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From Fukaya-Seidel Category to Constructible Sheaves

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Abstract

This work is concerned with the Laudau-Ginzburg A-model, or the Fukaya-Seidel category, associated with a Laurent polynomial $f : (\mathbb{C}^*)^n \to \mathbb{C}$. We use constructible sheaves on a real n-dimensional torus to describe the Lagrangian thimbles associated to f. Then we discuss the application to Homological Mirror Symmetry for smooth projective toric Fano variety. We also develope a general tool that allows one to deform a constructible sheaf as its singular support moves non-characteristically.

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CHAPTER 1

Introduction

Picard-Lefschetz theory is the complex analog of real Morse theory for holomorphic functions [AGV], where instead of studying the change of topology of the sublevel sets as level increases, one studies the regular fiber of f by moving it around or towards a singular fiber.

To fix idea, we consider the following example. Let $f = x^2 + y^2$ on $\mathbb{C}^2$. As we move the regular fiber $f^{-1}(\epsilon)$ towards the singular fiber $f^{-1}(0)$ along real positive ϵ, the circle $S_\epsilon = \{x^2 + y^2 = \epsilon, x, y \in \mathbb{R}\}$ in $f^{-1}(\epsilon)$ shrinks to the point $(0, 0)$. The circle S_ϵ is called a *vanishing sphere*, and the union of them $\bigcup_{0\le t\le\epsilon} S_t$ form a smooth disk $D_\epsilon = \{x^2 + y^2 \le \epsilon, x, y \in \mathbb{R}\}$ ending on $f^{-1}(\epsilon)$, called a *Lefschetz thimble*.

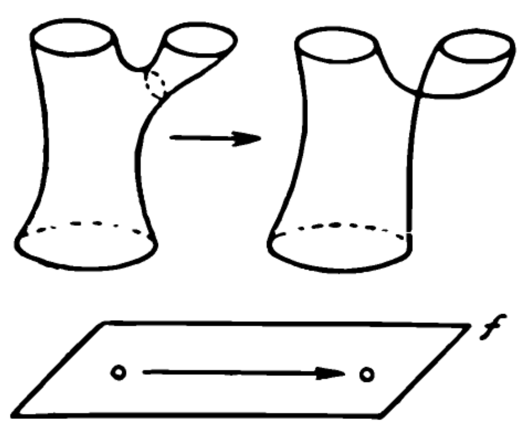


Figure 1.1. A smooth fiber of f and a singular one. (Figure from [AGV], p.7)

The symplectic Picard-Lefschetz theory [Do, Se1] is a refinement of the topological one, where in addition of requiring $f : X \to \mathbb{C}$ to be a holomorphic map, one give the total space X a Kahler structure, hence the the regular fibers are Kahler as well. Using symplectic parallel transport, i.e., lifting a tangent vector on the base to tangent vectors ω-perpendicular to the fiber, one can make the vanishing spheres and Lefschetz thimbles into Lagrangian submanifolds of the regular fibers and the total space, respectively. If one fix a regular value b in $\mathbb{C}$ as the base point, and a set of path from the critical values to b, then one can study the set of Lagrangian spheres in $f^{-1}(b)$, or equivalently corresponding the Lagrangian thimbles in X. The Fukaya-Seidel category for a holomorphic function on a Kähler manifold roughly is the Fukaya category generated by the Lagrangian thimbles. Although the study of Lefschetz thimble, vanishing cycles and intersection forms are classical topics of Picard-Lefschetz theory[AGV], the counting of J-holomorphic disks is computationally non-trivial.

Given a real analytic manifold M, Nadler-Zaslow theorem [NZ, N1] shows that the Fukaya category of the cotangent bundle $\mathrm{Fuk}(T^*M, \Lambda)$ is quasi-equivalent to the dg category of constructible sheaves $Sh(M, \Lambda)$ as A_∞ categories, where $\Lambda \subset T^*M$ is a conical Lagrangian, objects in $\mathrm{Fuk}(T^*M, \Lambda)$ are asymptotic conic Lagrangian tending to Λ, and objects in $Sh(M, \Lambda)$ are constructible complexes of sheaves with singular support in Λ.

Theorem (Nadler-Zaslow [NZ])**.** *The derived Fukaya category of the cotangent bundle T^*M is quasi-equivalent to differential graded category of constructible sheaves on the base manifold M as A_∞ categories.*

This work originates from the idea of applying Nadler-Zaslow theorem to the study of Fukaya-Seidel (FS) category for a holomorphic function on a complex torus $f : (\mathbb{C}^*)^n \to \mathbb{C}$. Along the way, we discovered the following related results and tools.

(1) Lagrangian skeleton of affine hypersurface in $(\mathbb{C}^*)^n$. Here we refine the previous construction of [**RSTZ**] and show that the combinatorially constructed skeleton of an affine hypersurface is not just a deformation retract of the hypersurface, but can be realized as the Liouville skeleton for some carefully chosen Kähler potential. We also define the notion of convexity of a Kähler potential with respect to a given polytope, which will be used again in controlling the asymptotic behavior of Lagrangian thimbles.

(2) Quantization of non-characteristic variation of singular support. Given a one-parameter family of conical Lagrangian $\Lambda_t \subset T^*M$ parameterized by $t \in [0, 1]$, we may ask if there is an analog of parallel transport of categories $Sh(M, \Lambda_t)$. We give a sufficient condition on the variations such that the categories remains invariant.

(3) As an application of the variation of singular support method, we consider constructible sheaves on a torus T^n, and prove the non-equivariant coherent-constructible correspondence for smooth projective variety.

(4) Finally, we prove our main theorem, that FS-category for a Laurent polynomial is quasi-equivalent to a category of constructible sheaves on a torus. One ingredient of the proof is the adapted Kähler potential, which allows us to control the thimble's behavior at infinity; another ingredient is the monodromy operator, which acts on the FS-category by varying the phase of the coefficients of the

monomial, and acts on the constructible sheaves category by varying the singular support constraint.

In the remaining part of the introduction, we give a brief summary of the results.

1.1. Notation

Here we establish some common notation that will be used through out this paper.

Let M, N be dual lattices of rank n. Let $T = \mathbb{R}/2\pi\mathbb{Z}$. For any abelian group A, e.g. $A = \mathbb{C}^*, \mathbb{R}, T$, we define $M_A := M \otimes_{\mathbb{Z}} A$ and similarly for N_A. If we fix a basis of M, then $M \cong \mathbb{Z}^n$, and $M_{\mathbb{C}^*} \cong (\mathbb{C}^*)^n, M_{\mathbb{R}} \cong \mathbb{R}^n, M_T \cong T^n$. We sometimes denote the torus M_T also as T_M.

Let $Q \subset N_{\mathbb{R}}$ be a integral convex polytope with 0 as an interior point. Let $\mathcal{T}$ be a coherent star triangulation of Q based at 0, and let $\partial\mathcal{T}$ be the induced triangulation of the boundary ∂Q. Let $\Sigma_{\mathcal{T}}$ be the simplicical fan spanned by the simplices in $\mathcal{T}$. Let A denote the vertices of $\mathcal{T}$, and ∂A that of $\partial\mathcal{T}$, then $A = \partial A \cup \{0\}$. Fix $(h, \Theta) : A \to \mathbb{R} \times T$, such that h induces the star triangulation of $\mathcal{T}$ and $h(0) = 0$. Let $\widehat{h} : Q \to \mathbb{R}$ denote the convex piecewise linear function extending h on A.

Let $g : M_{\mathbb{R}} \to \mathbb{R}$ be the *discrete Legendre transformation (DLT)* of h

$$g(y) = \sup_{x \in Q} \langle x, y \rangle - h(x), \tag{1.1.1}$$

and $g(y)$ is a convex piecewise affine-linear function on $M_{\mathbb{R}}$. The tropical amoeba $\mathcal{A}$ is defined as the singular loci of g.

We define conical Lagrangian $\Lambda_{\mathcal{T},\Theta} \subset T_M \times N_{\mathbb{R}} \cong T^*T_M$ by

$$\Lambda_{\mathcal{T},\Theta} = \bigcup_{\tau \in \partial\mathcal{T}} \{\theta \in T_M : \langle \alpha, \theta \rangle = \Theta(\alpha) \text{ for all vertices } \alpha \in \tau\} \times \operatorname{cone}(\tau) \tag{1.1.2}$$

where we used the pairing $\langle -, - \rangle : T_M \times N_{\mathbb{R}} \to T$ induced by the canonical pairing between M, N.

We also define the *generalized RSTZ-skeleton* [**RSTZ**] by

$$\Lambda^{\infty}_{\mathcal{T},\Theta} = \bigcup_{\tau \in \partial\mathcal{T}} \{\theta \in T_M : \langle \alpha, \theta \rangle = \Theta(\alpha) \text{ for all vertices } \alpha \in \tau\} \times \tau \tag{1.1.3}$$

For all large enough $R > 0$, we define the *tropical polynomial* as

$$f_{R,h,\Theta}(z) = \sum_{\alpha \in \partial A} e^{-i\Theta(\alpha)} R^{-h(\alpha)} z^{\alpha}. \tag{1.1.4}$$

where z^{α} is a monomial function on $M_{\mathbb{C}^*} \cong (\mathbb{C}^*)^n$.

If the function $\Theta \equiv 0$, we drop the Θ subscript everywhere.

1.2. Lagrangian Skeleton of Affine Hypersurface

A skeleton, generally speaking, keeps only the essential part without losing important information. A topological skeleton of a (non-compact) manifold is a strong deformation retract, hence remembers the homotopy type of the original manifold. A Lagrangian skeleton of a Weinstein manifold, i.e an exact symplect manifold $\omega = d\lambda$ with certain growth control, is the stable manifold of the expanding Liouville flow X_λ, defined by $\iota_{X_\lambda}\omega = \lambda$. Stein manifold, e.g. affine hypersurfaces, are Weinstein manifold, hence we may ask what the skeleton of any given affine hypersurface looks like.

Lagrangian skeleton is interesting for the study of Fukaya category of Weinstein manifold. Indeed by the contracting Liouville flow, every compact smooth Lagrangian is Lagrangian isotopic to one contained in a small tubular neighborhood of the skeleton, hence one has the hope of computing the Floer homology group $HF(L_1, L_2)$ of two Lagrangians, or even the Fukaya category, locally on the skeleton. In other words, it is believed that the Fukaya category of the Weinstein manifold can be computed as the global section of some sheaf of categories over the Lagrangian skeleton. A combinatorial recipe for computing this sheaf of category is given by Nadler in [**N2, N3**]. There are ongoing works by Ganatra-Pardon-Shende and Chantrain-Ghiggini-Rizell-Golovko to study Fukaya category using Liouville skeleton.

Topological skeleton, as strong deformation retract, for affine hypersurfaces has been studied by Ruddat-Sibilia-Treumann-Zaslow in [**RSTZ**]. They give a simple combinatorial description of the skeleton using Newton polytope of the defining polynomial.

Theorem (Ruddat-Sibilla-Treumann-Zaslow)**.** *Let $Q, \mathcal{T}, f_{R,h}$ as defined in the setup. The skeleton $\Lambda^\infty_{\mathcal{T}}$ embed into the hypersurface $\{f_{R,h}(z) = 0\}$ as a strong deformation retract.*

The proof there is algebraic and does not require a choice of the Weinstein structure, hence does not show the skeleton can be embedded as a Lagrangian skeleton, or the stable manifold of the Liouville flow.

Here we upgrade the topological skeleton into a Lagrangian skeleton. More precisely, we have the following theorem.

Theorem 1. *There exists an exact symplectic structure on* $(\mathbb{C}^*)^n$*, such that the skeleton* $\Lambda_{\mathcal{T}}^{\infty}$ *embed into the hypersurface* $\{f_{R,h}(z) = 0\}$ *as the Liouville skeleton with respect to the induced Liouville structure on the hypersurface.*

Our approach here is differential geometrical in flavor, using the idea of tropical degeneration in the sense of Mikhalkin [**Mi**], which roughly means introducing a parameter R and consider $R \to \infty$ limit (or fix R to be large enough).

1.3. Variation of Singular Support for Constructible Sheaves

Let M be a smooth manifold, $\Lambda^\infty \subset T^\infty M \cong S^*M$ a (possibly singular) Legendrian in the contact infinity of T^*M, and $Sh(M, \Lambda^\infty)$ the differential graded category of constructible sheaves with $SS^\infty(F) \subset \Lambda^\infty$. We are interested in the following question:

Given an initial Legendrian $\Lambda^\infty \subset T^\infty M$ *and a constructible sheaf* $F \in Sh(M, \Lambda^\infty)$*, for what kinds of deformation of* Λ^∞ *can we find a corresponding deformation of* F*, such that* $SS^\infty(F) \subset \Lambda^\infty$ *holds?*

Remark 1.3.1. If Λ^∞ is a smooth Legendrian and the deformation is an Legendrian isotopy, then there is a contactomorphism of the ambient space $T^\infty M$ that induces the deformation Λ^∞. The sheaf-quantization of contactomorphism of Guillermou-Kashiwara-Schapira [**GKS**] allows one to deform the sheaf F accordingly.

In [**N3**], Nadler introduced the notion of 'non-characteristic deformation' of Legendrian, which is defined using sheaf theory: a deformation of Legendrian is non-characteristic if the corresponding sheaf category is invariant. He also proved that any Legendrian can be deformed non-characteristically to one with standard singularity, called arboreal singularity [N2]. Here we are interested in finding geometric condition for the Legendrian deformation to be non-characteristic.

Let Λ_t^∞ be a family of Legendrian in $T^\infty M$. Our first theorem (Theorem 7) says that, if the complement of the Legendrian $T^\infty M \backslash \Lambda_t^\infty$ are contactomorphic to each other, then the corresponding sheaf categories $Sh(M, \Lambda_t^\infty)$ are equivalent to each other. However, the geometric condition is very hard to check in practice, though it can be useful to prove non-existence of contactomorphism.

We propose another easier to check condition on the Legendrian deformation, inspired by a remark in [N3]. Intuitively, we allow Legendrian to deform as long as there is no 'collision' with itself. One type of collision is detected by the appearance of short Reeb chord ending on the Legendrian. However, it is also possible that two components of Legendrian approach each other along the contact distribution, which is not detectable by Reeb flow. For example, consider the Legendrian in $J^1\mathbb{R}$ defined by $f_1(x) = 0$ and $f_2(x) = x^3 + tx$ for $t \in [0, 1]$. To remedy this, we introduce a notion of 'thickened' Legendrian (along the contact distribution direction), and convex tubular neighborhood around the Legendrian, generated by the Reeb flow-out of the thickened Legendrian. We prove that the sheaf categories is invariant if the Legendrian deformation admits a uniform convex tubular neighborhood (Theorem 9).

For application to Mirror symmetry, we also consider the special case of Legendrian deformation locally modeled by affine hyperplanes in a vector space (Theorem 10), and verify that if there is no short Reeb chords during the deformation then the sheaf category is invariant.

1.4. Non-equivariant Coherent-Constructible Correspondence

As a special case of the above non-characteristic variation of singular support method, we have the following theorem. Recall that $T_M \cong T^n$ is an n-dimensional torus.

Theorem 2. *Let $Q, \mathcal{T}$ be given as above. For all $\theta : A \to T$, the assignment $\theta \rightsquigarrow Sh(T_M, \Lambda_{\mathcal{T},\theta})$ gives a local system of categories over T^A. In particular, for any path $\gamma : [0,1] \to T^A$, there is a parallel transport functor $\Phi_\gamma : Sh(T_M, \Lambda_{\mathcal{T},\gamma(0)}) \to Sh(T_M, \Lambda_{\mathcal{T},\gamma(1)})$ inducing an equivalence of categories.*

This theorem is useful in that, it allows one to build new objects in the constructible sheaf category by applying monodromy operator. If we take the function $\theta : A \to T$ as identically zero, then the skyscraper sheaf $\mathbb{C}_0$ at $0 \in T_M$ is in the category. From this object, we may apply the various monodromy operator, indexed by elements in $\pi_1(T^A) = \mathbb{Z}^A$. It turns out that the monodromy action acts by convolution of constructible sheaves on T_M.

Definition 1.4.1. Let $\pi : \mathbb{R}^A \to T^A$ be the universal cover of T^A, and a point $\widetilde{\theta} \in \mathbb{R}^A$ represent a path $\gamma_{\widetilde{\theta}}$ from 0 to $\pi(\widetilde{\theta})$. For any $\widetilde{\theta} \in \mathbb{R}^A$ and $\theta = \pi(\widetilde{\theta})$, we define the twisted-polytope sheaf

$$P_{\widetilde{\theta}} = \Phi_{\gamma_{\widetilde{\theta}}}(\mathbb{C}_0) \in Sh(T_M, \Lambda_{\mathcal{T},\theta}).$$

And we denote the set of twisted polytope sheaves in $Sh(T_M, \Lambda_{\mathcal{T},\theta})$ by

$$\mathcal{P}_\theta = \{P_{\widetilde{\theta}} \mid \widetilde{\theta} \in \mathbb{R}^A, \text{ and } \pi_\Sigma(\widetilde{\theta}) = \theta\}.$$

Theorem 3. *For any $\theta \in T^A$, the set of twisted polytope sheaves $\mathcal{P}_\theta$ generates the category $Sh(T_M, \Lambda_{\mathcal{T},\theta})$.*

Example 1.4.2. Here is an example illustrating what a twisted-polytope sheaf looks like. $\triangle$

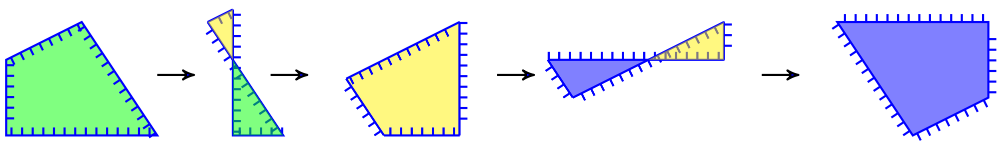


Figure 1.2. Twistings of a convex polytope. As one pushes the edges of the polytopes, a certain edge will shrink to zero-length then reappear on the other side. Note the co-directions of the edges, indicated by hairs, remain fixed. The corresponding twisted polytope sheaves have stalks on green, yellow and blue regions as $\mathbb{C}, \mathbb{C}[1], \mathbb{C}[2]$, respectively.

