty, and the module $M \otimes_{\mathbb{C}[\mathbf{s}]} R$ is relative holonomic over R and $(n+1)$-Cohen-Macaulay over $\mathscr{A}_R$.

Proof Since M is relative holonomic over $\mathbb{C}[\mathbf{s}]$, and since good filtrations localize by Remark 3.2.1, it follows that $M \otimes_{\mathbb{C}[\mathbf{s}]} R$ is relative holonomic over R, if Spec R is a non-empty open subset of $\mathbb{C}^r$.

Since $j(M) = n + 1$,

$$\mathrm{Ext}^k_{\mathscr{A}}(M, \mathscr{A}) = 0 \text{ for } k < n + 1.$$

By Auslander regularity of $\mathscr{A}$, if $\mathrm{Ext}^k_{\mathscr{A}}(M, \mathscr{A})) \neq 0$ for $k \geq n + 1$, then

$$j(\mathrm{Ext}^k_{\mathscr{A}}(M, \mathscr{A})) \geq k.$$

Note that since gl.dim$(\mathscr{A})$ is finite, there are only finitely many k with $\mathrm{Ext}^k_{\mathscr{A}}(M, \mathscr{A}) \neq 0$. By Lemma 3.2.4 (2), if $\mathrm{Ext}^k_{\mathscr{A}}(M, \mathscr{A}) \neq 0$, then $\mathrm{Ext}^k_{\mathscr{A}}(M, \mathscr{A}))$ is relative holonomic and

$$\mathrm{Ch}^{\mathrm{rel}}(\mathrm{Ext}^k_{\mathscr{A}}(M, \mathscr{A}))) \subset \mathrm{Ch}^{\mathrm{rel}}(M).$$

By Lemma 3.2.2 (1), when $k > n + 1$,

$$\dim(\mathrm{Ch}^{\mathrm{rel}}(\mathrm{Ext}^k_{\mathscr{A}}(M, \mathscr{A}))) < n + r - 1. \tag{3.7}$$

By relative holonomicity, the irreducible components of $\mathrm{Ch}^{\mathrm{rel}}(M)$ are $\Lambda_i \times Z_i$ with i in some finite index set I, $\Lambda_i \subset T^*X$ irreducible conic Lagrangian, and Z_i irreducible closed in $\mathbb{C}^r$. Then the irreducible components of $\mathrm{Ch}^{\mathrm{rel}}(\mathrm{Ext}^k_{\mathscr{A}}(M, \mathscr{A}))$ are $\Lambda_i \times Z'_i$ with i in some subset $J \subset I$, and Z'_i irreducible closed in Z_i. By Lemma 3.4.1 applied to M and $\mathrm{Ext}^k_{\mathscr{A}}(M, \mathscr{A})$, respectively, we have that $Z(B_F) = \cup_{i \in I} Z_i$, and the support in $\mathbb{C}^r$ of $\mathrm{Ext}^k_{\mathscr{A}}(M, \mathscr{A})$ is $\cup_{i \in J} Z'_i$. Then $\dim Z(B_F) = r - 1$, and $\dim Z'_i < r - 1$ for each $k > n + 1$ by (3.7). Therefore the support in $\mathbb{C}^r$ of $\mathrm{Ext}^k_{\mathscr{A}}(M, \mathscr{A})$ is a proper algebraic subset of $Z(B_F)$ not containing any top-dimensional component of $Z(B_F)$ if $k > n + 1$. Choose $V = \mathrm{Spec}\, R$ to be an open affine subset of $\mathbb{C}^r$ away from these proper subsets of $Z(B_F)$ for all $k > n + 1$. Then for any good filtration we have

$$(\mathrm{gr}\, \mathrm{Ext}^k_{\mathscr{A}}(M, \mathscr{A})) \otimes_{\mathbb{C}[\mathbf{s}]} R = 0$$