As an application to Homological Mirror symmetry, we have the following result of non-equivariant coherent-constructible correspondence, enhancing the previous result of [**FLTZ1**, **FLTZ2**, **Tr**, **SS**, **Ku1**]. This result was first obtained by Kuwagaki in [**Ku2**] using a different method.

Theorem 4 (Non-equivariant Coherent-Constructible Correspondence)**.** *Let $\Sigma \subset N_{\mathbb{R}}$ be the fan of a smooth projective toric variety, X_Σ, $Q \subset N_{\mathbb{R}}$ be the polytope generated by primitive vectors of rays in Σ, and $\mathcal{T}$ is the unique star triangulation generated by faces of Q. Then there is an quasi-equivalence of category*

$$Coh(X_\Sigma) \xrightarrow{\sim} Sh(T_M, \Lambda_{\mathcal{T}})$$

where $\Lambda_{\mathcal{T}}$ is given by (1.1.2) for $\theta \equiv 0$. Under this equivalence, the trivial line bundle $\mathcal{O}_{X_\Sigma}$ is sent to the skyscraper sheaf $\mathbb{C}_0$, and there is a canonical one-to-one correspondence between line bundle to the twisted-polytope sheaves.

1.5. Lagrangian Thimbles and Constructible Sheaves

Given any lattice polytope $Q \subset \mathbb{Z}^n$ with 0 in the interior. Let $\mathcal{T}$ be a regular star triangulation of Q based at 0 and A the vertices of $\partial\mathcal{T}$. Let $h : A \to \mathbb{R}$ be any function, such that $h(0) = 0$ and h induces the triangulation $\mathcal{T}$. Let $f_{R,h}$ be tropical polynomial defined in 1.1.4.

Theorem 5. *The Fukaya-Seidel category for $f_{R,h,\theta}$ is equivalent to the category of constructible sheaves on the n-torus*

$$\Phi_{\mathcal{T},\theta} : FS((\mathbb{C}^*)^n, f_{R,h,\theta}) \xrightarrow{\sim} Sh(T^n, \Lambda_{\mathcal{T},\theta}).$$

Here we give the sketch of the proof. First, we choose a special Kahler potential on $(\mathbb{C}^*)^n$, and extends the thimbles ending on $f^{-1}_{R,h,\theta}(0)$ for large R to aymptotically conical Lagrangians close to the conical Lagrangian $\Lambda_{\mathcal{T},\theta}$. This embeds $FS((\mathbb{C}^*)^n, f_{R,h,\theta})$ to $\mathrm{Fuk}(T^*T^n, \Lambda_{\mathcal{T},\theta}) \xrightarrow{\sim} Sh(T^n, \Lambda_{\mathcal{T},\theta})$.

To study the image of the embedding when $\theta = 0$, we consider monodromy of θ on T^A. On the Fukaya-Seidel side, the monodromy of θ mutates the thimbles. On the constructible sheaf side, the monodromy acts by changing the singular support.

We apologize for the lack of details in this section, they will appear in future work.

CHAPTER 2

Review of Fukaya-Seidel category

In this section, we give a rather informal overview of the Fukaya-Seidel category [**Au**, **Sm**, **Se1**]. Then we present the definition and properties following the comprehensive monograph [**Se1**] Chapter 1, as well as [**NZ**] Section 2.

Fukaya category of a compact symplectic manifold (M^{2n}, ω) is a symplectic invariant valued in an A_∞-category, where roughly speaking, the objects are Lagrangian submanifolds, morphisms are Floer chain complex $CF(L_1, L_2)$, and (higher) compositions are given by counting pseudo-holomorphic disks with boundaries on the Lagrangians.

Fukaya category is an A_∞-category, that is, the composition is not strictly associative but only associative up to corrections from higher order products, hence it is not an 'honest' category. At first sight, the non-associativity seems a shortcoming that complicates many manipulations. However, one can easily cure this, by taking homology of the morphism complex, that is, using Floer homology groups $HF(L_1, L_2)$ rather than Floer chain complex $CF(L_1, L_2)$, then the composition is associative on the nose, and in addition $HF(L_1, L_2)$ themselves become invariant under Hamiltonian isotopies of L_i. The reason for not choosing to pass to homology is that, it will 'lose information'. Just as the de Rham complex $(\Omega^*(X), d, \wedge)$ of a smooth compact manifold X as a differential graded algebra (dga) remembers the real homotopy type of the manifold X rather than just the cohomologies, the A_∞ structure maps

$$\mu^k : CF^\bullet(L_0, L_1) \otimes CF^\bullet(L_1, \cdots, L_2) \otimes \cdots \otimes CF^\bullet(L_{k-1}, L_k) \to CF^{(\sum \bullet)+2-k}(L_0, L_k)$$

remembers more information than $HF(L_i, L_j)$. These μ^k satisfies A_∞ relations, where the first fews are

$$\mu^1(\mu^1(a)) = 0, \quad \mu^1(\mu^2(a \otimes b)) = \pm\mu^2(\mu^1(a) \otimes b) \pm \mu^2(a \otimes \mu^1(b)),$$

$$\begin{aligned} & \mu^1 \cdot \mu^3(a \otimes b \otimes c) \pm \cdot\mu^3((\mu^1 a) \otimes b \otimes c) \pm \cdot\mu^3(a \otimes \mu^1 b \otimes c) \pm \mu^3(a \otimes b \otimes \mu^1 c) \\ = \ & \mu^2(\mu^2(a \otimes b) \otimes c) \pm \mu^2(a \otimes \mu^2(b \otimes c)) \end{aligned}$$

and the precise definitions are given in the next sections.

Fukaya-Seidel category assigns a Fukaya category to a holomorphic function $f : M \to \mathbb{C}$ on a Kähler manifold M. Given a smooth fiber of f and paths from singular fiber to smooth fibers, we may construct an ordered sequence of vanishing spheres $L_1, \cdots, L_k$ in the smooth fiber, as Lagrangian submanifolds. The Fukaya-Seidel category is the triangulated envelope generated by an directed A_∞-category, where 'directed' means $\mathrm{Hom}(L_i, L_j) = 0$ if $i > j$.

Fukaya-Seidel category relates the symplectic topology of the ambient space, to that of a smooth fiber, hence reduces the dimension of the problem. If repeated by taking Lefschetz fibration of the fiber again and again, one can reduce the problem to complex one-dimensional case, where counting holomorphic disks becoming counting polygons on a Riemann surface.

Another motivation to study the Fukaya-Seidel category is from Mirror symmetry. One version of Mirror symmetry conjecture predicts certain Fano manifold X should be mirror dual to a certain Landau-Ginzburg(LG) model $f : X^\vee \to \mathbb{C}$, in particular
(1) the B-model $D^bCoh(X)$ of X should be dual to the A-model $FS(X^\vee, f)$ of $(X^\vee, f)$, and
(2)the A-model $Fuk(X)$ of X should be dual to the B-model of $D^b_{sing}(X^\vee, f)$.
Hence the Fukaya-Seidel category is the study of the LG A-model. Our work here focus on the special case where X is a smooth projective toric Fano variety, and $(X^\vee, f)$ is the Hori-Vafa mirror dual of X.

We briefly explain the organization of this section. We first review the homological setup of A_∞-category, explaining what is an A_∞-category and its triangulated envelope, also introduce the notion of a directed category. Then we focus on the general setup and properties of Symplectic Lefschetz fibration, explaining the results of Seidel, that monodromy around a singular fiber corresponds to the Dehn twist of the corresponding Lagrangian vanishing sphere.

2.1. Triangulated A_∞-categories

An A_∞-category (not necessarily unital) $\mathcal{C}$ consists of a set of objects $\mathrm{Ob}\,\mathcal{C}$, together with a $\mathbb{Z}$-graded $\mathbb{C}$-vector space $hom_{\mathcal{C}}(X_0, X_1)$ for any two objects, and multilinear maps

$$\mu^k_{\mathcal{C}} : hom_{\mathcal{C}}(X_0, X_1) \otimes \cdots hom_{\mathcal{C}}(X_{k-1}, X_k) \to hom_{\mathcal{C}}(X_0, X_1)[2-k]$$

for all $k \geq 1$ and all $(k+1)$-tuples of objects $(X_0, \cdots, X_{k+1})$, satisfying the A_∞ associativity relations. Roughly speaking, if we define the path space from X to Y be

$$Path(X,Y) := \bigoplus_{k \geq 1, \{X_i\}} hom(X_0, X_1)[1] \otimes \cdots hom(X_{k-1}, X_k)[1], \quad X_0 = X, X_k = Y$$

and define the 'merge' operation μ on $Path(X,Y)$ of degree 1 by merging subsegments using μ^d in all possible ways and sum them up, then the A_∞-relation is simply $\mu \circ \mu = 0$.

The underlying cohomological category $H(\mathcal{C})$ has the same objects as $\mathcal{C}$, with

$$hom_{H(\mathcal{C})}(X_0, X_1) = H(hom_{\mathcal{C}}(X_0, X_1), \mu^1_{\mathcal{C}})$$

There is also the subcategory $H^0(\mathcal{C})$ which has the same objects, but only keep the degree zero cohomologies. One says $H(\mathcal{C})$ is *c-unital* (cohomological unital) if $H(\mathcal{C})$ has identity morphisms.

Remark 2.1.1. A differential graded (dg) category is an A_∞-category with $\{\mu^k\}_{k>2}$ vanishing. An A_∞ algebra is an A_∞-category with one object.

Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be an A_∞-functor between A_∞-categories with map on objects $\mathcal{F} : \operatorname{Ob} \mathcal{A} \to \operatorname{Ob} \mathcal{B}$, and morphism maps

$$\mathcal{F}^d : hom_{\mathcal{A}}(X_0, X_1) \otimes \cdots hom_{\mathcal{A}}(X_{d-1}, X_d) \to hom_{\mathcal{B}}(\mathcal{F}X_0, \mathcal{F}X_d)[1-d],$$

or a degree-0 map from $Path(X,Y)$ to $Path(\mathcal{F}X, \mathcal{F}Y)$ commuting with the 'merge' operators $\mu_{\mathcal{A}}, \mu_{\mathcal{B}}$. An A_∞-functor $\mathcal{F}$ is said to be *c-unital*, if $H(\mathcal{F})$ is unital. The category of A_∞-functors from $\mathcal{A}$ to $\mathcal{B}$ forms an A_∞-category, where a morphism T from functor $\mathcal{F}$

to functor $\mathcal{G}$ allows one to take in a path in $\mathcal{A}$ and output a path in $\mathcal{B}$ where the first half is convert usnig $\mathcal{F}$ and the second half is convert using $\mathcal{G}$ with the jump given by T.

Assumptions: We will assume all A_∞-categories to be c-unital, and all A_∞-functors to be c-unital too. We say an A_∞-functor $\mathcal{F}$ is a *quasi-equivalence* if the induced functor $H(\mathcal{F})$ is an equivalence. We say $\mathcal{F}$ is *quasi-embedding* if $H(\mathcal{F})$ is full and faithful. It can be shown that if $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is a quasi-equivalence, then there is a quasi-equivalence $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ such that $H(\mathcal{G})$ is an inverse equivalence to $H(\mathcal{F})$.

2.1.1. A_∞-modules and Yoneda embedding

Let Ch denote the dg category of chain complexes (over $\mathbb{C}$), considered as an A_∞-category. Given an A_∞-category $\mathcal{A}$, a (right) A_∞-module over $\mathcal{A}$ is an A_∞-functor $\mathcal{M} : \mathcal{A}^{op} \to Ch$. Let $mod(\mathcal{A})$ denote the A_∞-category of A_∞-modules of $\mathcal{A}$. Since Ch is a dg category, $mod(\mathcal{A})$ is actually a dg category as well, and its cohomological category $H^0(mod(\mathcal{A}))$ is a triangulated category.

For any object $Y \in Ob\mathcal{A}$, we have the A_∞-module $\mathcal{Y}$ defined as

$$\mathcal{Y}(X) = hom_{\mathcal{A}}(X, Y).$$

Let $\mathcal{J}$ denote the A_∞ Yoneda embedding $\mathcal{A} \hookrightarrow mod(\mathcal{A})$, then $H(\mathcal{J})$ is full and faithful. Since the ambient category $mod(\mathcal{A})$ is a dg category, the image $\mathcal{J}(\mathcal{A})$ of the embedding is as well. Thus each A_∞-category is canonically quasi-equivalent to a dg category $\mathcal{J}(\mathcal{A})$.

2.1.2. Exact Triangles

Let Y_0, Y_1 be objects of $\mathcal{A}$, and $c \in hom^0_{\mathcal{A}}(Y_0, Y_1)$ a degree zero cocycle. The *abstract mapping cone* of c is the A_∞-module $\mathscr{C} = \mathscr{C}one(c)$ defined by

$$\mathscr{C}(X) = hom_{\mathcal{A}}(X, Y_0)[1] \oplus hom_{\mathcal{A}}(X, Y_1)$$

There is an exact triangle diagram in $H(mod(\mathcal{A}))$

$$\mathcal{J}(Y_0) \to \mathcal{J}(Y_1) \to \mathscr{C}one(c) \xrightarrow{[1]} .$$

We call any triangles in $H(\mathcal{A})$ exact, if its Yoneda embedding image is isomorphic to one of such triangles. A shift SX of an object X is any object which ecome isomorphic to the shift in $H(mod(\mathcal{A}))$ under the Yoneda embedding.

A non-empty A_∞-category $\mathcal{A}$ is said to be triangulated if the following hold:

(1) Every morphism in $H^0(\mathcal{A})$ can be completed to an exact triangle in $H(\mathcal{A})$. In particular, every object X has a shift SX.
(2) For each object X, there is an object $\widetilde{X}$, such that $S\widetilde{X} \cong X$ in $H^0(\mathcal{A})$.

If $\mathcal{A}$ is a triangulated A_∞-category, then $H^0(\mathcal{A})$ is a triangulated category in the usual sense. Furthermore, if $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is an A_∞-functor between triangulated A_∞-categories, then $H^0(\mathcal{F})$ is an exact functor.

The *triangulated envelope* of a nonempty A_∞-category $\mathcal{A}$ is a pair $(\overline{\mathcal{A}}, \mathcal{F})$ consisting of a triangualted A_∞-category $\overline{\mathcal{A}}$ and a cohomologically full and faithful functor $\mathcal{F} : \mathcal{A} \to \overline{\mathcal{A}}$, such that the objects in the image of $\mathcal{F}$ generate $\overline{\mathcal{A}}$, in the sense that $\overline{\mathcal{A}}$ is the smallest full subcategory in $\overline{\mathcal{A}}$ that contains $\mathcal{A}$, is closed under cohomological isomorphism, and is itself triangulated.

2.1.3. Twisted Complex

There are two standard construction of a triangulated envelope for $\mathcal{A}$, one is taking the full subcategory of $mod(\mathcal{A})$ generated by the image fo the Yoneda embedding, and another the A_∞-category of twisted complex $Tw\mathcal{A}$. Here we first state the construction of twisted complex, then state their properties.

The construction (modulo signs) proceeds in two steps: (1) The additive enlargement $\Sigma\mathcal{A}$ has objects with are formal sums

$$X = \bigoplus_{f \in F} X_f[\sigma_f],$$

where F is some finite set, $X_f \in Ob\mathcal{A}$, and $\sigma_f \in \mathbb{Z}$. The morphisms between any two objects are

$$hom_{\Sigma\mathcal{A}}(\bigoplus_{f \in F} X_f[\sigma_f], \bigoplus_{g \in G} Y_g[\tau_g]) = \bigoplus_{f,g} hom_{\mathcal{A}}(X_f, Y_g)[\tau_g - \sigma_f]$$

and compositions are defined using those of $\mathcal{A}$ and the obvious 'matrix multiplication' rule.

(2) A twisted complex is a pair (C, δ) consisting of $C \in Ob\Sigma\mathcal{A}$ and $\delta \in hom^1_{\Sigma\mathcal{A}}(C, C)$, with the strict upper triangularity property that the index set F for $C = \oplus_{f \in F} C_f$ can be ordered in such a way that all components δ_{fg} with $f \geq g$ are zero, and subject to the

generalized Maurer-Cartan equation

$$\mu^1_{\Sigma\mathcal{A}}(\delta) + \mu^2_{\Sigma\mathcal{A}}(\delta, \delta) + \cdots = 0.$$

The upper triangularity ensures this is a finite sum. We define an A_∞-category $Tw\mathcal{A}$ of which the (C, δ) are the objects. The spaces of morphisms are the same as for $\Sigma\mathcal{A}$, but composition in $Tw\mathcal{A}$ involves the δs padded in the slots of compositions for $\Sigma\mathcal{A}$

$$\mu^d_{Tw\mathcal{A}}(a_d, \cdots, a_1) = \sum_{j_0,\cdots,j_d \geq 0} \mu^{d+\sum j_k}_{\Sigma\mathcal{C}}(\delta_d, \cdots, \delta_d, a_d, \delta_{d-1}, \cdots, \delta_{d-1}, a_{d-1}, \cdots)$$

where δ_k are repeated j_k times.

Given two twisted complexes $(C_k, \delta_k), k = 0, 1$, and a morphism $a \in hom^0_{Tw\mathcal{C}}(C_0, C_1)$ such that $\mu^1_{Tw\mathcal{C}}(a) = 0$, one can define the mapping cone as

$$Cone(a) = \{C_0 \xrightarrow{a} C_1\} := \left(C_0[1] \oplus C_1, \begin{bmatrix} \delta_0 & 0 \\ a & \delta_1 \end{bmatrix}\right) \in Ob\, Tw\mathcal{A}.$$

2.2. Symplectic Lefschetz fibration

Here we follow Seidel [Se4] about the defintions of exact symplectic fibration. The main results here is that the geometric Dehn twist around a Lagrangian sphere corresponds to the mutation. The fibration we will consider has non-compact fibers, hence the symplectic parallel transport has the danger of running off to fiberwise-infinity. The usual treatment, as adopted in [Se4, Se1], is to cut-off the fiber, and consider a compact fiber with boundary, and trivialize the fibration near the boundary. It is somewhat awkard to implement this condition here, instead we will prove by hand that the symplectic parallel transport does not 'run to infinity'. For the present discussion, we will still state the setup using compactified fiber with boundary, in accordance with the literature, and introduce the necessary modification later.

Here we follow the presentation in [Se5] closely. Let (M, ω, θ) be an exact symplectic manifold of dimension $2n$, where M is compact with boundary ∂M, $\omega \in \Omega^2(M)$ is a symplectic form, and $\theta \in \Omega^1(M)$ satisfies $d\theta = \omega$. We consider *exact* symplectomorphism $Symp^e(M)$ ϕ of M which are equal to the identity near ∂M, where exact means $[\phi^*\theta - \theta] = 0$ in $H^1(M, \partial M; \mathbb{R})$. Note that any isotopy within this subgroup is Hamiltonian.

Let S be a smooth manifold with boundary. An *exact symplectic fibration* over S consists of data (E, π, Ω, Θ) as follows:

(1) $\pi : E \to S$ is a proper differentiable fiber bundle with $2n$-dimensional fiber.

(2) $\Omega \in \Omega^2(E)$ and $\Theta \in \Omega^1(E)$ such that $\Omega = d\Theta$, the vertical part $\Omega| \ker(d\pi)$ is non-degenerate everywhere.

(3) The boundary $\partial E = \partial_h E \cup \partial_v E$, where the horizontal part $\partial_h E \to S$ is again a fiber bundle (see [Se5] for precise control of the boundary), and the vertical part $\partial_v E = \pi^{-1}(\partial S)$.

The form Ω defines a canonical connection on $\pi : E \to S$, with structure group $Symp^e(E_z)$. We denote the parallel transport maps of the canonical connection $\rho_c : E_{c(a)} \to E_{c(b)}$ for $c : [a, b] \to S$.

The exact Lefschetz fibration is a modification of above construction by allowing Morse type singularities. Assuming S is two-dimensional and oriented, an exact Lefschetz fibration over S consists of the data $(E, \pi, \Omega, \Theta, J_0, j_0)$, where (E, π, Ω, Θ) are as before except

π is allowed to have finitely many critical points living in the interior of E and in distinct fibers. J_0 is an integrable complex structure defined in a neighborhood of the set $E^{crit} \subset E$ of critical points, where Ω is a Kähler form on it, and similarly j_0 is a positively oriented complex structure defined in the neighborhood of critical values $S^{crit} \subset S$, such that π is (J_0, j_0)-holomorphic and with non-degenerate Hessian near every critical point. An exact Lefschetz fibration is denoted as (E, π) for short.

A *framed Lagrangian sphere* is a Lagrangian submanfiold L togehter with an equivalence class $[f]$ of diffeomorphisms $f : S^n \to L$. Here f_1, f_2 are equivalent, if $f_2^{-1} \circ f_1$ is isotopic to some element in $O(n+1) \subset Diff(S^n)$. One can associated to any $(L, [f])$ a Dehn twist $\tau_{(L,[f])} \in Symp(M)$ which is unique up to Hamiltonian isotopy. If L is exact, so is the Dehn twist along it. See ([Se4], 1.2) for the precise construction of it. We will omit the framing $[f]$ from the notation in the following.

Let $c : [0,1] \to S$ be a path such that $c(0)$ is a critical value for critical point x, and $c'(0) \neq 0$. Then the stable manifold

$$B = \{x\} \cup \{y \in E_{c(s)}, s \in (0,1], \lim_{t \to 0} \rho_c|_{[t,s]}^{-1}(y) = x\}$$

is a smoothly embedded $(n+1)$-ball, on which Ω vanishes. The intersection $V = B \cap E_{s(1)}$ is then a Lagrangian sphere, with a prefered framing $[f]$ induced by the framing on the sphere $B \cap E_{s(\epsilon)}$ for small enough positive ϵ. The monodromy of along a path counterclockwise around the critical value is isotopic to $\tau_{(V,[f])}$.

Suppose $S = D$ is the closed unit disk in $\mathbb{C}$, and $\pi : E \to S$ is an exact Lefschetz fibration with critical values $c_1, \cdots, c_m$, then we may fix any regular value $b \in S$ and draw non-intersecting (except at the ends) paths from the critical values to the regular value.

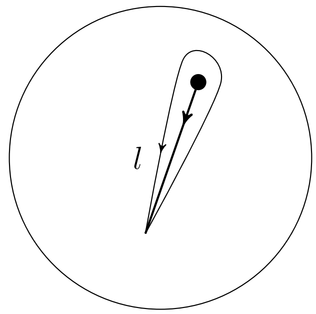


Figure 2.1

Counting the paths coming into b clockwise gives a cyclic ordering on the paths, one may fix a linear ordering by choosing the 'first' path. In the case when b is on the boundary, one may choose the first path as the 'leftmost' one viewed from b. The ordered collection of exact framed Lagrangian spheres in E_b is called a *distinguished basis* of vanishing cycles.

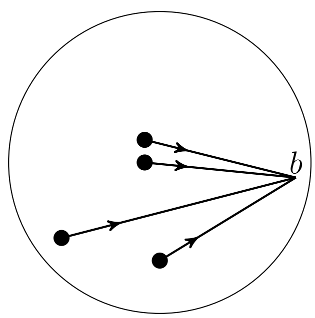


Figure 2.2

Definition 2.2.1. A Lagrangian configuration in an exact symplectic manifold M is an ordered family $\Gamma = (L_1, \cdots, L_m)$ of Lagrangian spheres. Two configurations are called Hurwitz equivalent if they are connected by a sequence of the following moves and their inverses:

(a) $\Gamma = (L_1, \cdots, L_m) \rightsquigarrow (L_1, \cdots, \phi(L_i), \cdots, L_m)$ for some $1 \leq i \leq m$ and some $\phi \in$

$Symp^e(M)$ isotopic to identity.

(b) For $1 \leq i < m$, $\Gamma \rightsquigarrow \beta_i\Gamma := (L_1, \cdots, L_{i-1}, \tau_i L_{i+1}, L_i, L_{i+2}, \cdots, L_m)$.

The $\{\beta_1, \cdots, \beta_{m-1}\}$ are generators of a the braid group on m strands.

CHAPTER 3

Review of Constructible Sheaves

Here we give a quick review on constructible sheaves, following the introduction of [**N1**] and the Appendix of [**STW**] very closely. For a thorough account on constructible sheaf theory and its relation with Fukaya category, see [**KS, S**] and [**NZ**].

We first give the categorical background, especially the definition for dg enhancement of the triangulated derived categories. Then we give some useful formula for practical computations. Next, we take a brief detour in symplectic geometry to define conical Lagrangian in cotangent bundles, so that we can define the singular support $SS(F)$ of a constructible sheaf F. Then, we discuss some *non-characteristic* deformations results, where the sections $\Gamma(U_t, F)$ is invariant up to quasi-isomorphism as U_t vary. Then, we recall quantization of Hamiltonian contactomorphism of [**GKS**].

3.1. Classical and differential graded derived categories of sheaves

Let X be a topological space. The poset (viewed as a category) $Open(X)$ has objects of open subsets, and partial orderings (morphisms) are given by inclusions of open subsets. Let Vect be the abelian category of $\mathbb{C}$-vector spaces.

- A presheaf F valued in Vect is a functor

$$F : Open(X)^o \to \mathrm{Vect}\,.$$

A presheaf F is a sheaf, if for any collection of open subsets $\{U_i\}_{i\in I}$, we have an exact sequence

$$0 \to F(\bigcup_i U_i) \to \prod_i F(U_i) \to \prod_{i,j} F(U_i \cap U_j).$$

- Let $C(X)$ be the abelian category of complexes of sheaves on X, with morphisms being degree-zero chain maps.
- Let $K(X)$ be the homotopy category of $C(X)$, where objects are the same as $C(X)$, ands morphisms are chain maps upto homotopy equivalences

$$\mathrm{Hom}_{K(X)}(F^\bullet, G^\bullet) := \mathrm{Hom}_{C(X)}(F^\bullet, G^\bullet)/\sim$$

  where $\varphi_1 \sim \varphi_2$ if $\varphi_1 - \varphi_2 = d \circ h - h \circ d$ for some degree -1 map $h : F^\bullet \to G^{\bullet-1}$.
- The derived category $D(X)$ is obtained from $K(X)$ by inverting quasi isomorphisms. The bounded derived category $D^b(X)$ is defined to be the full subcategory of complexes with bounded cohomologies.

To define constructibility, let X be a real analytic manifold. We fix an algebro-geometric category $\mathcal{C}$, e.g., the category of subanalytic sets. A *Whitney stratification* $\mathcal{S} = \{\mathcal{S}_\alpha\}_{\alpha\in A}$ of X by $\mathcal{C}$-submanifolds, is a decomposition $X = \sqcup_{\alpha\in A}\mathcal{S}_\alpha$ into disjoint strata $\{\mathcal{S}_\alpha\}$ indexed by A, where $\mathcal{S}_\alpha$ are locally closed $\mathcal{C}$-submanifolds and $\mathcal{S}_\alpha \cap \overline{\mathcal{S}_\beta} \neq \emptyset$ if and only if $\mathcal{S}_\alpha \subset \overline{\mathcal{S}_\beta}$, and any pair of distinct strata $(\mathcal{S}_\alpha, \mathcal{S}_\beta)$ satisfies the *Whitney condition*, that is, if a sequence $\{x_n \in \mathcal{S}_\alpha\}$ and a sequence $\{y_n \in \mathcal{S}_\beta\}$ converges to a point $y \in \mathcal{S}_\beta$, such that in some local coordinate patch, the secant lines $\overline{x_i y_i}$ converges to a line l and the tangent planes $T_{x_i}\mathcal{S}_\alpha$ converges to a plane τ, then $l \subset \tau$.

Let $\mathcal{S} = \{\mathcal{S}_\alpha\}$ be a Whitney stratification of X. An object $F^\bullet \in D(X)$ is said to be $\mathcal{S}$-constructible, if the restrictions $H^i(F^\bullet)|_{\mathcal{S}_\alpha}$ of its cohomology sheaves to the strata of $\mathcal{S}$ are finite-rank and locally constant. We denote by $D_{\mathcal{S}}(X)$ the full subcategory of $D(X)$ spanned by $\mathcal{S}$-constructible objects, and denote by $D_c(X)$ the full subcategory of $D(X)$ spanned by constructible sheaves for some Whitney stratification. If the stratification $\mathcal{S}$ is finite, then by the finite rank condition on cohomology sheaves, $D_c(X) \subset D^b(X)$.

Next, we define the differential graded(dg) derived category. For a review on dg category and dg quotient construction, see [**Ke**] and [**Dr**].

- The (naive) dg category $Sh_{naive}(X)$ has objects as chain complexes of sheaves $F^\bullet$, same as $C(X)$, and morphisms are chain complexes, with the degree n element as
$$hom^n_{Sh_{naive}(X)}(F^\bullet, G^\bullet) := \prod_{i \in \mathbb{Z}} hom_X(F^i, G^{n+i})$$
and differentials are given by
$$d^n : hom^n_{Sh_{naive}(X)}(F^\bullet, G^\bullet) \to hom^{n+1}_{Sh_{naive}(X)}(F^\bullet, G^\bullet), \quad \varphi \mapsto d_F \circ \varphi - (-1)^n \varphi \circ d_G.$$
- The dg derived category $Sh(X)$ is the dg quotient of $Sh_{naive}(X)$ by the full subcategory spanned by acyclic objects [**Dr**]. This is a triangulated dg category whose cohomology category $H^0(Sh(X))$ is canonically equivalent to the derived category $D(X)$ as a triangulated category.
- The dg derived category of bounded constructible categories $Sh^b_c(X)$ is the full subcategory of $Sh(X)$, whose objects $F^\bullet$ have bounded constructible cohomology sheaves. For a fixed Whitney stratification $\mathcal{S}$ of X, the $\mathcal{S}$-constructible dg

derived category $Sh_{\mathcal{S}}(X)$ is the full dg subcategory of $Sh(X)$ spanned by objects projecting to $D_{\mathcal{S}}(X)$.

- A dg functor $F : \mathcal{C} \to \mathcal{D}$ of dg categories is a quasi-embedding (resp. quasi-equivalence) if and only if the induced cohomological functor $H(F) : H(\mathcal{C}) \to H(\mathcal{D})$ is an embedding (resp. equivalence).

In this paper, we will only work with constructible sheaves, and will omit the subscript c for constructibility. To simplify notation, we use sheaf F to mean complex of sheaves $F^\bullet$, hom_X to mean hom-complex $hom^\bullet_{Sh(X)}$.

3.2. Useful Formulae for Computations

Inspite of the abstract categorical definitions, constructible sheaves enjoy many functorial properties which faciliates actual computations. Here we give some useful formulae and examples.

We use $f^*, f_*, f^!, f_!, \underline{hom}, \otimes$ to mean the corresponding dg derived functors:

$$\begin{aligned} - \otimes F : \quad & Sh(X) \leftrightarrow Sh(X) \quad : \underline{hom}(F, -) \\ f^* : \quad & Sh(X) \leftrightarrow Sh(Y) \quad : f_* \\ f_! : \quad & Sh(Y) \leftrightarrow Sh(X) \quad : f^! \end{aligned}$$

where $f : Y \to X$ is a map of real analytic manifolds.

The Verdier duality $\mathbb{D} : Sh(X)^o \to Sh(X)$ is an anti-involution. It interchanges shriek with star

$$\mathbb{D}\mathbb{D} = id, \quad f_! = \mathbb{D} f_* \mathbb{D}, \quad f^! = \mathbb{D} f^* \mathbb{D}.$$

The shrieks and stars are directly related in two cases: when f is proper $f_! = f_*$; when f is a smooth morphism of relative dimension d_f, $f^!(-) \cong f^*(-) \otimes \omega_{Y/X} \cong f^*(-) \otimes \mathfrak{or}_{Y/X}[d_f]$, where $\mathfrak{or}_{Y/X}$ is the orientation sheaf of the fiber.

Given an open subset U of X and its closed complement Z,

$$\text{open inclusion:} \quad U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z, \quad \text{closed inclusion,}$$

we have $j^* = j^!$ and $i_* = i_!$. Furthermore, there are exact triangles

$$i_! i^! \to id \to j_* j^* \xrightarrow{[1]}, \quad j_! j^! \to id \to i_* i^* \xrightarrow{[1]} .$$

These are sheaf-theoretic incarnations of excisions: applied to the constant sheaf on X and taking global sections, we get

$$H^*(Z, i^! \mathbb{C}) \to H^*(X, \mathbb{C}) \to H^*(U, \mathbb{C}) \xrightarrow{[1]}, \quad H^*_c(U, \mathbb{C}) \to H^*_c(X, \mathbb{C}) \to H^*_c(Z, \mathbb{C}) \xrightarrow{[1]} .$$

If Y is a locally closed $\mathcal{C}$-submanifold of X, we use $j_Y : Y \hookrightarrow X$ to denote the inclusion. Let $\mathbb{C}_Y \in Sh(Y)$ denote the constant sheaf on Y, and $\omega_Y = \mathbb{D}\mathbb{C}_Y$ be the Verdier dualizing complex of Y, then ω_Y is the canonically isomorphic to the shifted orientation sheaf $\mathfrak{or}_Y[\dim Y]$ on Y. The standard sheaf on Y is $j_{Y*}\mathbb{C}_Y$, and the costandard sheaf on Y is $j_{Y!}\omega_Y$.

The constructible sheaves can be 'constructed' by taking shifts and mapping cones of certain finite collection of sheaves. Let $\mathcal{T} = \{\tau_\alpha\}$ be a triangulation of X by simplices $j_\alpha : \tau_\alpha \hookrightarrow X$, where each τ_α is the embedding image of some open simplex. We denote by $\mathcal{C}_*(\mathcal{T})$ the the full dg subcategory of $Sh_{\mathcal{T}}(X)$ spanned by *standard objects* $j_{\alpha *}\mathbb{C}_{\tau_\alpha}$. The morphisms between standard objects are quasi-isomorphic to complexes concentrated at degree zero.

$$hom_{Sh_{\mathcal{T}}(X)}(j_{\beta_*}\mathbb{C}_{\tau_\beta}, j_{\alpha *}\mathbb{C}_{\tau_\alpha}) \cong \begin{cases} \mathbb{C} & \text{if } \beta \geq \alpha \\ 0 & \text{else} \end{cases}$$

where $\alpha \leq \beta$ if $\tau_\alpha \subset \overline{\tau_\beta}$.

Applying Verdier duality to the standard sheaves, we get the costandard sheaves $\mathbb{D}(j_{\alpha *}\mathbb{C}_{\tau_\alpha}) = j_{\alpha !}\omega_{\tau_\alpha}$. We have

$$hom_{Sh_{\mathcal{T}}(X)}(j_{\alpha !}\omega_{\tau_\alpha}, j_{\beta_1}\omega_{\tau_\beta}) \cong \begin{cases} \mathbb{C} & \text{if } \alpha \leq \beta \\ 0 & \text{else.} \end{cases}$$

We denote by $\mathcal{C}_!(\mathcal{T})$ the the full dg subcategory of $Sh_{\mathcal{T}}(X)$ spanned by *costandard objects* $j_{\alpha !}\omega_{\tau_\alpha}$

Lemma 3.2.1 ([N1], Lemma 2.3.1)**.** *$Sh_{\mathcal{T}}(X)$ is the triangulated envelope of $\mathcal{C}_*(\mathcal{T})$ (resp. $\mathcal{C}_!(\mathcal{T})$).*

Example 3.2.2. Let $Y = (0,1) \subset \mathbb{R}$, then $j_{Y*}\mathbb{C}_Y$ is the constant sheaf with stalk $\mathbb{C}$ supported on the closed interval $[0,1]$, and $j_{Y!}\omega_Y$ is the costandard sheaf with stalk isomorphic to $\mathbb{C}[1]$ supported on the open interval $(0,1)$. △

Example 3.2.3. Let $f : \{0\} \hookrightarrow \mathbb{R}^n$, then

$$f^!(\mathbb{C}_{\mathbb{R}^n}) = \mathbb{D}f^*\mathbb{D}\mathbb{C}_{\mathbb{R}^n} = \mathbb{D}f^*(\mathbb{C}_{\mathbb{R}^n}[n]) = \mathbb{D}(\mathbb{C}_{\{0\}}[n]) = \mathbb{C}_{\{0\}}[-n]$$

where we have identified the orientation sheaf on $\mathbb{R}^n$ with the constant sheaf by chosing an orientation on $\mathbb{R}^n$. $\triangle$

3.3. Conical Lagrangian and Singular Support

In this subsection, we define singular supports of constructible sheaves. Roughly speaking, singular supports encode the 'positions and directions' where sections 'fail to propagate'. We first need to introduce notations from symplectic and contact geometry.