for all $k > n+1$. Since R is the localization of $\mathbb{C}[\mathbf{s}]$ with respect to some multiplicatively closed subset S, and since good filtrations localize, cf. Remark 3.2.1, we have

$$\mathrm{gr}\,(S^{-1}\mathrm{Ext}^k_{\mathscr{A}}(M, \mathscr{A})) = 0,$$

and so

$$S^{-1}\mathrm{Ext}^k_{\mathscr{A}}(M, \mathscr{A}) = 0.$$

Since S is also a multiplicatively closed subset of $\mathscr{A}$, in the center of $\mathscr{A}$, and M is finitely generated over the noetherian ring $\mathscr{A}$, the Ext module localizes

$$0 = S^{-1}\mathrm{Ext}^k_{\mathscr{A}}(M, \mathscr{A}) = \mathrm{Ext}^k_{S^{-1}\mathscr{A}}(S^{-1}M, S^{-1}\mathscr{A}),$$

cf. [26, Lemma 3.3.8] and the proof of [26, Proposition 3.3.10], where one identifies the localization functor $S^{-1}(\_)$ on $\mathscr{A}$-modules with the flat extension $(\_) \otimes_{\mathscr{A}} \mathscr{A}_R = (\_) \otimes_{\mathbb{C}[\mathbf{s}]} R$. Thus $S^{-1}M = M \otimes_{\mathbb{C}[\mathbf{s}]} R$ is $(n+1)$-Cohen-Macaulay over $S^{-1}\mathscr{A} = \mathscr{A}_R$. □

Now Lemma 3.5.2 and Proposition 3.4.3 immediately imply:

Theorem 3.5.3 *For every irreducible component H of codimension one of $Z(B_F)$ and for every general point $\boldsymbol{\alpha}$ on H,*

$$M \otimes_{\mathbb{C}[\mathbf{s}]} \mathbb{C}_{\boldsymbol{\alpha}} \neq 0.$$

3.6 Analytic case

Theorem 3.5.3 holds also in the local analytic case. We indicate now the necessary changes in the arguments. The smooth affine algebraic variety X is replaced by the germ (X, x) of a complex manifold of dimension n. The rings R stay as before and we let Y denote the complex manifold underlying the smooth affine complex algebraic variety Spec (R). The rings and modules from the algebraic case $\mathscr{D}_X$, $\mathscr{A}_R = \mathscr{D}_X \otimes_{\mathbb{C}} R$, N, etc., have natural analytic versions as sheaves on the complex manifold X, but their role from the previous arguments will be played by the stalks of these sheaves, $\mathscr{D}_{X,x}$, $\mathscr{A}_{R,x} = \mathscr{D}_{X,x} \otimes_{\mathbb{C}} R$, N_x, etc. The role of $\mathrm{Ch}^{\mathrm{rel}}(N)$ from the algebraic case will be played by $\mathrm{Ch}^{\mathrm{rel}}(N) \cap \pi^{-1}(\Omega \times Y)$, for a very small open ball Ω in X centered at x. Recall that for a coherent sheaf of $\mathscr{A}_R$-modules N on the complex manifold X, the relative characteristic variety $\mathrm{Ch}^{\mathrm{rel}}(N)$ is the analytic subspace of $T^*X \times Y$ defined as the zero locus of the radical of the annihilator of N in $\mathscr{A}_R$. With these changes, all the statements in this section hold in the local analytic case as well.

There are however a few special issues arising in this case, since (partial) analytifications of $\mathscr{A}_R$ and N are needed in order for the module theory as in the Appendix to capture the analytic structure of $\mathrm{Ch}^{\mathrm{rel}}(N)$. For a sheaf of $\mathscr{O}_X \otimes_{\mathbb{C}} R$-modules L on the complex manifold X, one defines the (partial) analytification

$$\widetilde{L} = \mathscr{O}_{X\times Y} \otimes_{p^{-1}(\mathscr{O}_X \otimes_{\mathbb{C}} R)} p^{-1}(L),$$