Let X be a smooth manifold, T^*X its cotangent bundle with the canonical one-form $\lambda = pdq$ and the canonical sympletic two form $\omega = d\lambda = dp \wedge dq$. Let $\dot{T}^*X = T^*X \backslash X$, where X is identified with the zero section in T^*X. Let $T^\infty X = \dot{T}^*X/\mathbb{R}_{>0}$, where $\mathbb{R}_{>0}$ acts by fiberwise dilation. There is a natural fiberwise compactifiation of T^*X to $\overline{T}^*X$, where $T^\infty X$ corresponds to the divisor at infinity $\overline{T}^*X \backslash T^*X$ ([NZ], §5.1.1).

A contact manifold (M, ξ) is a smooth manifold of odd dimension $2m+1$, with a smooth rank $2m$ subbundle $\xi \subset TM$, called a *hyperplane distribution*, such that locally $\xi = \ker(\alpha)$ for some one-form α and $\alpha \wedge (d\alpha)^m \neq 0$. Such a one-form α is called a *contact form*. The *Reeb vector field* with respect to a contact form α is the unique vector field R such that $\iota_R \alpha = 1$ and $\iota_R d\alpha = 0$. A contactomorphism between contact manifolds is a diffeomorphism that preserves the hyperplane distributions. A Legendrian submanifold $\mathcal{L}$ of M is an m-dimensional submanifold such that $T\mathcal{L} \subset \ker(\alpha) \cap \ker(d\alpha)$.

The divisor $T^\infty X$ at infinity of the compactification $\overline{T}^*X$ has a natural contact structure defined in the following way: Fix any smooth section H of the $\mathbb{R}_{>0}$-bundle $\dot{T}^*X \to T^\infty X$, then $T^\infty X$ is diffeomorphic to H by the section map; the canonical one-form λ of T^*X restricts to a contact form α on H, hence induces a contact structure ξ on

$T^\infty X$. If we fix a Riemmanian metric on X, then the section H can be taken as the *unit cosphere bundle*

$$S^*X = \{(x,\eta) \in T^*X \mid \|\eta\| = 1\}.$$

The Reeb flow on S^*X is the unit geodesic flow. We will identify S^*X and $T^\infty X$.

Example 3.3.1. The simplest example contact manifold is the 1-jet bundle on $\mathbb{R}^n$: $J^1\mathbb{R}^n := T^*_{(x,y)}\mathbb{R}^n \times \mathbb{R}_z$, and one choice of the contact form can be taken as $\alpha = z - \sum_{i=1}^n y_i dx_i$ and the corresponding Reeb flow is ∂_z. $\triangle$

A conical Lagrangian $\Lambda \subset T^*X$ is a Lagrangian (possibly singular) invariant under the $\mathbb{R}_{>0}$-action. A homogenous conical Lagrangian is a one contained in $\dot{T}^*X$. Given a conical Lagrangian Λ, we define the associated Legendrians as

$$\mathrm{Leg}(\Lambda) = \Lambda^\infty = (\Lambda \backslash X)/\mathbb{R}_{>0} \subset T^\infty X.$$

Conversely, given a Legendrian $\mathcal{L} \subset T^\infty X$, we use $\mathrm{Lag}(\mathcal{L})$ to denote the homogeneous conical Lagrangian in $\dot{T}^*X$ as the preimage of the quotient $\dot{T}^*X \to T^\infty X$.

Let $\mathcal{S} = \{\mathcal{S}_\alpha\}_{\alpha\in A}$ be a Whitney stratification of X, then there is a canonical conical Lagrangian associated to $\mathcal{S}$,

$$\Lambda_{\mathcal{S}} := \bigcup_{\alpha\in A} T^*_{\mathcal{S}_\alpha}X$$

where $T^*_N X = \{(x,\eta) \in T^*X \mid x \in N, \eta|_{TN} = 0\}$ denotes the conormal bundle of a submanifold $N \subset X$.

Let F be a $\mathcal{S}$-constructible sheaf in $Sh_{\mathcal{S}}(X)$ for a Whitney stratification $\mathcal{S}$. The singular support $SS(F)$ is a (singular) conical Lagrangian contained in $\Lambda_{\mathcal{S}}$ defined in the following way: a point $(x,\eta) \in T^*X$ is **not** in the singular support $SS(F)$, if there is a small open ball $B(x,\epsilon)$ around x, and a Morse function $f : B(x,\epsilon) \to \mathbb{R}$, with $f(x) = 0$ and $df(x) = \eta$, such that for any $0 < \delta \ll 1$, the canonical restriction morphism

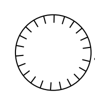
$$\Gamma(f^{-1}(-\infty,\delta),F) \to \Gamma(f^{-1}(-\infty,-\delta),F)$$

is a quasi-isomorphism. We use

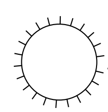
$$SS^{\infty}(F) := (SS(F))^{\infty} = (SS(F)\backslash X)/\mathbb{R}_{>0}$$

to denote the Legendrian in $T^{\infty}X$ associated to the conical Lagrangian $SS(F)$ in T^*X.

Example 3.3.2. Let $j : U = B(0,1) \hookrightarrow \mathbb{R}^2$ be the inclusion of an open unit ball in $\mathbb{R}^2$. Then $j_*\mathbb{C}_U$ is supported on the closed set $\overline{U}$, with singular support at infinity as

$$SS^{\infty}(j_*\mathbb{C}_U) = \{(x,\eta) \in S^*\mathbb{R}^2 \mid x \in \partial U, \eta = -d|x|\} =$$ .

And $j_!\mathbb{C}_U$ is supported on the open set U, with singular support at infinity as

$$SS^{\infty}(j_!\mathbb{C}_U) = \{(x,\eta) \in S^*\mathbb{R}^2 \mid x \in \partial U, \eta = d|x|\} =$$ .

Here the Legendrians are represented by co-oriented hypersurfaces in $\mathbb{R}^2$ with hairs indicating the co-orientation. $\triangle$

The following Lemma from [KS] is useful in characterising the singular support under $\otimes$ and $\underline{hom}$.

Proposition 3.3.3 (Proposition 5.4.14 of [KS])**.** *Let F and G belong to $Sh^b(X)$, then*

(1) if $SS^{\infty}(F) \cap (SS(G)^a)^{\infty} = \emptyset$, then $SS(F \otimes G) \subset SS(F) + SS(G)$.

(2) if $SS^\infty(F) \cap SS^\infty(G) = \emptyset$, then $SS(\underline{hom}(F,G)) \subset SS(G) - SS(F)$.

where $(-)^a$ is the fiberwise anti-podal map in T^*X and $\pm$ is the fiberwise sum/substraction in T^*X.

For a version without assuming $SS^\infty(F) \cap SS^\infty(G) = \emptyset$, see Corollary 6.4.5 and 6.2.4 in loc.cit.

3.4. Kernel and Functors

Let X_i, $i = 1, 2$, be spaces, and $K \in Sh(X_2 \times X_1)$. We define the following pair of adjoint functors

$$K_! : Sh(X_1) \leftrightarrow Sh(X_2) : K^! \tag{3.4.1}$$

$$K_! : F \mapsto (\pi_2)_!(K \otimes \pi_1^* F), \qquad K^! : G \mapsto (\pi_1)_*(\underline{hom}(K, \pi_2^! G)) \tag{3.4.2}$$

Indeed

$$\begin{aligned} & hom(K_! F, G) = hom((\pi_2)_!(K \otimes \pi_1^* F), G) = hom((K \otimes \pi_1^* F), (\pi_2)^! G) \\ = \ & hom(\pi_1^* F, \underline{hom}(K, (\pi_2)^! G)) = hom(F, (\pi_1)_* \underline{hom}(K, (\pi_2)^! G)) = hom(F, K^! G) \end{aligned}$$

Similarly, we may define

$$K^* : Sh(X_2) \leftrightarrow Sh(X_1) : K_* \tag{3.4.3}$$

$$K^* : G \mapsto (\pi_1)_!(K \otimes \pi_2^* G), \qquad K_* : F \mapsto (\pi_2)_*(\underline{hom}(K, \pi_1^! F)) \tag{3.4.4}$$

If we define transposition $K^t \in Sh(X_1 \times X_2)$ be the pullback of K under the transposition $t : X_1 \times X_2 \to X_2 \times X_1$, then

$$(K^t)_* = K^!, (K^t)^* = K_! \tag{3.4.5}$$

Proposition 3.4.1. *Let $f : X_1 \to X_2$ be a continuous map, and let $\mathbb{C}(f) = \mathbb{C}_{\Gamma(f)} \in Sh(X_2 \times X_1)$ be the constant sheaf on the graph $\Gamma(f)$ of f. Then, $\mathbb{C}(f)^* = f^*, \mathbb{C}(f)_* = f_*, \mathbb{C}(f)^! = f^!, \mathbb{C}(f)_! = f_!$.*

The Verdier duality also works as expected from the notation,

$$\mathbb{D}K_*\mathbb{D}\mathcal{F} = \mathbb{D}(\pi_2)_*(\underline{hom}(K, \pi_1^!\mathbb{D}\mathcal{F})) = (\pi_2)_!\mathbb{D}(\underline{hom}(K, \mathbb{D}\pi_1^*\mathcal{F})) = (\pi_2)_!(K \otimes \pi_1^*\mathcal{F}) \tag{3.4.6}$$

where we have used ([**KS**], Proposition 3.4.6)

$$\underline{hom}(G, F) = \underline{hom}(\mathbb{D}F, \mathbb{D}G) = \mathbb{D}(\mathbb{D}(F) \otimes G) \tag{3.4.7}$$

The functors can be composed as well. Let $K_{21} \in Sh(X_2 \times X_1)$ and $K_{32} \in Sh(X_3 \times X_2)$. Let $\pi_{ij} : X_3 \times X_2 \times X_1 \to X_i \times X_j$ be the projection map. Then the composition is defined as

$$K_{32} \circ K_{21} := (\pi_{31})_!(\pi_{32}^* K_{32} \otimes \pi_{21}^* K_{21}) \tag{3.4.8}$$

The composition has the property

$$(K_{32} \circ K_{21})_! = K_{32,!} \circ K_{21,!}, \quad (K_{32} \circ K_{21})_* = (K_{32})_* \circ (K_{21})_* \tag{3.4.9}$$

3.5. Non-characteristic Deformation Lemma

Just as in Morse theory, where a level sets $f^{-1}(t)$ of a Morse function f on M has constant diffeomorphism type when t varies in the connected components of the complement of the critical values of f, the non-characteristic deformation results for constructible sheaves are about the invariance of the hom-complexes $hom(F_t, G_t)$ for families of sheaves $\{F_t\}$ and $\{G_t\}$, when $SS^\infty(F)$ and $SS^\infty(G)$ are disjoint.

We first state the version regarding sections of a sheaf over an increasing sequence of open sets.

Proposition 3.5.1 (Proposition 2.7.2 in [KS]). *Let X be a real analytic manifold, F a bounded complex of constructible sheaves in $Sh(X)$, and let $\{U_t\}_{t\in\mathbb{R}}$ be a family of open subsets of X. We assume the following conditions:*
(1) $U_t = \bigcup_{s<t} U_s$ for all $t \in \mathbb{R}$.
(2) For all pairs (s,t) with $s \leq t$, the set $\overline{U_t \backslash U_s} \cap \mathrm{Supp}(F)$ is compact.
(3) Setting $Z_s = \cap_{t>s}\overline{U_t \backslash U_s}$ [1]*, we have for all pairs (s,t) with $s \leq t$ and all $x \in Z_s \backslash U_t$, that*

$$\underline{hom}(j_{X\backslash U_t *}\mathbb{C}_{X\backslash U_t}, F)|_x \cong 0.$$

Then we have for all $t \in \mathbb{R}$, the quasi-isomorphism

$$\Gamma(U, F) \xrightarrow{\sim} \Gamma(U_t; F), \qquad \textit{where } U = \bigcup_{s\in\mathbb{R}} U_s.$$

Remark 3.5.2. The section functor can be viewed as $\Gamma(U_t, F) = hom(j_{U_t!}\mathbb{C}_{U_t}, F)$. Hence this is a special case for $hom(G_t, F_t)$. The advantage for this version is that the

[1] See errata at https://webusers.imj-prg.fr/~pierre.schapira/books/Errata.pdf for the need of closure.

results holds for the section over union of the open sets $\{U_s\}$, instead of just between pairs of open sets U_t, U_s for some finite t, s.

Proposition 3.5.3 (Corollary 2.10, [?]). *Let I be an open interval of $\mathbb{R}$, let $q : M\times I \to I$ be the projection, and let ι_s be the embedding $M \times \{s\} \hookrightarrow M \times I$. Let $F \in Sh(M \times I)$, such that $SS^\infty(F) \cap (T^*_M M \times T^*I)^\infty = \emptyset$ and q is proper on $\mathrm{Supp}(F)$. Set $F_s = \iota_s^* F$. Then we have isomorphisms*

$$\Gamma(M, F_s) \cong \Gamma(M, F_t) \quad \textit{for all} \;\; s, t \in I.$$

Since the hom-complex can be obtained by taking the global section of hom-sheaf, we have a non-characteristic deformation result for $hom(F_t, G_t)$. First we state a lemma:

Lemma 3.5.4 (Petrowsky theorem for sheaves, Corollary 4.6 [?]). *Let F, G be bounded constructibles sheaves in $Sh(X)$. If $SS^\infty(F) \cap SS^\infty(G) = \emptyset$, then the natural morphism*

$$\underline{hom}(F, \mathbb{C}_X) \otimes G \to \underline{hom}(F, G)$$

is an isomorphism.

3.6. Quantization of Contactomorphism

For any contactomorphism $\varphi : S^*M \to S^*M$, Hamiltonian isotopic to identity, GKS constructed a kernel $K_\varphi \in Sh(M \times M)$, such that the functor $K_{\varphi !}$ satisfies

$$SS^\infty(K_{\varphi !}\mathcal{F}) = \varphi(SS^\infty(\mathcal{F})) \tag{3.6.1}$$

One may achieve the same effect of moving the singular support, using the 'lower-star' push-forard. Let $a : S^*M \to S^*M$ be the anti-podal map $(q,p) \mapsto (q,-p)$, and $\varphi^a := a \circ \varphi \circ a$, then

Lemma 3.6.1.

$$SS^\infty((K_{\varphi^a})_*\mathcal{F}) = \varphi(SS^\infty(\mathcal{F})) \tag{3.6.2}$$

Hence

$$(K_{\varphi^a})_* = K_{\varphi!} \tag{3.6.3}$$

Proof. Using Verdial duality $\mathbb{D}$, we have

$$K_{\varphi_*} = \mathbb{D}K_{\varphi!}\mathbb{D}, \quad SS^\infty(K_{\varphi_*}\mathcal{F}) = SS^\infty(\mathbb{D}K_{\varphi!}\mathbb{D}\mathcal{F}) = a \circ \varphi \circ a(SS^\infty(\mathcal{F})) \tag{3.6.4}$$

Replacing φ by φ^a finishes the proof of the first line. The equality of the two functors can be seen when $\varphi = \varphi_0$ is the identity map and the equality persist as φ_t interpolates between the identity map and the contactomorphism φ. $\square$

There are several ways to express the inverse of $(K_\varphi)_!$.

(1) By the commutativity of quantization and composition, we have

$$(K_{\varphi^{-1}})_! \circ (K_\varphi)_! = (K_{\varphi^{-1}\circ\varphi})_! = (K_{id})_! = id \tag{3.6.5}$$

and

$$(K_{(\varphi^a)^{-1}})_* \circ (K_{\varphi^a})_* = id \tag{3.6.6}$$

(2) By the property of adjoint functor, we have

$$id \xrightarrow{\sim} (K_\varphi)^!(K_\varphi)_!, \quad (K_\varphi)_!(K_\varphi)^! \xrightarrow{\sim} id \tag{3.6.7}$$

(3) By the general construction of the inverse kernel ([GKS], Proposition 1.14), for any $K \in Sh(X_2 \times X_1)$, we may define

$$K^{-1} := [\underline{hom}(K, \pi_1^! \mathbb{C}_{X_1})]^t \tag{3.6.8}$$

then we have a canonical map

$$K^{-1} \circ K \to \mathbb{C}_{\Delta_{X_1}} \tag{3.6.9}$$

Under suitable condition (see loc.cit), the above map is an isomorphism.

Hence, we have three candidate inverse functors to $(K_\varphi)_!$,

$$(K_{\varphi^{-1}})_! = (K_{(\varphi^a)^{-1}})_*, \quad (K_\varphi)^! = (K_\varphi^t)_*, \quad (K_\varphi^{-1})_! \tag{3.6.10}$$

In the case where φ is the geodesic flow on $\mathbb{R}^n$ for time t, we have

$$K_\varphi = \begin{cases} \mathbb{C}_{\{|x_2 - x_1| < t\}}[n] & t > 0 \\ \mathbb{C}_{\{|x_2 - x_1| \leq |t|\}} & t \leq 0 \end{cases} \tag{3.6.11}$$

Here $\varphi^a = \varphi^{-1}$, and $K_\varphi = K_\varphi^t$, we do have

$$K_\varphi^{-1} = K_{\varphi^{-1}}, \quad K_\varphi^t = K_\varphi = K_{(\varphi^a)^{-1}} \tag{3.6.12}$$

CHAPTER 4

Lagrangian Skeleton of Hypersurface in $(\mathbb{C}^*)^n$

A skeleton L of a smooth non-compact manifold M is a minimal deformation retracts of M. If M is an exact symplectic manifold, $\omega = d\alpha$, then there is canonically-defined retracting flow ξ ($\iota_\xi \omega = -\alpha$) on M that preserves the symplectic structure (upto rescaling), and the retraction core is a singular Lagrangian submanifold, called the *Lagrangian skeleton.* (The precise definition will be given later.)