a sheaf of $\mathscr{O}_{X\times Y}$-modules, where $p : X \times Y \to X$ is the first projection. Thus $\widetilde{\mathscr{A}_R}$ is the sheaf of relative differential operators $\mathscr{D}_{X\times Y/Y}$, locally isomorphic to $\mathscr{O}_{X\times Y}[\partial_1, \dots, \partial_n]$. The analytification of the filtration on $\mathscr{A}_R$ is the natural filtration on $\widetilde{\mathscr{A}_R}$, and $\mathrm{gr}\,\mathscr{A}_R$ is locally isomorphic to $\mathscr{O}_{X\times Y}[\xi_1, \dots, \xi_n]$, a sheaf of subrings of $\mathscr{O}_{T^*X\times Y}$, where ξ_i are coordinates of the fibers of the natural projection $\pi : T^*X \times Y \to X \times Y$. If N is a coherent sheaf of $\mathscr{A}_R$-modules, then $\widetilde{N}$ is a coherent sheaf of $\widetilde{\mathscr{A}_R}$-modules. Since $\widetilde{(\_)}$ is an exact functor, it is is compatible with good filtrations, $\mathrm{gr}\,\widetilde{N} = \widetilde{\mathrm{gr}\,N}$, the annihilator in $\mathrm{gr}\,\widetilde{\mathscr{A}_R}$ of $\mathrm{gr}\,\widetilde{N}$ is the analytification of the annihilator of $\mathrm{gr}\,N$ in $\mathscr{A}_R$, and the radical $J(\widetilde{N})$ of the former is the analytification $\widetilde{J(N)}$ of the radical of the latter. Then $\mathrm{Ch}^{\mathrm{rel}}(N)$ is the analytic subspace of $T^*X \times Y$ defined by the ideal generated by $J(\widetilde{N})$ in $\mathscr{O}_{T^*X\times Y}$, the full analytification, cf. [7, I.6.21].

Note that there is a natural isomorphism of $\mathbb{C}$-algebras

$$\mathrm{gr}\,\mathscr{A}_{R,x} \simeq \mathbb{C}\{x_1, \dots, x_n\}[\xi_1, \dots, \xi_n] \otimes_{\mathbb{C}} R$$

after choosing local coordinates $x_1, \dots, x_n$ on X at x. This ring is a regular commutative integral domain of dimension $2n+\dim(R)$. Thus all the results in the Appendix apply to this ring, except Proposition 4.5.1 (ii). Indeed, $\mathrm{gr}\,\mathscr{A}_{R,x}$ has maximal ideals of height less than $\dim(\mathrm{gr}\,\mathscr{A}_{R,x})$. (For example, the ideal $(1 - x\xi)$ of $\mathbb{C}\{x\}[\xi]$ is maximal of height 1.) On the other hand, our modules are special: $\mathrm{gr}\,N_x$ is a graded module if $\mathrm{gr}\,\mathscr{A}_{R,x}$ is given the natural grading in the coordinates $\xi_1, \dots, \xi_n$. The exact functor $\widetilde{(\_)}$ is also faithful on the category of coherent graded $\mathrm{gr}\,\mathscr{A}_R$-modules:

Proposition 3.6.1 (Maisonobe [20, Lemme 1]) *If M is a coherent $\mathrm{gr}\,\mathscr{A}_R$-module and $x \in X$, then $M_x = 0$ if and only if there exists an open neighborhood Ω of x in X such that $\widetilde{M}|_{\Omega\times Y} = 0$.*

Thus one obtains, cf. [20, Proposition 2]: for a small enough Ω,

$$j_{\mathrm{gr}\,\mathscr{A}_{R,x}}(\mathrm{gr}\,N_x) = \inf_{(x',y)\in\Omega\times Y} j_{(\mathrm{gr}\,\widetilde{\mathscr{A}_R})_{(x',y)}}((\mathrm{gr}\,\widetilde{N})_{(x',y)}).$$