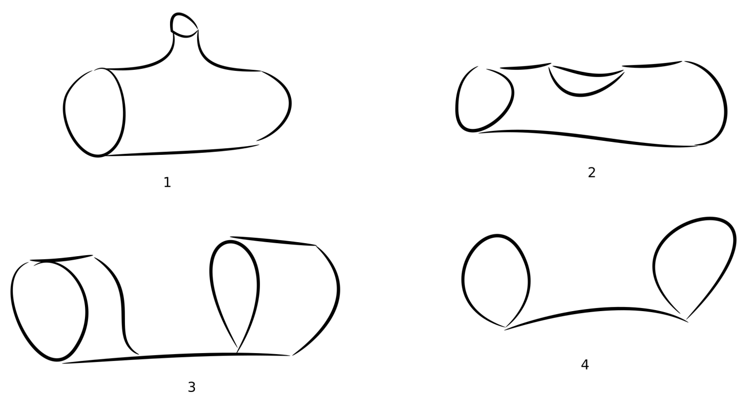


Figure 4.1. Deformation retraction of a pair-of-pants.

Just as a skeleton knows the homotopy type of the ambient manifold, a Lagrangian skeleton (or rather, its tubular neighborhood) remembers the symplectic topology of the ambient exact symplectic manifold. In particular, under the retracting flow, compact Lagrangians in M will flow into the tubular neighborhood of the Lagrangian skeleton L, and Floer theoretic computation might be turned into local computation on the skeleton.

If M is an exact Kähler manifold, that is $\omega = dd^c\varphi$ globally for some pluri-subharmonic function $\varphi : M \to \mathbb{R}$, then (M, ω, α) is also an exact symplectic manifold, with $\alpha = d^c\varphi$, and the retracting flow is the (negative) gradient flow for φ. Hence, if we equip $(\mathbb{C}^*)^n$ with a Kähler structure (ω, φ), then any complex hypersurface Y in $(\mathbb{C}^*)^n$ is an exact symplectic manifold coming from the induced exact Kähler structure $(\omega|_Y, \varphi|_Y)$.

Skeletons of affine hypersurfaces in $(\mathbb{C}^*)^n$ have previously been studied by [RSTZ], where a skeleton is constructed combinatorially using the Newton polytope of the defining Laurent polynomial of the hypersurface. Here we show that the skeleton can be improved to be a Lagrangian skeleton. One technique is Abouzaid's semi-tropicalization, which simplifies the defining local defining equation of a hypersurface by omitting non-dominating terms. Another technique is the construction of an adapted Kähler potential φ on $(\mathbb{C}^*)^n$ for the Newton polytope, so that the induced skeleton on the hypersurface is like that constructed by [RSTZ].

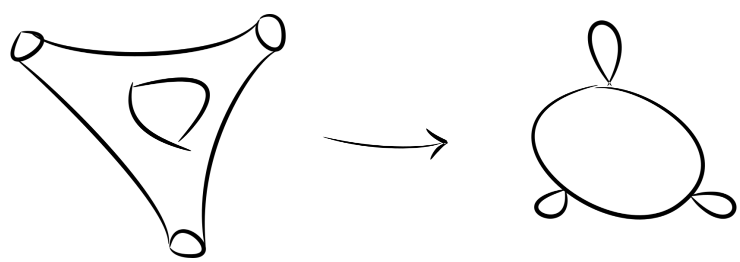


Figure 4.2. Deformation retraction of an three-times punctured torus.

We will first give the necessary background on Weinstein manifold following [CE]. Then we review the construction of the RSTZ-skeleton. Then we introducing the adapted Kähler potential for the Newton polytope, and Abouzaid's tropical localization method. Finally, we prove the Lagrangian skeleton agrees with the RSTZ-skeleton.

4.1. Review of Weinstein Manifold and Skeleton

Here we recall the relevant definitions and properties from [CE], Chapter 11. An *exact symplectic structure* on a symplectic manifold (M, ω) is a one-form λ, called *Liouville form* such that $\omega = d\lambda$. The vector field X that is ω-dual to λ, i.e., $\iota_X = d\lambda$, is called the *Liouville vector field* for λ. X is an outgoing vector field, in that

$$\mathcal{L}_X \omega = (\iota_X d + d\iota_X)\omega = d\lambda = \omega. \tag{4.1.1}$$

Also note that

$$\iota_X \lambda = \iota_X \iota_X \omega = 0, \quad \mathcal{L}_X \lambda = \iota_X d\lambda = \lambda. \tag{4.1.2}$$

A map $\psi : (M_1, \omega_1, \lambda_1) \to (M_2, \omega_2, \lambda_2)$ between exact symplectic manifolds is called *exact symplectic* if $[\psi^* \lambda_2 - \lambda_1] = 0 \in H^1(M_1, \mathbb{R})$.

Definition 4.1.1. A *Liouville manifold* is an exact symplectic manifold (M, ω, λ, X), or denoted as (M, ω, λ) or (M, ω, X), such that

(1) the expanding vector field X is *complete*, and
(2) the manifold is *convex* in the sense that there exists an exhaustion $M = \cup_{k=1}^{\infty} M^k$ by compact domains $M^k \subset M$ with smooth boundaries along with X is outward pointing.

We denote the time-t flow generated by X as $\Phi_X^t : M \to M$, for all $t \in X$. Since $\mathcal{L}_X \omega = \omega$, $(\Phi_X^t)^* \omega = e^t \omega$, in other words, the ω-area of a surface increases as the surface flows with X.

Note that the sets M^k are invariant under the contracting flow Φ_X^{-t}, $t > 0$. The sets

$$\mathrm{Skel}(M, \omega, X) := \bigcup_{k=1}^{\infty} \bigcap_{t>0} \Phi_X^{-t}(M^k) \tag{4.1.3}$$

is independent of the choice of the exhausting sequence of compact sets M^k and is called the *skeleton* of the Liouville manifold (M, ω, λ).

Definition 4.1.2. A *Weinstein manifold* (M, ω, X, φ) is a Liouville manifold (M, ω, X) with a complete Liouville vector field X which is gradient-like for an exhausting Morse function $\varphi : M \to \mathbb{R}$.

We recall that a function $\varphi : M \to \mathbb{R}$ is exhausting if φ is proper and bounded from below, and a vector field X is gradient-like for φ if it satisfies

$$\langle X, d\varphi \rangle > \delta(|X|_g^2 + |d\varphi|_g^2) \tag{4.1.4}$$

for some $\delta > 0$ and some Riemannian metric g on M.

Remark 4.1.3. For our application, it is necessary and convenient to allow for Morse-Bott exhausting function φ. We expect most results about Weinstein manifold would carry over with little modification to the Morse-Bott setup.

Definition 4.1.4. A *Stein manifold* is a properly embedded complex submanifold of $\mathbb{C}^N$ for some N. Equivalently, a complex manifold (M, J) is Stein if and only if it admits an exhausting pluri-subharmonic (psh) function.

Remark 4.1.5. Let (M, J) be a Stein manifold, with φ an exhausting psh function, then $\omega_\varphi = 2i\partial\bar{\partial}\varphi = -dd^c\varphi$ is an exact symplectic two-form, with a Liouville one-form

$\lambda_\varphi = -d^c\varphi$, where $d^c\varphi = d\varphi \circ J$. The data $(M, \omega_\varphi, \lambda_\varphi, \varphi)$ gives M a Weinstein structure. The Liouville vector field $X_\varphi = \nabla\varphi$, since for any vector field Y

$$\omega(X_\varphi, Y) = \lambda_\varphi(Y) = -d^c\varphi(Y) = d\varphi(-JY) = g(\nabla\varphi, -JY) = \omega(\nabla\varphi, Y) \tag{4.1.5}$$

where we used the relation $g(X, Y) = \omega(X, JY)$ among (J, ω, g) for Kähler manifold.

We recall the following results of Lagrangian skeleton.

Proposition 4.1.6 ([CE], Lemma 11.1)**.** *The interior of a skeleton* $\mathrm{Skel}(M, \omega, X)$ *is empty.*

Proof. For each compact set M^k, we have

$$\mathrm{Vol}(\Phi_X^{-t}(M^k)) = e^{-nt}\,\mathrm{Vol}\, M^k \xrightarrow{t\to\infty} 0, \tag{4.1.6}$$

hence $\mathrm{Vol}(\cap_{t>0}\Phi_X^{-t}(M^k)) = 0$ for all $k \in \mathbb{N}$. □

We say that a Liouville manifold is *of finite type* if its skeleton is compact. In this case, let $W \subset M$ be a compact domain containing the skeleton with smooth boundary $\Pi = \partial W$ along which X is outward pointing (e.g. $W = M^k$ for large k). Then the forward flow of X starting from Π defines a diffeomorphism $M \backslash \mathrm{Int}\, W \cong \Pi \times [0, \infty)$. Under this identification, the Liouville form λ corresponds to $e^t\alpha$ for $t \geq 0$ and $\alpha = \lambda|_\Pi$. The form α is a contact one-form on Π, and $M \backslash \mathrm{Int}\, W$ is identified with the positive half of the symplectization of (Π, α), and

$$M \backslash \mathrm{Skel}(M) \cong \bigcup_{t\in\mathbb{R}} \Phi_X^t(\Pi). \tag{4.1.7}$$

The following lemma shows that for finite type Liouville manifolds, symplectomorphism can be made into exact symplectomorphism.

Lemma 4.1.7 ([CE], Lemma 11.2)**.** *Any symplectomorphism* $f : (M_1, \omega_1, \lambda_1) \to (M_2, \omega_2 \lambda_2)$ *between finite type Liouville manifolds is diffeotopic to an exact symplectomorphism.*

Lemma 4.1.8 ([CE], Lemma 11.4)**.** *Let* Π_1, Π_2 *be hypersurfaces in a Liouville manifold* (M, ω, X) *such that flowlines of* X *defines a diffeomorphism* $\Gamma : \Pi_1 \to \Pi_2$*. Then* Γ *is a contactomorphism.*

Proof. Use the Liouville flow to embed the symplectization $\mathbb{R} \times \Pi_1 \hookrightarrow M$, such that Π corresponds to $\{0\} \times \Pi_1$, $\lambda = e^r \alpha$ and $X = \partial_r$, where $\alpha = \lambda|_{\Pi_1}$. Then Π_2 is given as the graph $r = f(x)$ for some function $f : \Pi_1 \to \mathbb{R}$, and $\Gamma^*(\lambda|_{\Pi_2} = e^f \alpha$. $\square$

Next, we study the local properties of stable and unstable manifolds of flow X. We recall the notation first ([CE], Section 9.2) for a general smooth vector field X. Let p be a zero of X, the differential $D_p X : T_p M \to T_p M$ induces a splitting into invariant subspaces

$$T_p M = E_p^+ \oplus E_p^0 \oplus E_p^- \tag{4.1.8}$$

where E_p^+ (resp. E_p^0, E_p^-) are spanned by generalized eigenvectors of $D_p X$ with positive (negative, zero) real part of eigenvalues. Locally near a fix point p of X, there exists local stable manifold W_p^- and local unstable manifold W_p^+, that is invariant under the flow X, and $W_p^\pm$ tangent to $E_p^\pm$. $W_p^\pm$ are unique and smooth. [1]

A zero p of a vector field p is *non-degenerate*, if all its eigenvalue are non-zero. It is called *hyperbolic*, if all the real parts of eigenvalue are non-zero. If p is hyperbolic, one can define *global* stable and unstable manifolds as

$$W_p^\pm = \{x \in M \mid \lim_{s\to\mp\infty} \Phi_X^s(x) = p\}. \tag{4.1.9}$$

They are injectively immerse (but not necessarily embedded) in V.

Proposition 4.1.9 ([CE], Proposition 11.9)**.** *Let (M, ω, X) be an exact symplectic manifold, and p be a (possibly degenerate) zero of X. Then:*
(a) the local stable manifold W_p^- is isotropic;
(b) the local unstable manifold W_p^+ is coisotropic.

If M is a Weinstein manifold (see Definition 4.1.2), then we can say more.

Proposition 4.1.10 ([CE], Lemma 11.13)**.** *Let (M, ω, X, φ) be a Weinstein manifold.*
(a) The stable manifold W_p^- of any critical point p of φ (with respect to X) satisfies $\lambda|_{W_p^-} \equiv 0$. In particular, W_p^- is isotropic and the intersection $W_p^- \cap \varphi^{-1}(c)$ with any regular level set is isotropic for the contact structure induced by λ on $\varphi^{-1}(c)$.
(b) Suppose φ has no critical value in $[a, b]$, then the image of any isotropic submanfiold $\Lambda^a \subset \varphi^{-1}(a)$ under the flow of X intersects $\varphi^{-1}(b)$ in an isotropic submanifold.

[1]One can also construct W_p^0 as C^r manifold for arbitrarily large r, such that W_p^0 is tangent to E_p^0, however W_p^0 in general is not unique and need not be smooth.

In particular, every zero p in a Weinstein manifold is hyperbolic. Thus, the skeleton of (M, ω, X) is the union of all stable manifolds, which is isotropic.

Proposition 4.1.11. *If (M, ω, X, φ) is a Weinstein manifold with φ a Morse-Bott function, and P is a connected component of a critical manifold of φ, then*
(a) the local stable manifold W_P^- is isotropic;
(b) the local unstable manifold W_P^+ is coisotropic;
(c) the skeleton $\mathrm{Skel}(M, \omega, X)$ is a union of stable manifolds for the vector field X.

Proof. The proof is exactly the same as for the Morse case. □

4.2. Review of RSTZ-skeleton

Here we do not present the most general case where the RSTZ construction applies. We follow the presentation of the introduction in [RSTZ].

Let Z be an affine hypersurface in $(\mathbb{C}^*)^n$, with defining equation $f = 0$ and

$$f = \sum_{\alpha \in A \subset \mathbb{Z}^n} c_\alpha z^\alpha, \quad c_\alpha \neq 0 \tag{4.2.1}$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. The convex hull Q of A is called the *Newton polytope* of f. By multiplying f by a monomial, we may assume without loss of generality, that Q is a polytope containing 0. For generic choices of coefficients c_α, the topological type of the hypersurface depends only on this polytope. However, just like the Morse-Smale CW-decomposition of a Riemannian manifold depends on a choice of a Morse function, the skeleton depends on a choice of a triangulation of the polytope Q.

First, we recall some notion of triangulations from [**GKZ**]. A *triangulation* of a convex polytope Q is a decomposition of Q into a finite number of simplices such that the intersection of any two of these simplices is a common face of them both (maybe empty). If $Q = \mathrm{conv}(A)$ for a finite subset A, then a *triangulation of* (Q, A) means a triangulation of Q with vertices in A. Note that we do not require every element of A to appear as a vertex of a simplex. A star triangulation for Q with base at $q \in Q$, is a triangulation of Q where every maximal simplex contains q.

A continuous function $g : Q \to \mathbb{R}$ is convex, if for any $x, y \in Q$, $g(tx + (1-t)y) \leq tg(x) + (1-t)g(y)$; g is T-piecewise-linear, if it is affine linear on every simplex of T. The domain of linearity of a convex function $g : Q \to \mathbb{R}$ is a subset $U \subset Q$, such that $g|_U$ is affine-linear and which is maximal with this property.

Definition 4.2.1. A triangulation T of Q is *regular (or coherent)*, if there exists a convex T-piecewise-linear function whose domains of linearity are precisely (maximal) simplices of T.

Next we give the definition of RSTZ-skeleton.

Definition 4.2.2 ([**RSTZ**], Definition 1.1)**.** Let $Q \subset \mathbb{R}^n$ be a lattice polytope with $0 \in Q$. Let T' be a regular star triangulation of Q based at 0, whose vertices are lattice points, and define T to be the set of simplices of T' not meeting 0. Let $|T|$ denote the union of the simplices in T, and for each $x \in |T|$, let $\tau(x)$ denote the smallest simplex in T containing x.

We define the candidate skeleton $S_{Q,T} \subset |T| \times \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, S^1)$ by

$$S_{Q,T} := \{(x, \phi) \mid \phi(v) = 1, \text{ if } v \text{ is a vertex of } \tau(x)\}. \tag{4.2.2}$$

Theorem 6 ([RSTZ], Main Theorem)**.** *Let Q and T be as in Definition 4.2.2. Let Z_Q be a generic smooth hypersurface whose Newton polytope is Q. Then $S_{Q,T}$ embeds into Z_Q as a deformation retract.*

We give some examples for illustration.

Example 4.2.3. Let $f = x + y + 1/xy$. The Newton polytope Q and its star triangulations are shown below.

(4.2.3)

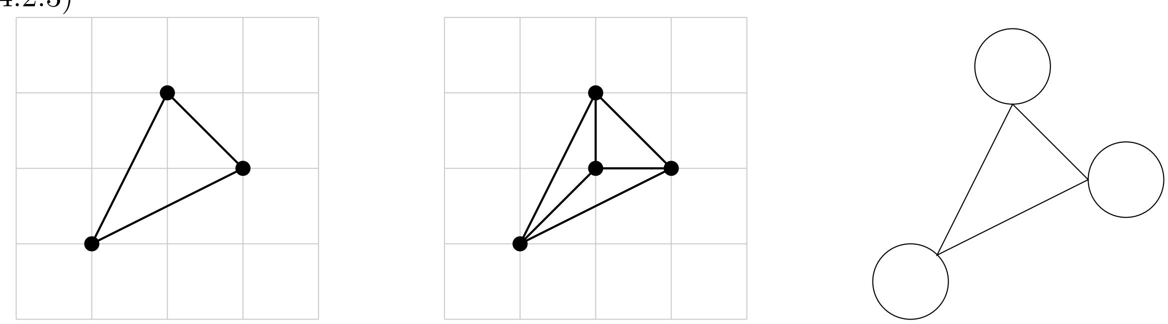


Figure 4.3. $f = x+y+1/xy$. The Newton polytope Q; the star triangulation and the RSTZ skeleton.

Since Q contains 0 as an interior point, we have $|T| = \partial Q$ in Definition 4.2.2. We work out the detail of $S_{Q,T}$ for illustration. Fix a the standard basis e_x, e_y for $\mathbb{Z}^2$, then a homomorphism $\varphi : \mathbb{Z}^2 \to S^1$ is determined by the images of e_x and e_y, denote them by

$e^{i\phi_x}, e^{i\phi_y}$ respectively. Then we have the following components in $S_{Q,T}$:

$$\begin{cases} x = (1,0) & \phi_x = 0 \\ x = (0,1) & \phi_y = 0 \\ x = (-1,-1) & \phi_x + \phi_y = 0 \\ x \in \operatorname{conv}\{(1,0),(0,1)\} & \phi_x = \phi_y = 0 \\ x \in \operatorname{conv}\{(1,0),(-1,-1)\} & \phi_x = \phi_y = 0 \\ x \in \operatorname{conv}\{(0,1),(-1,-1)\} & \phi_x = \phi_y = 0 \end{cases} \tag{4.2.4}$$

The first three lines define three circles, over the vertices of Q; the last three lines defines three segments, over the edges over Q. The fiber of f is a three-times punctured torus, hence the degeneration is the same as Figure 4.2. △

Remark 4.2.4. Here is an heuristic way to understand why the skeleton for smooth fiber of $f = x + y + 1/xy$ is as above. Consider a very large positive real number R, and the fiber $f^{-1}(R)$. The three circles in Figure 4.3 correspond to one-term-domination, i.e. places where one term in f is much larger than the other two terms, and the circle represent the irrelevant term's choice of the argument. The three edges corresponds to two-term-domination, e.g. the edge between $(0,1)$ and $(1,0)$ corresponds to $x > 0, y > 0$ and $x + y \approx R$ and $|x|, |y| \gg |1/xy|$, and since dominant terms needs to be real, there is no freedom in the choice of the argument. We will see in the next section how to make this rigorous.

4.3. Convex function and Legendre transformation.

In this section we use Legendre transform to define a diffeomorphism between $(\mathbb{C}^*)^n \cong T^*T^n$. We will also prove a few properties about homogeneous convex functions that will be used later. We will use the same notation about $M, M_{\mathbb{R}}, N, \cdots$ as in the introduction.

4.3.1. Legendre transformation

Let V be a real vector space of dimension n, and $V^\vee$ be its dual space. There is a natural identification of symplectic space

$$T^*V \cong V \times V^\vee \cong T^*V^\vee.$$

Let π_V and $\pi_{V^\vee}$ denote the projection of $V \times V^\vee$ to its first and second factor, respectively.

Let φ be a smooth strictly convex function on V. The Legendre transformation for φ is defined as

$$\Phi_\varphi : V \to V^\vee, \quad x \mapsto d\varphi(x).$$

We will always assume φ satisfies some growth condition such that the Legendre transformation is surjective. The Legendre dual ψ of φ is also a convex function defined as

$$\psi : V^\vee \to \mathbb{R}, \quad y \mapsto \sup_{x \in V} \langle x, y \rangle - \varphi(x) = \langle \Phi_\varphi^{-1}(y), y \rangle - \varphi(\Phi_\varphi^{-1}(y)).$$

If we fix a linear coordinate $\rho = (\rho_1, \cdots, \rho_n)$ on V and dual coordinate $p = (p_1, \cdots, p_n)$ on $V^\vee$, then the Legendre transformation can be written as

$$p_i = \partial_{\rho_i}\varphi(\rho).$$

If $p = d\varphi(p)$, then Legendre dual function

$$\psi(p) = \sum_i \rho_i p_i - \varphi(\rho).$$

And the two matrices $\operatorname{Hess}\varphi(\rho) = \partial_{ij}\varphi(\rho)$ and $\operatorname{Hess}\psi(p) = \partial_{ij}\psi(p)$ are inverse of each other. There is a metric on V induced by φ:

$$g_\varphi = \partial_{ij}\varphi(\rho) d\rho_i \otimes d\rho_j.$$

The above contruction can be interpreted symplectically. Consider the graph Lagrangian $\Gamma_{d\varphi}$ in T^*V

$$\Gamma_{d\varphi} = \{(x, y) \in V \times V^\vee \mid y = d\varphi(x)\}.$$

Let $L = \Gamma_{d\varphi}$. Then the Legendre transform is $\Phi_\varphi = \pi_{V^\vee}|_L \circ \pi_V|_L^{-1}$

$$\begin{tikzcd} & L \arrow[dl, "\pi_V"'] \arrow[dr, "\pi_{V^\vee}"] & \\ V & & V^\vee \end{tikzcd} .$$

L as a section in $T^*V^\vee$ is the graph of $\Gamma_{d\psi}$ for the Legendre dual function ψ of φ.

We record the following result for future reference

Proposition 4.3.1. *Let φ be any smooth convex function on V, inducing a Legendre transform $\Phi_\varphi : V \to V^\vee$ and a metric g_φ on V. Let $f : V \to \mathbb{R}$ be any smooth function.*

Let $\rho \in V$, $p = \Phi_\varphi(\rho) \in V^\vee$, then

$$(\Phi_\varphi)_*(\nabla f|_\rho) = df(\rho),$$

under the identification of $\in T^*_\rho V \cong V^\vee \cong T_p V^\vee$.

Proof. We work with linear coordinates $(\rho_1, \cdots, \rho_n)$ on V and dual coordinate $(p_1, \cdots, p_n)$ on $V^\vee$. Let $g_{ij} = (g_\varphi)_{ij} = \partial_{ij}\varphi$ and g^{ij} be the matrix inverse of g_{ij}.

$$\begin{aligned}
(\Phi_\varphi)_*\nabla f(\rho) &= \sum_{i,j,k} \partial_{\rho_k} f \cdot g^{jk} \cdot \frac{\partial p_i(\rho)}{\partial \rho_j} \cdot \partial_{p_i} \\
&= \sum_{i,j,k} \partial_{\rho_k} f \cdot g^{jk} \cdot g_{ij} \cdot \partial_{p_i} \\
&= \sum_{i,k} \partial_{\rho_k} f \cdot \delta^i_k \cdot \partial_{p_i} = df.
\end{aligned}$$

□

4.3.2. Identification between $M_{\mathbb{C}^*}$ and T^*T_M.

There is a canonical complex structure on $M_{\mathbb{C}^*} \cong M_{\mathbb{R}} \times T_M$, and a canonical symplectic structure on $T^*T_M \cong N_{\mathbb{R}} \times T_M$. We will use notation $\theta \in T_M, \rho \in M_{\mathbb{R}}$ and $p \in N_{\mathbb{R}}$. If we fix a $\mathbb{Z}$-basis for M, then we have $M_{\mathbb{C}^*} \cong (\mathbb{C}^*)^n = \{(e^{\rho_i + i\theta_i})_i\}$ and $T^*T_M \cong T^*T^n = \{(\theta_i, p_i)_i\}$.

Let $\varphi : M_{\mathbb{R}} \to \mathbb{R}$ be a smooth strictly convex function such that the Legendre transformation $\Phi_\varphi : M_{\mathbb{R}} \to N_{\mathbb{R}}$ is surjective. We abuse notation and also denote by φ the pullback via $M_{\mathbb{C}^*} \to M_{\mathbb{R}}$, and call φ a Kähler potential on $M_{\mathbb{C}^*}$. Then we may define Liouville one-form and symplectic two-form on $M_{\mathbb{C}^*}$

$$\lambda = -d^c\varphi, \quad \omega = -dd^c\varphi.$$

In coordinate form, we have

$$\lambda_\varphi = \sum_i \partial_i \varphi(\rho) d\theta_i, \quad \omega_\varphi = \sum_{i,j} \partial_{ij}\varphi(\rho)\, d\rho_i \wedge d\theta_j.$$

The Riemannian metric can also be obtained by $g_\varphi(X, Y) = \omega_\varphi(X, JY)$, where $J\partial_{\rho_i} = \partial_{\theta_i}, J\partial_{\theta_i} = -\partial_{\rho_i}$, or in coordinate form

$$g = \sum_{i,j} \partial_{ij}\varphi(\rho)(d\rho_i \otimes d\rho_j + d\theta_i \otimes d\theta_j).$$

If we equip T^*T_M with the standard exact symplectic structure (ω, λ):

$$\lambda_{std} = \sum_i p_i d\theta_i, \quad \omega_{std} = \sum_i dp_i \wedge d\theta_i,$$

then by Legendre transformation $\Phi_\varphi \times \mathrm{id} : M_{\mathbb{C}^*} = M_{\mathbb{R}} \times T_M \to N_{\mathbb{R}} \times T_M = T^*T_M$, we have

$$(\Phi_\varphi \times \mathrm{id})^*(\lambda_{std}) = \lambda_\varphi, \quad (\Phi_\varphi \times \mathrm{id})^*(\omega_{std}) = \omega_\varphi.$$

4.3.3. Homogeneous Kähler potential

Next we will restrict ourselves to homogenous convex functions as Kähler potential.

Definition 4.3.2. A convex function φ on $M_{\mathbb{R}}$ is said to be *homogeneous of degree d* for some $d \geq 1$, if for any $0 \neq x \in M_{\mathbb{R}}$ and any $\lambda > 0$, we have

$$\varphi(\lambda x) = \lambda^d \varphi(x), \tag{4.3.1}$$

and $\Omega = \{x : \varphi(x) \leq 1\}$ is a bounded strictly convex closed set with smooth boundary.

Remark 4.3.3. Any positive definite quadratic form on $M_{\mathbb{R}}$ is a homogeneous degree two convex function. More generally, any bounded strictly convex subset $\Omega \subset M_{\mathbb{R}}$ with smooth boundary and containing 0 as an interior point determines a homogeneous degree d convex function $\varphi_{\Omega,d}$ such that $\Omega = \{x : \varphi(x) \leq 1\}$.

Proposition 4.3.4. *For any homogeneous convex function φ of degree d with $d \geq 1$, we have*

(1) φ is smooth on $M_{\mathbb{R}} \backslash \{0\}$.

(2) φ is C^k at 0 where k is the largest integer less than d.

(3) If $d > 1$, then φ is strictly convex.

Proof. (1) and (3) are easy to verify. We only prove (2). Fix a linear coordinate $x_1, \cdots, x_n$ on $M_{\mathbb{R}}$. For multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, any point $0 \neq x \in M_{\mathbb{R}}$ and $\lambda > 0$, we have $\partial_x^\alpha \varphi(\lambda x) = \lambda^{d-|\alpha|} \partial_x^\alpha \varphi(x)$. Hence if in addition $|\alpha| \leq k < d$, then $\lim_{\lambda \to 0} \partial_x^\alpha \varphi(\lambda x) = 0$. Hence all k-th derivative can be continuated to $x = 0$. □

Proposition 4.3.5. *If φ is a homogeneous degree two convex function, the Legendre transformation Φ_φ is homogeneous of degree 1, i.e.*

$$\Phi_\varphi(\lambda \rho) = \lambda \Phi_\varphi(\rho).$$

Proof. This follows from definition. □

Definition 4.3.6. Let $M_{\mathbb{R}}^\infty := (M_{\mathbb{R}} \backslash 0)/\mathbb{R}_{>0}$ and $N_{\mathbb{R}}^\infty := (N_{\mathbb{R}} \backslash 0)/\mathbb{R}_{>0}$. Then we define the projective Legendre transformation

$$\Phi_\varphi^\infty : M_{\mathbb{R}}^\infty \to N_{\mathbb{R}}^\infty.$$

It is easy to check that Φ_φ^∞ is an orientation perserving diffeomorphism from S^{n-1} to itself. Intuitively, if we take the level set $S = \varphi^{-1}(1)$, then each element in $M_\mathbb{R}^\infty$ corresponds to a point on S, and the outward conormal of S at the point is the element in $N_\mathbb{R}^\infty$ obtained by Φ_φ^∞.

Proposition 4.3.7. *Let φ be any homogeneous convex function on $M_\mathbb{R}$ of degree $k > 1$, and equip $M_\mathbb{R}$ with metric g_φ induced from Hessian of φ. Then the integral curves in $M_\mathbb{R} \backslash \{0\}$ of the gradient of φ are rays.*

Proof. For any nonzero $\rho \in M_\mathbb{R}$, we have $\Phi_\varphi(\rho) = d\varphi(\rho)$, also by Proposition 4.3.1 we have $(\Phi_\varphi)_*(\nabla\rho) = d\varphi(\rho)$, hence the gradient field of φ on $M_\mathbb{R}$ when pushed-forward to $N_\mathbb{R}$ is exactly the radial vector field $p\partial_p$ whose integral curves are rays. Since φ is homogeneous, hence Φ_φ takes ray to ray, hence the integral curve of $\nabla\varphi$ is the pull-back of integral curve of $p\partial_p$, i.e. rays. $\square$

4.3.4. Kähler potentials Adapted to a Polytope

Let P be a convex polytope in $M_\mathbb{R}$ containing 0 as an interior point. We define a notion of convexity with respect to P.

Definition 4.3.8. A homogeneous convex function φ on $M_\mathbb{R}$ is *convex with respect to* P, if for each face F of P of positive dimension, the restriction $\varphi|_F$ has a unique minimum point in the interior of F. A *Kähler potential adapted to* P is a homogeneous degree two convex function $\varphi : M_\mathbb{R} \to \mathbb{R}$ that is convex with respect to P.

Remark 4.3.9. A homogeneous convex function φ on $M_{\mathbb{R}}$ is convex with respect to P, if the increasing sequence of level sets $\{\varphi(\rho) < c\}$ meet the faces of P in the interior first.

Proposition 4.3.10. *For any convex polytope P in $M_{\mathbb{R}}$ containing 0 as an interior point, there exists a non-empty contractible set of Kähler potential adapted to P.*

Proof. First, we show the existence of such potential φ. We will build the level set $S = \{\varphi = 1\}$, and show that as we rescale S to λS, for λ from 0 to ∞, S will meet the interior of each face F first. We will proceed by first build a polyhedral approximation of S, then smooth it.

For each face F of P, we pick a point x_F in the interior of F if $\dim F > 0$, or $x_F = F$ if F is a point. Let T be the simplicial triangulation of P with vertices of F, then T is also a barycentric subdivision of P. Let $\phi_T : P \to \mathbb{R}$ a piecewise linear convex function on P, with maximal convex domain the top-dimensional simplices of T, and such that for any $0 \leq d \leq n-1$, and any face x_F of dimension d, $\phi_T(x_F) = c_d$ are the same for all such F. Such ϕ_T can be constructed inductively from x_F with $\dim F$ from 0 to $n-1$. Let ϕ_T be extended to $M_{\mathbb{R}}$ by linearity. Thus ϕ_T has a unique minium point in each face F.

Let $\eta \in C_c^\infty(\mathbb{R}^n)$ be a bump function with $\int \eta = 1$, and let $\eta_\epsilon(x) = \eta(x/\epsilon)/\epsilon^n$. Let $\phi_{T,\epsilon} = \eta_\epsilon \star \phi_T$, then $\phi_{T,\epsilon}$ is a linear combination of convex function hence still convex. Since $\phi_{T,\epsilon} \to \phi_T$ as $\epsilon \to 0$. For ϵ small enough, $\phi_{T,\epsilon}$ still a unique minimum point in each face F. And $S_{T,\epsilon} = \{\phi_{T,\epsilon} = 1\}$ is a convex smooth boundary, such that $S_{T,\epsilon} \to S_T = \{\phi_T = 1\}$ as $\epsilon \to 0$. Then, for small enough ϵ, we can use $S_{T,\epsilon}$ as the contour of the homogeneous degree two convex function $\{\varphi(x) = 1\}$.

(2) Let $\mathcal{K}$ be the set of homogeneous degree two potential adapted to P. Then there is surjective continuous map $\pi : \mathcal{K} \to \prod_{F, \dim F > 0} \mathrm{Int}(F)$, by sending φ to its critical points on each face. Since if two convex functions φ_1, φ_2 have the same critical points, then their convex linear combination $t\varphi_1 + (1-t)\varphi_2$ for $t \in [0,1]$ are still homogeneous degree two and with the same critical points, we see the fiber of map π is convex hence contractible. Since the base of the fibration Cr is contractible as well, we have $\mathcal{K}$ contractible.

□

Let P be a convex polytope in $M_{\mathbb{R}}$ containing 0 as an interior point. Recall the definition of the dual polytope $P^\vee \subset N_{\mathbb{R}}$

$$P^\vee = \{p \in N_{\mathbb{R}} \mid \langle p, x\rangle \leq 1 \; \forall x \in P\}. \tag{4.3.2}$$

For any face $F \subset P$, there is dual face $F^\vee \subset P^\vee$, and $\dim_{\mathbb{R}} F + \dim_{\mathbb{R}} F^\vee = n-1$. We define three subsets of $M_{\mathbb{R}} \times N_{\mathbb{R}}$

$$L_P = \bigcup_F \mathrm{cone}(F) \times F^\vee, \quad L_{P^\vee} = \bigcup_F F \times \mathrm{cone}(F^\vee), \quad \Lambda_P = \bigcup_F F \times F^\vee, \tag{4.3.3}$$

where F runs over the faces of P, and $\mathrm{cone}(F) = \mathbb{R}_{>0} \cdot F$.

Remark 4.3.11. L_P and $L_{P^\vee}$ are piecewise Lagrangians, and $\Lambda_P = L_P \cap L_{P^\vee}$ is piecewise isotropic. L_P is the exterior conormal of $P^\vee$ in $T^*N_{\mathbb{R}}$, and $L_P^\vee$ is the exterior conormal of P in $T^*M_{\mathbb{R}}$. If we let $\varphi_{P,1}$ be the piecewise linear function on $M_{\mathbb{R}}$, such that $P = \{x : \varphi_{P,1}(x) \leq 1\}$, then L_P morally is $\Gamma_{d\varphi_{P,1}}$.

Lemma 4.3.12. *Let φ be a homogeneous degree two convex function on $M_{\mathbb{R}}$. $P, P^\vee$ be dual convex polytopes in $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ as above. Let F be a face of P. Then there is a* *natural bijection*

$$\operatorname{cone}(F) \times F^\vee \cap \Gamma_{d\varphi} \leftrightarrow F \times \operatorname{cone}(F^\vee) \cap \Gamma_{d\varphi}. \tag{4.3.4}$$

Proof. If $(\lambda x, p) \in \operatorname{cone}(F) \times F^\vee \cap \Gamma_{d\varphi}$, where $\lambda > 0$ and $x \in F, p \in F^\vee$, then by conic invariance of $\Gamma_{d\varphi}$, we have

$$(x, p/\lambda) = \frac{1}{\lambda}(\lambda x, p) \in F \times \operatorname{cone}(F^\vee) \cap \Gamma_{d\varphi}. \tag{4.3.5}$$

Sending $(\lambda x, p)$ to $(x, p/\lambda)$ is the desired bijection. □

Next, we give some equivalent characterization for convexity with respect to a polytope.