The stalks $(\mathrm{gr}\,\widetilde{N})_{(x',y)}$ determine the local analytic structure at $(x', 0, y)$ of the conical set $\mathrm{Ch}^{\mathrm{rel}}(N)$, since the extension functor from the category of graded coherent sheaves over $\mathrm{gr}\,\widetilde{\mathscr{A}_R}$ into the category of coherent sheaves over $\mathscr{O}_{T^*X\times Y}$ is also faithful besides being exact, by the Nullstellensatz for conical analytic sets, cf. [7, Remark I.1.6.8]. In particular, there is a 1-1 correspondence between conical analytic sets in $T^*X \times Y$ and radical graded coherent ideals in $\mathrm{gr}\,\widetilde{\mathscr{A}_R}$. Therefore the ring $(\mathrm{gr}\,\widetilde{\mathscr{A}_R})_{(x',y)}$ and the module $(\mathrm{gr}\,\widetilde{N})_{(x',y)}$ can be replaced by their localization at the unique graded maximal ideal (cf. [9, 1.5]) and in this context Proposition 4.5.1 (ii) does apply. A consequence is that

Lemma 3.2.2 (1) holds indeed with the changes we have mentioned: for a small neighborhood Ω of x,

$$j_{\mathscr{A}_{R,x}}(N_x) + \dim(\mathrm{Ch}^{\mathrm{rel}}(N) \cap \pi^{-1}(\Omega \times Y)) = 2n + \dim(R).$$

This is [20, Proposition 2, Théorème 1], where $R = \mathbb{C}[\mathbf{s}]$ but the proof applies in general, and we used semicontinuity of the dimension function [13, p.94] to rephrase the statement slightly.

Next, in keeping up with the changes indicated, the condition "regular holonomic" will be replaced by the condition that a coherent module N over $\mathscr{A}_R$ is *regular holonomic at* x, that is, there exists a neighborhood Ω of x such that $\mathrm{Ch}^{\mathrm{rel}}(N) \cap \pi^{-1}(\Omega \times Y)$ is as in Definition 3.2.3.

The condition "j-Cohen-Macaulay" will be replaced by the condition that N is *j-Cohen-Macaulay at* x, that is, N_x is j-Cohen-Macaulay. This is equivalent to N being j-Cohen-Macaulay on some neighborhood Ω of x, that is, j-Cohen-Macaulay at all points x' in $\Omega \cap \mathrm{supp}(N)$. Note that the support of N is an analytic subset of X by Proposition 3.6.1, since the support of $\widetilde{N}$ is an analytic subset of $X \times Y$ by the conical property of $\mathrm{Ch}^{\mathrm{rel}}(N)$. Moreover, N is j-Cohen-Macaulay on Ω if and only if one of the following two equivalent conditions hold for $k \neq j$: $\mathscr{E}xt^k_{\mathscr{A}_R}(N, \mathscr{A}_R)|_\Omega = 0$; $\mathscr{E}xt^k_{\mathscr{A}_R}(N, \mathscr{A}_R)_{x'} = 0$ for all $x' \in \Omega$. Also, N is j-Cohen-Macaulay at x if and only if $\widetilde{N}$ is j-Cohen-Macaulay on $\Omega \times Y$ for some $\Omega \ni x$, by Proposition 3.6.1. This implies, by applying Proposition 4.5.1 in the context mentioned above, that Remark 3.3.2 holds in the local analytic case; in particular, if N is j-Cohen-Macaulay at x, then $\mathrm{Ch}^{\mathrm{rel}}(N) \cap \pi^{-1}(\Omega \times Y)$ is equidimensional of codimension j.

With the changes we have indicated, the rest of the arguments remain as before, and all statements in this section are true in this case.

3.7 Proof of Theorem 1.5.2.

By Theorem 3.5.3 and Proposition 2.5.4, the image under Exp of a non-empty open subset of each irreducible component of codimension one of $Z(B_F)$ lies in $\mathcal{S}(F)$. By the description of $Z(B_F)$ from Theorem 3.5.1 and the paragraphs preceding it, it follows that $\mathrm{Exp}(Z(B_F))$ is included in $\mathcal{S}(F)$. $\square$

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4 Appendix

We recall some facts for not-necessarily commutative rings from [7, A.III and A.IV] that we use in the proof of the main theorem.