Proposition 4.3.13. *Let P be a convex polytope in $M_\mathbb{R}$ containing 0 as an interior point. Let φ be a homogeneous degree two convex function on $M_\mathbb{R}$. The following conditions are equivalent:*

(1) φ is adapted to P.

(2) For each face F of P, the smooth component $\operatorname{Int}(F \times \operatorname{cone}(F^\vee))$ of $L_P^\vee$ has a unique intersection with $\Gamma_{d\varphi}$.

(3) For each face F of P, the smooth component $\operatorname{Int}(\operatorname{cone}(F) \times F^\vee)$ of L_P has a unique intersection with $\Gamma_{d\varphi}$.

Proof. (2) is equivalent to (3) by Lemma 4.3.12.

(2) $\Rightarrow$ (1): since $\varphi|_F$ is still convex, hence as at most one minimum point in the interior, and any interior critical point is a minimum. Since

$$\emptyset \neq F \times \mathrm{cone}(F^\vee) \cap \Gamma_{d\varphi} \subset T^*_F M_{\mathbb{R}} \cap \Gamma_{d\varphi} \tag{4.3.6}$$

we see $\varphi|_F$ has a critical point.

(1) $\Rightarrow$ (2): for each face F of P, let x_F be the critical point of $\varphi|_F$, and let $H_F \subset M_{\mathbb{R}}$ be the affine hyperplane tangent to the contour of φ at x_F. We claim that H_F is a supporting hyperplane for P, and $P \cap H_F = F$. Then $p = d\varphi|_{x_F} \in T^*_{x_F} M_{\mathbb{R}} \cong N_{\mathbb{R}}$ is in the exterior conormal of H_F (exterior with respect to P), hence $p \in \mathrm{cone}(F^\vee)$. Thus, $(x, p) \in F \times \mathrm{cone}(F^\vee)$. $\square$

A consequence of the proposition is the compatibility of the 'adaptedness' with Legendre transformation.

Corollary 4.3.14. *Let P be a convex polytope in $M_{\mathbb{R}}$ containing 0 as an interior point and $P^\vee$ the dual polytope. Let φ be homogeneous degree two convex function, and ψ the Legendrian dual of φ. Then φ is adapted to P if and only if ψ is adapted to $P^\vee$.*

4.4. Deformation of Tropical Hypersurface and Amoeba

In this section, we modify the tropical polynomial of Mikhalkin and Abouzaid, to turn off the terms in f when they become sufficiently small compared with 1.

Recall that $Q, \mathcal{T}, \Theta, h, P, f$ from the introduction, that is Q is the Newton polytope containing 0 in its interior, $(\mathcal{T}, h)$ is a coherent star triangulation of Q with vertices on

∂Q denoted ∂A, and P is defined by

$$P = \{\rho \in M_{\mathbb{R}} : \langle \rho, \alpha \rangle - h(\alpha) \leq 0, \forall \alpha \in A\}.$$

For simplicity of presentation, we set $\Theta : A \to \mathbb{R}$ to be identically zero. The polynomial is

$$f_\beta(z) = \sum_{\alpha \in \partial A} e^{-\beta h(\alpha)} z^\alpha = \sum_{\alpha \in \partial A} e^{i\langle \theta, \alpha \rangle} e^{\langle \rho, \alpha \rangle - \beta h(\alpha)}$$

for large real number β.

To define the tropicalize hypersurface, we fix a smooth convex function e_0^x such that

$$e_0^x = \begin{cases} e^x & x \in [0, +\infty) \\ 0 & x \in (-\infty, -2] \end{cases}.$$

and $e_0^x = e^{-\frac{1}{x+2}}$ in a small neighborhood $(-2, -2+\epsilon)$ for some ϵ. And for any positive number b, we define

$$e_b^x := e^{-b} e_0^{x+b}, \quad E_b^x := \frac{d}{dx} e_b^x \tag{4.4.1}$$

We also define the linear interpotation between e_b^x and e^x as

$$e_{b,s}^x := (1-s)e^x + s e_b^x, \quad e_b^x = e_{b,1}^x$$

Proposition 4.4.1. *e_b^x is a smooth convex function of x, and*

$$e_b^x = \begin{cases} e^x & x \in [-b, +\infty) \\ 0 & x \in (-\infty, -2-b] \end{cases}.$$

Moreover, for each positive integer k, there exists a constant C_k independent of b, such that

$$\|e_b^x - e^x\|_{C^k(\mathbb{R})} \leq C_k e^{-b}.$$

Proof. The smooth function $e_0^x - e^x$ vanishes for $x > 0$ and has exponential decay for $x \ll 0$, hence has finite C^k norm for all k. Let $C_k = \|e_0^x - e^x\|_{C^k}$. Then

$$\|e_b^x - e^x\|_{C^k} = e^{-b}\|e_0^{x+b} - e^{x+b}\|_{C^k} = e^{-b}\|e_0^x - e^x\|_{C^k} = C_k e^{-b}.$$

□

Our family of tropical localized hypersurface is defined by, in log coordinates

$$f_{\beta,s}(\rho,\theta) = \sum_{\alpha \in \partial A} e^{i\langle\theta,\alpha\rangle} e_{\sqrt{\beta},s}^{\langle\rho,\alpha\rangle - \beta h(\alpha)}, \quad \mathcal{H}_{\beta,s} = f_{\beta,s}^{-1}(1).$$

In short, we turn off a summand $z^\alpha e^{-\beta h(\alpha)}$, if its modulus is less than $e^{-\sqrt{\beta}}$.

Remark 4.4.2. Our approach differs from that of Abouzaid's in that we only modify the defining equation $\{f_\beta(z) = 1\}$ when the term 1 dominate.

Proposition 4.4.3. *There exists β_0, such that for all $\beta > \beta_0$ and $s \in [0,1]$, $\mathcal{H}_{\beta,s}$ is a symplectic hypersurface.*

Proof. We compute the norm of ∂f and $\bar{\partial} f$. First, we note that $g_{ij}(\rho) = \partial_{\rho_i}\partial_{\rho_j}\varphi_P(\rho)$ is homogeneous degree 0, hence $|d\rho_i|$ and $|d\theta_i|$ are uniformly bounded.

Let $E^x_{b,s} = \frac{d}{dx} e^x_{b,s}$, then

$$\begin{aligned} 2\partial f_{\sqrt{\beta},s} &= \sum_i (\partial_{\rho_i} - i\partial_{\theta_i}) f_{\beta,s}(\rho,\theta) d(\rho_i + i\theta_i) \\ &= \sum_i \sum_{\alpha\in\partial A} (e^{\langle\rho,\alpha\rangle-\beta h(\alpha)}_{\sqrt{\beta},s} + E^{\langle\rho,\alpha\rangle-\beta h(\alpha)}_{\sqrt{\beta},s}) e^{i\langle\theta,\alpha\rangle} \alpha_i d(\rho_i + i\theta_i) \\ &= \sum_i \sum_{\alpha\in\partial A} (2e^{\langle\rho,\alpha\rangle-\beta h(\alpha)} + O(e^{-\sqrt{\beta}})) e^{i\langle\theta,\alpha\rangle} \alpha_i d(\rho_i + i\theta_i) \end{aligned}$$

and similarly

$$\begin{aligned} 2\bar{\partial} f_{\sqrt{\beta},s} &= \sum_i (\partial_{\rho_i} + i\partial_{\theta_i}) f_{\beta,s}(\rho,\theta) d(\rho_i - i\theta_i) \\ &= \sum_i \sum_{\alpha\in\partial A} (-e^{\langle\rho,\alpha\rangle-\beta h(\alpha)}_{\sqrt{\beta},s} + E^{\langle\rho,\alpha\rangle-\beta h(\alpha)}_{\sqrt{\beta},s}) e^{i\langle\theta,\alpha\rangle} \alpha_i d(\rho_i - i\theta_i) \\ &= \sum_i \sum_{\alpha\in\partial A} (O(e^{-\sqrt{\beta}})) e^{i\langle\theta,\alpha\rangle} \alpha_i d(\rho_i - i\theta_i) = O(e^{-\sqrt{\beta}}). \end{aligned}$$

where we use the uniform bound that

$$\|e^x_{b,1} - e^x\|_{C^1} = \|e^x_{b,1} - e^x\|_{C^0} + \|E^x_{b,1} - e^x\|_{C^0} < C_1 e^{-b}$$

to replace both $e^{\langle\rho,\alpha\rangle-\beta h(\alpha)}_{\sqrt{\beta},1}$ and $E^{\langle\rho,\alpha\rangle-\beta h(\alpha)}_{\sqrt{\beta},1}$ by $e^{\langle\rho,\alpha\rangle}$ at a cost of $O(e^{-\sqrt{\beta}})$ error. □

Definition 4.4.4. We define the compact component of the complement of the amoeba as

$$P_{\beta,s} = \{\rho \in M_{\mathbb{R}} : \hat{F}_{\beta,s} \leq 1\}$$

where the defining function $\hat{F}_{\beta,s}$ is

$$\hat{F}_{\beta,s}(\rho) = \sum_{\alpha\in\partial A} e^{\beta(\langle\rho,\alpha\rangle - h(\alpha))}_{\sqrt{\beta},s}.$$

We also define $F_{\beta,s}(\rho) = \hat{F}_{\beta,s}(\rho/\beta)$.

Lemma 4.4.5. *For any β, s, $\partial P_{\beta,s}$ is a convex hypersurface, i.e. smooth boundary of a convex set.*

Proof. This follows directly by the fact that $e^{x}_{b,s}$ is a convex function in x for all b, s, hence $\hat{F}_{\beta,s}(\rho)$ is a convex function in ρ. $\square$

Fix any linear inner product on $M_{\mathbb{R}}$ and call the induce metric g_0. Since g_0 and g_φ induced by φ_P are comparable in the sense that there exists $c > 0$, such that $c^{-1}g_0 < g_\varphi < cg_0$, we will use g_0 in the definition of distance function $\mathrm{dist}_{M_{\mathbb{R}}}$ and $\mathrm{dist}_{S^*M_{\mathbb{R}}}$.

Lemma 4.4.6. *The Gromov-Hausdorff (GH) distance between $\partial P_{\beta,s}$ and ∂P is bounded from above by $O(1/\sqrt{\beta})$. Even better, the Legendrian lift of $P_{\beta,s}$ and ∂P as co-oriented hypersurface into $S^*M_{\mathbb{R}}$ have GH distance bounded from above by $O(1/\sqrt{\beta})$.*

Proof. For $\rho \in \partial P_{\beta,s}$, let $\alpha_1, \cdots, \alpha_k$ be the set of elements in ∂A, such that $e^{\beta(\langle\rho,\alpha\rangle - h(\alpha))}_{\sqrt{\beta},s} \neq 0$. Since P is a vertex simplicial polytope, there are at most n non-zero terms, hence $k \leq n$. We will find an approximation of ρ denoted by $\hat{\rho}$, such that $\widehat{\rho}$ is contained in a face τ defined by $\langle\hat{\rho}, \alpha_i\rangle - h(\alpha_i) = 0$. This is always possible, e.g. let $\widehat{\rho}$ be orthogonal projection of ρ to τ. Since we know

$$-1/\sqrt{\beta} < \langle\rho, \alpha_i\rangle - h(\alpha) < 0, \quad \forall i = 1, \cdots, k$$

Thus the distance between ρ and $\widehat{\rho}$, in any fixed metric, can be bounded by $O(1/\sqrt{\beta})$.

To prove the second statement, we note any the exterior unit conormal p of the hypersurface at ρ is spanned by positive linear combination of $\alpha_1, \cdots, \alpha_k$, hence the co-vector p is also orthogonal to the face τ, thus $(\widehat{\rho}, p)$ is contained in the Legendrian lift of ∂P. Hence $\mathrm{dist}_{S^*_{\mathbb{R}}M}((\rho, p); (\widehat{\rho}, p)) = \mathrm{dist}_{M_{\mathbb{R}}}(\rho, \widehat{\rho}) = O(1/\sqrt{\beta})$. $\square$

4.5. Gradient flow on Tropical Amoeba

In this section we prepare for the Liouville flow on the tropical hypersurface, by considering the gradient flow of φ_P on the boundary $\partial P_{\beta,1}$ (see Definition 4.4.4). Our goal is to describe unstable manifolds for the downward gradient flow $-\nabla\varphi_P$.

First, we show that the critical points are approximately the stratified Morse critical points of φ_P on ∂P, i.e., for each face τ of P, the minimum ρ_τ of φ_P restricted on τ.

We fix an identification of $V \cong \mathbb{R}^n$ and take Euclidean metric on V and the induced metric on T^*V and S^*V. We identify the sphere compactification boundary $T^\infty V = (T^*V - V)/\mathbb{R}_{>0}$ with the unit cosphere bundle S^*V. If $U \subset V$ open set with smooth boundary, then $S^*_U V$ is the one-sided unit conormal bundle of ∂U with covectors pointing outward. The generalization to open convex set U with piecewise smooth boundary is also straightforward.

Proposition 4.5.1. *Let $V \cong \mathbb{R}^n$ be a real vector space of dimension n, $P \subset V$ a convex polytope containing the origin, $\varphi : V \to \mathbb{R}$ a potential adapted to P. Let $\{P_j\}$ be a sequence of convex bounded domains with smooth boundaries, such that the exterior conormals $L_j := S^*_{P_j} V$ converges to $L := S^*_P V$ in the cosphere bundle S^*V in the Gromov-Hausdorff sense. Then there exists a number $j_0 > 0$ large enough, such that for each face*

τ of P and $j > j_0$, there is a critical point $\rho_{\tau,j} \in \partial P_j$ of Morse index $n-1-\dim\tau$, and $\rho_{\tau,j} \to \rho_\tau$.

Proof. (1) We express the critical point condition in terms of Legendrian intersection. Define the projection image of $\Gamma_{d\varphi}$ in $T^\infty V$ as

$$\Gamma^\infty_{d\varphi} = (\mathbb{R}_{>0} \cdot \Gamma_{d\varphi})/\mathbb{R}_{>0} \subset T^\infty V.$$

Then $\Gamma^\infty_{d\varphi}$ is also the union of unit conormal for level sets of φ:

$$\Gamma^\infty_{d\varphi} = \bigcup_{c\in\mathbb{R}} S^*_{\{\varphi(\rho)\leq c\}}V.$$

The Legendrian $L = S^*_P V$ is a piecewise smooth C^1 manifold, where the smooth components L_τ are labelled by faces τ of P. If ρ_τ is a critical point of φ on τ, then there is a unique unit covector $p_\tau \in L_\tau$, such that $x_\tau = (\rho_\tau, p_\tau) \in L \pitchfork \Gamma^\infty_{d\varphi}$, and the intersection is transversal.

(2) Consider the unit speed geodesic flow Φ^t_R on the unit cosphere bundle S^*V. Fix any small flow time $1 \gg \epsilon > 0$, since $\Phi^\epsilon_R : S^*V \to S^*V$ is a diffeomorphism, $\Phi^\epsilon_R(L_j)$ still converges to $\Phi^\epsilon_R(L)$ in GH sense. For any subset $A \subset V$, define

$$A^\epsilon := \{x : \mathrm{dist}(x, A) < \epsilon\}$$

to be the ϵ-fattening of A. If A is a convex set, we have $\Phi^t_R(S^*_A V) = S^*_{A^\epsilon}V$. Hence ∂P^ϵ is a C^1 hypersurface, and $\partial P^\epsilon_j \to \partial P^\epsilon$ in GH sense as $j \to \infty$. Define

$$L^t = \Phi^t_R(L), \quad L^t_j = \Phi^t_R(L_j).$$

The geodesic flow applied to $\Gamma^\infty_{d\varphi}$ can be understood as follow

$$\Phi^\epsilon_R(\Gamma^\infty_{d\varphi}) = \bigcup_{c\in\mathbb{R}} \Phi^\epsilon_R(S^*_{\{\varphi(\rho)\le c\}}V) = \bigcup_{c\in\mathbb{R}} S^*_{\{\varphi(\rho)\le c\}^\epsilon}V.$$

Define function $\widetilde{\varphi}^\epsilon$, such that $\{\widetilde{\varphi}^\epsilon(\rho) < c\} = \{\varphi(\rho) \le c\}^\epsilon$, then $\widetilde{\varphi}^\epsilon$ is a levelset convex function. By Lemma 2.7 of [CE], there exists a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$, such that $\varphi^\epsilon = f \circ \widetilde{\varphi}^\epsilon$ is a convex function. Thus, we have

$$\Phi^\epsilon_R(\Gamma^\infty_{d\varphi}) = \Gamma^\infty_{d\varphi^\epsilon} \quad \varphi^\epsilon \text{ is convex .}$$

Let $x^\epsilon_\tau = \Phi^\epsilon_R(x_\tau)$, $\rho^\epsilon_\tau = \pi(x^\epsilon_\tau)$ in the expanded face $\tau^\epsilon = \pi(\Phi^\epsilon_R(L_\tau))$. Then x^ϵ_τ is still the intersection of $\Gamma^\infty_{d\varphi^\epsilon}$ and $S^*_{P^\epsilon}V$, and ρ^ϵ_τ is the unique Morse critical points of φ^ϵ restricted on τ^ϵ, and ρ^ϵ_τ is in the interior of τ^ϵ. One may easily check that the Morse index of ρ^ϵ_τ is $n - 1 - \dim\tau$.

(3) We now prove that for large enough j, for each τ, there is a unique critical points $\rho^\epsilon_{\tau,j}$ of φ^ϵ on ∂P^ϵ_j approaching ρ^ϵ_τ.

Fix a small neighborhood $W_\tau \subset \partial P^\epsilon$ near ρ^ϵ_τ, and for small enough δ, let $\widetilde{W}_\tau \cong W_\tau \times (-\delta, \delta)$ be the flow-out of W_τ under the Reeb flow for time in $(-\delta, \delta)$, with projection map $\pi_W : \widetilde{W}_\tau \to W_\tau$. We claim that for large enough j, $\partial P^\epsilon_j \cap \widetilde{W}_\tau$ projects bijectively to W_τ, since otherwise this contradicts with P^ϵ_j being convex and the fiber of π_W being straight-line segments Reeb trajectories. Thus, we have a sequence of smooth sections $\iota_j : W_\tau \to \widetilde{W}_\tau$ for large enough j, such that ι_j converges to the zero section in C^1.