4.1.

Let A be a ring, by which we mean an associative ring with a unit element. Let $\mathrm{Mod}_f(A)$ be the abelian category of finitely generated left A-modules.

We say that A is a *positively filtered ring* if A is endowed with a $\mathbb{Z}$-indexed increasing exhaustive filtration $\{F_i A\}_{i\in\mathbb{Z}}$ of additive subgroups such that $F_i A \cdot F_j A \subset F_{i+j} A$ for all i, j in $\mathbb{Z}$, and $F_{-1} A = 0$. The associated graded object $\mathrm{gr}^F A = \oplus_i (F_i A/F_{i-1} A)$ has a natural ring structure. When we do not need to specify the filtration, we write $\mathrm{gr}\, A$ for $\mathrm{gr}^F A$.

If A is a positively filtered ring such that $\mathrm{gr}\, A$ is noetherian, then A is noetherian, [7, A.III 1.27]. Here, noetherian means both left and right noetherian.

4.2.

Let A be a noetherian ring, positively filtered. A *good filtration* on $M \in \mathrm{Mod}_f(A)$ is an increasing exhaustive filtration $F_\bullet M$ of additive subgroups such that $F_i A \cdot F_j M \subset F_{i+j} M$ for all i, j in $\mathbb{Z}$, and such that its associated graded object $\mathrm{gr}\, M$ is a finitely generated graded module over $\mathrm{gr}\, A$, cf. [7, A.III 1.29].

Proposition 4.2.1 ([7, A.III 3.20–3.23]) *Let A be a noetherian ring, positively filtered.*

(1) Let M be in $\mathrm{Mod}_f(A)$ with a good filtration. Then the radical of the annihilator ideal in $\mathrm{gr}\, A$

$$J(M) := \sqrt{\mathrm{Ann}_{\mathrm{gr}\, A}(\mathrm{gr}\, M)}$$

and the multiplicities $m_\mathfrak{p}(M)$ of $\mathrm{gr}\, M$ at minimal primes $\mathfrak{p}$ of $J(M)$ do not depend on the choice of a good filtration.

(2) If

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence in $\mathrm{Mod}_f(A)$ then

$$J(M) = J(M') \cap J(M'')$$

and if $\mathfrak{p}$ is a minimal prime of $J(M)$ then

$$m_\mathfrak{p}(M) = m_\mathfrak{p}(M') + m_\mathfrak{p}(M'').$$

Note that the last assertion is equivalent to the existence of a $\mathbb{Z}$-valued additive map $m_{\mathfrak{p}}$ on the Grothendieck group generated by the finitely generated modules N over $\operatorname{gr} A$ with $J(M) \subset \sqrt{\operatorname{Ann}_{\operatorname{gr} A} N}$, as it is phrased in *loc. cit.*

Proposition 4.2.2 ([7, A.IV 4.5]) *Let A be a noetherian ring, positively filtered. Let M be in* $\operatorname{Mod}_f(A)$ *with a good filtration. For every $k \geq 0$, there exists a good filtration on the right A-module* $\operatorname{Ext}^k_A(M, A)$ *such that* $\operatorname{gr}(\operatorname{Ext}^k_A(M, A))$ *is a subquotient of* $\operatorname{Ext}^k_{\operatorname{gr} A}(\operatorname{gr} M, \operatorname{gr} A)$.

4.3.

Let A be a noetherian ring. The smallest $k \geq 0$ for which every M in $\operatorname{Mod}_f(A)$ has a projective resolution of length $\leq k$ is called the *homological dimension* of A and it is denoted by $\operatorname{gl.dim}(A)$.

Definition 4.3.1 For a nonzero M in $\operatorname{Mod}_f(A)$, the smallest integer $k \geq 0$ such that $\operatorname{Ext}^k_A(M, A) \neq 0$ is denoted

$$j_A(M)$$

and it is called the *grade number* of M. If $M = 0$ the grade number is taken to be ∞.

The ring A is *Auslander regular* if it has finite homological dimension and, for every M in $\operatorname{Mod}_f(A)$, every $k \geq 0$, and every nonzero right submodule N of $\operatorname{Ext}^k_A(M, A)$, one has $j_A(N) \geq k$. This implies the similar condition phrased for right A-modules M, see [7, A.IV 1.10] and the comment thereafter.

Theorem 4.3.2 ([7, A.IV 5.1]) *If A is a positively filtered ring such that* $\operatorname{gr} A$ *is a regular commutative ring, then A is an Auslander regular ring.*

Proposition 4.3.3 ([7, A.IV 1.11]) *Let A be an Auslander regular ring. Then*

$$\operatorname{gl.dim}(A) = \sup\{j_A(M) \mid 0 \neq M \in \operatorname{Mod}_f(A)\}.$$

Definition 4.3.4 A nonzero module M in $\operatorname{Mod}_f(A)$ is j-*pure* (or simply, *pure*) if $j_A(N) = j_A(M) = j$ for every nonzero submodule N.

Lemma 4.3.5 ([7, A.IV 2.6]) *Let A be an Auslander regular ring, M nonzero in* $\operatorname{Mod}_f(A)$, *and $j = j_A(M)$. Then:*

(1) $\operatorname{Ext}^j_A(M, A)$ *is a j-pure right A-module;*
(2) M is pure if and only if $\operatorname{Ext}^k_A(\operatorname{Ext}^k_A(M, A), A) = 0$ *for every $k \neq j$.*

4.4.

We assume now that A is a positively filtered ring such that $\mathrm{gr}\, A$ is a regular commutative ring. Then A is also Auslander regular by Theorem 4.3.2. Moreover, with these assumptions one has the following two results.

Proposition 4.4.1 ([7, A.IV 4.10 and 4.11]) *If M in $\mathrm{Mod}_f(A)$ is j-pure, there exists a good filtration on M such that $\mathrm{gr}\, M$ is a j-pure $\mathrm{gr}\, A$-module.*

Proposition 4.4.2 ([7, A.IV 4.15]) *For any M in $\mathrm{Mod}_f(A)$ and any good filtration on M,*

$$j_A(M) = j_{\mathrm{gr}\, A}(\mathrm{gr}\, M).$$

4.5.

Lastly, we consider a regular commutative ring A. Then $\mathrm{gl.dim.}(A) = \sup\{\mathrm{gl.dim.}(A_{\mathfrak{m}}) \mid \mathfrak{m} \subset A \text{ maximal ideal}\}$, cf. [6, Ch. 2, 5.20]. We let $\dim(A)$ denote the Krull dimension. For a module $M \in \mathrm{Mod}_f(A)$, $\dim_A(M)$ denotes $\dim(A/\mathrm{Ann}_A(M))$. If A is a regular local commutative ring, then $\dim(A) = \mathrm{gl.dim.}(A)$, cf. [7, A.IV 3.5].

Proposition 4.5.1 *Let A be a regular commutative ring and M nonzero in* $\mathrm{Mod}_f(A)$. *Then:*

(i) ([7, A.IV 3.4]) A is Auslander regular;
(ii) ([6, Ch. 2, Thm. 7.1]) if $\dim(A_{\mathfrak{m}}) = m$ for every maximal ideal $\mathfrak{m}$ of A,

$$j_A(M) + \dim_A(M) = m\,;$$

(iii) ([7, A.IV 3.7 and 3.8]) M is a pure A-module if and only if every associated prime of M is a minimal prime of M and $j_A(M) = \dim(A_{\mathfrak{p}})$ for every minimal prime $\mathfrak{p}$ of M.

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