Let $f_j = \iota_j^*\varphi_\epsilon|_{\widetilde{W}_\tau} \in C^\infty(W_\tau, \mathbb{R})$, and a smooth function $f_\infty = \iota_\infty^*\varphi_\epsilon|_{\widetilde{W}_\tau}$, where $\iota_\infty : W_\tau \hookrightarrow \widetilde{W}_\tau$ is the identity map of zero section. Since $\iota_j \to \iota_\infty$ in C^1, $f_j \to f_\infty$ in C^1.

Since f_∞ has a non-degenerate critical point, by stability of critical points under C^1-perturbation, f_j has a unique critical point of the same index as f_∞.

(4) Finally, we show that there are no other critical points. Let U_τ be the preimage of $\widetilde{W}_\tau$ under $S^*V \to V$. Let U be the union of all such $\widetilde{U}_\tau$. If $\delta > 0$ is small enough, such that

$$\operatorname{dist}(\Gamma^\infty_{d\varphi^\epsilon} \backslash U, L^\epsilon) > 3\delta.$$

Then by GH convergence from L^ϵ_j to L^ϵ, we make take j_0 large enough, such that for all $j > j_0$ and all $x \in L^\epsilon_j$, $\operatorname{dist}(x, L^\epsilon) < \delta$. This shows

$$\operatorname{dist}(\Gamma^\infty_{d\varphi^\epsilon} \backslash U, L^\epsilon_j) \geq \operatorname{dist}(\Gamma^\infty_{d\varphi^\epsilon} \backslash U, L^\epsilon) - \operatorname{dist}(L^\epsilon_j, L^\epsilon) > 2\delta,$$

hence there is no intersection between L^ϵ_j and $\Gamma^\infty_{d\varphi^\epsilon}$ away from U.

(5) Since Φ^ϵ_R is a diffeomorphism, the result about $L^\epsilon_j \cap \Gamma^\infty_{d\varphi^\epsilon}$ implies the same result about $L_j \cap \Gamma^\infty_{d\varphi}$, and we finish the proof of the proposition. □

Proposition 4.5.2. *There exists β_0 large enough, such that for all $\beta > \beta_0$, the critical points of φ_P on $\partial P_{\beta,1}$ are given by $\{\rho_{\tau,\beta}\}_\tau$ where τ runs over the faces of P, such that $\lim_{\beta\to\infty} \rho_{\tau,\beta} \to \rho_\tau$.*

Proof. This follows from the convergences result from Lemma 4.4.6 and Proposition 4.5.1. □

Next, we prove that the unstable manifold $W^-_{\beta,\tau}$ for critical point $\rho_{\beta,\tau}$ are cells of a dual polyhedral decomposition of ∂P. This is true not only in the combinatorial sense, but in a more refined geometrical sense.

Proposition 4.5.3. *Let β_0 be large enough as in Proposition 4.5.2. Then for all $\beta > \beta_0$, the unstable manifold $W^-_{\beta,\tau}$ is a smooth manifold of dimension $n - 1 - \dim \tau$, and contains critical point $\rho_{\beta,\tau'}$ in the boundary if and only if τ is in the boundary of τ'.*

Proof. The statement dimension follows from the Morse index result. For any critical point $\rho_{\beta,\tau}$, take a small enough ball B of radius ϵ around it, then B can be stratified by the destination of the downward gradient flow. For each facet F adjacent to τ, there is an open ball U_F in ∂B whose points flows to critical point $\rho_{\beta,F}$. If a face τ' adjacent to τ can be written as $\tau' = F_1 \cap \cdots \cap F_k$ for facets F_i, then points in the relative interior of $\cap_{i=1}^k \overline{U_{F_i}}$ will flow to $\rho_{\beta,\tau'}$. Since $\cap_{i=1}^k \overline{U_{F_i}}$ contains $\rho_{\beta,\tau}$, hence there exists flowline from $\rho_{\beta,\tau}$ to $\rho_{\beta,\tau'}$. $\square$

Now we give a more refined description of the unstable manifold. Recall from Definition 4.3.6, that we have identified rays in $M_\mathbb{R}$ with rays in $N_\mathbb{R}$ by $\Phi_\varphi^\infty : M_\mathbb{R}^\infty \xrightarrow{\sim} N_\mathbb{R}^\infty$. We may also make the following identification

$$\partial P \cong \partial P_\beta \cong M_\mathbb{R}^\infty, \quad \partial P^\vee \cong \partial Q \cong N_\mathbb{R}^\infty.$$

and define

$$\Phi_{\varphi,\beta}^{PP} : \partial P_\beta \xrightarrow{\sim} \partial P^\vee, \quad \Phi_{\varphi,\beta}^{PQ} : \partial P_\beta \xrightarrow{\sim} \partial Q$$

If τ is a k-face of P, let $\tau^\vee$ denote the dual $(n-1-k)$-face in $P^\vee$, and $\tau_Q^\vee \subset \partial Q$ is cell of the star triangulation $\mathcal{T}$. If $\tau_Q^\vee$ has vertices $\{\alpha_1, \cdots, \alpha_k\}$, we define a submanifold with boundary in ∂P_β:

$$U_\tau = \{\rho \in \partial P_\beta : F_{\beta,\alpha} > 0 \text{ if and only if } \alpha = \alpha_1, \cdots, \alpha_k\}.$$

where $F_{\beta,\alpha}(\rho) = e^{\frac{\beta(\langle\rho,\alpha\rangle - h(\alpha))}{\sqrt{\beta}}}$ are summands of function $F_\beta = F_{\beta,1}$. The following lemma is easy to check.

Lemma 4.5.4. *If $\rho \in U_\tau$, then $dF(\rho) \in \operatorname{Int}\operatorname{cone}(\tau^\vee)$.*

Proposition 4.5.5. *Let β_0 be large enough as in Proposition 4.5.2. Then for all $\beta > \beta_0$, and τ any face of P, the unstable manifold for $\rho_{\beta,\tau}$ goes to $\tau^\vee$ under the mapping of $\Phi^{PP}_{\varphi,\beta}$.*

$$\Phi^{PP}_{\varphi,\beta}(W^-_{\beta,\tau}) = \operatorname{Int}\tau^\vee.$$

Proof. Let $\rho_{\beta,\tau}$ be the critical point corresponding to τ, then $\rho_{\beta,\tau} \in U_\tau$, and the unstable manifold $W^-_{\beta,\tau}$ is contained in the union of $U_{\tau'}$ for those τ' such that $\overline{\tau'} \supset \tau$.

(1) We first prove that $\Phi^{PP}_{\varphi,\beta}(\rho_{\beta,\tau})$ is contained in the interior of the cell $\tau^\vee$, or equivalently

$$\Phi_\varphi(\rho_{\beta,\tau}) \in \operatorname{Int}\operatorname{cone}(\tau^\vee).$$

Since $d\varphi$ is positively proportional to dF_β at the critical point $\rho_{\beta,\tau} \in U_\tau$, hence by the previous Lemma, we have

$$\Phi_\varphi(\rho_{\beta,\tau}) = d\varphi(\rho_{\beta,\tau}) = C \cdot dF_\beta(\rho_{\beta,\tau}) \in \operatorname{Int}\operatorname{cone}(\tau^\vee).$$

(2) Next, we claim that for any point ρ on the unstable manifold $W^-_{\beta,\tau}$,

$$(\Phi_\varphi)_*(\iota_*(-\nabla(\varphi|_{\partial P_\beta}))) \subset \overline{\operatorname{cone}(\tau^\vee)} + \mathbb{R}\cdot p\partial_p.$$

where $\iota : \partial P_\beta \hookrightarrow M_{\mathbb{R}}$ is the embedding of ∂P_β. For $\rho \in W^-_{\beta,\tau}$, $\rho \subset U_{\tau'}$ for some $\tau' \supset \tau$, thus

$$dF_\beta(\rho) \subset \overline{\mathrm{cone}(\tau^\vee)}.$$

On the other hand

$$-\iota_*(\nabla(\varphi|_{\partial P_\beta})) = -\nabla\varphi + c_1 \nabla F_\beta = -c_2 \rho \partial_\rho + c_1 \nabla F_\beta$$

for some $c_1, c_2 > 0$. Hence

$$(\Phi_\varphi)_*(\iota_*(-\nabla(\varphi|_{\partial P_\beta}))) \subset \mathbb{R} \cdot p\partial_p + \mathbb{R}_{>0} \cdot dF_\beta(\rho) \subset \overline{\mathrm{cone}(\tau^\vee)} + \mathbb{R} \cdot p\partial_p.$$

(3) If a curve $\gamma : \mathbb{R} \to N_{\mathbb{R}}$, satisfies that $\lim_{t\to-\infty} \gamma(t) \in \mathrm{Int}\,\mathrm{cone}(\tau^\vee)$, and $\dot\gamma(t) \in wb\mathrm{cone}(\tau^\vee) + \mathbb{R} \cdot p\partial_p$, then the image of the curve is contained in $\mathrm{Int}\,\mathrm{cone}(\tau^\vee)$. Hence, we have shown $\Phi_\varphi(W^-_{\beta,\tau}) \subset \mathrm{Int}\,\mathrm{cone}(\tau^\vee)$, or equivalently,

$$\Phi^{PP}_{\varphi,\beta}(W^-_{\beta,\tau}) \subset \mathrm{Int}\,\tau^\vee.$$

Since the boundary of the closure of unstable manifolds $W^-_{\beta,\tau}$ are union of other unstable manifolds

$$\partial(W^-_{\beta,\tau}) = \bigcup_{\tau' \sup \tau} W^-_{\beta,\tau'}.$$

By induction on the dimension of $W^-_{\beta,\tau}$ from 0 to $n-1$, we can prove that $\Phi^{PP}_{\varphi,\beta}(W^-_{\beta,\tau}) = \mathrm{Int}\,\tau^\vee$. This finishes the proof of the proposition. $\square$

4.6. Liouville Flow on Tropical Hypersurface

Recall our tropical polynomial is

$$f_{\beta,1}(\rho,\theta)=\sum_{\alpha\in\partial A} e^{i\langle\theta,\alpha\rangle-i\Theta(\alpha)}e_{\sqrt{\beta},1}^{\langle\rho,\alpha\rangle-\beta h(\alpha)},\quad (\rho,\theta)\in M_{\mathbb{R}}\times T_M\cong M_{\mathbb{C}^*}$$

and the tropical hypersurface is

$$\mathcal{H}_{\beta,1}=f_{\beta,1}^{-1}(1)\subset M_{\mathbb{C}^*},$$

where we modified the exponential function e^x to e_b^x, such that if $x<-b$, then we cut-off it while keeping e_b^x convex. The exact symplectic structure on $M_{\mathbb{C}^*}$ is given in subsection 4.3.2 $\lambda=-d^c\varphi_P,\omega=-dd^c\varphi_P$. The hypersurface $\mathcal{H}_{\beta,1}$ has induced Liouville one-form $\lambda_{\mathcal{H}}$, symplectic two-form $\omega_{\mathcal{H}}$ and the Liouville vector field $X_{\mathcal{H}}$.

Our skeleton candidate is defined as

$$\mathcal{S}_{\beta,h,\Theta}=\bigcup_{\tau}(\beta\cdot W_{\beta,\tau}^{-})\times T_{\tau,\theta}\subset M_{\mathbb{R}}\times T_M.$$

where $T_{\tau,\Theta}$ is the sub tori of T_M defined by

$$T_{\tau,\Theta}=\{\theta\in T_M:\langle\theta,\alpha\rangle=\Theta(\alpha),\quad \text{for all }\ \alpha \text{ as vertex of } \tau_Q^\vee\}$$

First, we state some basic properties of the Liouville vector field. Take any point $x=(\rho,\theta)\in\mathcal{H}_{\beta,1}$, we have

$$X_\lambda(x)=X_\lambda^{\parallel}(x)+X_\lambda^{\perp}(x).$$

where $X_\lambda^\perp(x)$ is symplectically orthogonal to $T\mathcal{H}_{\beta,1}$. We note that $X_\lambda^\|(x) = X_{\lambda_\mathcal{H}}(x)$, since for any $v \in T_x\mathcal{H}_{\beta,1}$,

$$\omega_\mathcal{H}(X_\lambda^\|(x), v) = \omega(X_\lambda(x) - X_\lambda^\perp(x), v) = \omega(X_\lambda(x) - X_\lambda^\perp(x), v) = \lambda(v) = \lambda_\mathcal{H}(v).$$

And $X_\lambda^\perp(x)$ is the horizontal lift of $(f_{\beta,1})_*(X_\lambda(x)) \in T_1\mathbb{C}$.

Definition 4.6.1. The *positive loci* of $\mathcal{H}_{\beta,1}$ is the subset $\mathcal{H}_{\beta,1}^\mathbb{R}$, where the summand of $f_{\beta,1}$ are either zero or positive.

Lemma 4.6.2. *If $x \in \mathcal{H}_{\beta,1}^\mathbb{R}$, then $(f_{\beta,1})_*(X_\lambda(x))$ is in the positive real direction.*

Proof. We check explicitly

$$\begin{aligned} df_{\beta,1} &= \sum_{\alpha\in\partial A} e_{\sqrt{\beta},1}^{\langle\rho,\alpha\rangle-\beta h(\alpha)} \cdot de^{i\langle\theta,\alpha\rangle - i\Theta(\alpha)} + e^{i\langle\theta,\alpha\rangle-i\Theta(\alpha)} \cdot de_{\sqrt{\beta},1}^{\langle\rho,\alpha\rangle-\beta h(\alpha)} \\ &= \sum_{\alpha\in\partial A} e_{\sqrt{\beta},1}^{\langle\rho,\alpha\rangle-\beta h(\alpha)} \cdot \sqrt{-1}d\langle\theta,\alpha\rangle + E_{\sqrt{\beta},1}^{\langle\rho,\alpha\rangle-\beta h(\alpha)} \cdot d\langle\rho,\alpha\rangle \end{aligned} \tag{4.6.1}$$

Then using $X_\lambda(x) = c(x)\rho\partial_\rho$ for some $c(x) > 0$, we have

$$\begin{aligned} \langle df_{\beta,1}, \rho\partial_\rho\rangle &= \sum_{\alpha\in\partial A} E_{\sqrt{\beta},1}^{\langle\rho,\alpha\rangle-\beta h(\alpha)} \cdot \langle\rho,\alpha\rangle \\ &= \beta\left(\sum_{\alpha\in\partial A} E_{\sqrt{\beta},1}^{\langle\rho,\alpha\rangle-\beta h(\alpha)} \cdot (\langle\frac{\rho}{\beta},\alpha\rangle - h(\alpha)) + \sum_{\alpha\in\partial A} E_{\sqrt{\beta},1}^{\langle\rho,\alpha\rangle-h(\alpha)} h(\alpha)\right) \\ &\geq \beta\left(\frac{-1-\sqrt{\beta}}{\beta}\sum_{\alpha\in\partial A} E_{\sqrt{\beta},1}^{\langle\rho,\alpha\rangle-\beta h(\alpha)} + (\inf_{\alpha\in\partial A} h(\alpha))\sum_{\alpha\in\partial A} E_{\sqrt{\beta},1}^{\langle\rho,\alpha\rangle-h(\alpha)}\right) \\ &\geq \beta(\inf_{\alpha\in\partial A} h(\alpha) + O(\beta^{-1/2}))(\sum_{\alpha\in\partial A} e_{\sqrt{\beta},1}^{\langle\rho,\alpha\rangle-\beta h(\alpha)} + O(e^{-\sqrt{\beta}})) \\ &= \beta(\inf_{\alpha\in\partial A} h(\alpha) + O(\beta^{-1/2})) \end{aligned}$$

Hence for large enough β,

$$\langle df_{\beta,1}, \rho\partial_\rho\rangle = \langle d\mathrm{Re} f_{\beta,1}, \rho\partial_\rho\rangle > C\beta \tag{4.6.2}$$

where constant C is independent of x. Hence $\langle df_{\beta,1}, X_\lambda\rangle$ is positive. □

Lemma 4.6.3. *Let X_{Imf} be the Hamiltonian vector field of Imf, then $X_{Imf} \perp T\mathcal{H}_{\beta,1}$. If $x \in \mathcal{H}^{\mathbb{R}}_{\beta,1}$, then $(f_{\beta,1})_*(X_{Imf})$ is in the positive real direction.*

Proof. Denote $f_{\beta,1}$ by f for short. Take any $v \in T\mathcal{H}_{\beta,1}$, we have

$$\omega(X_{\mathrm{Im}f}, v) = d(\mathrm{Im}f)(v) = \mathrm{Im}(df(v)) = 0.$$

And we have

$$\langle df, X_{\mathrm{Im}f}\rangle = \langle d\mathrm{Re}f, X_{\mathrm{Im}f}\rangle = g(\nabla\mathrm{Re}f, X_{\mathrm{Im}f}) \geq \frac{g(\nabla\mathrm{Re}f, \rho\partial_\rho)g(\rho\partial_\rho, X_{\mathrm{Im}f})}{g(\rho\partial_\rho, \rho\partial_\rho)}, \tag{4.6.3}$$

If $x = (\rho, \theta) \in \mathcal{H}^{\mathbb{R}}_{\beta,1}$, then we can compute $d\mathrm{Im}f$ using (4.6.1),

$$d\mathrm{Im}f = \sum_{\alpha\in\partial A} e^{\langle\rho,\alpha\rangle-\beta h(\alpha)}_{\sqrt{\beta},1} \cdot d\langle\theta,\alpha\rangle, \quad d\mathrm{Re}f = \sum_{\alpha\in\partial A} E^{\langle\rho,\alpha\rangle-\beta h(\alpha)}_{\sqrt{\beta},1} \cdot d\langle\rho,\alpha\rangle, \tag{4.6.4}$$

Hence

$$X_{\mathrm{Im}f} = \sum_{\alpha\in\partial A} e^{\langle\rho,\alpha\rangle-\beta h(\alpha)}_{\sqrt{\beta},1} \sum_{i,j} \alpha_i g^{ij}(\rho)\partial_{\rho_j} \tag{4.6.5}$$

$$\nabla\mathrm{Re}f = \sum_{\alpha\in\partial A} E^{\langle\rho,\alpha\rangle-\beta h(\alpha)}_{\sqrt{\beta},1} \sum_{i,j} \alpha_i g^{ij}(\rho)\partial_{\rho_j} \tag{4.6.6}$$

Since $\|E_{\sqrt{\beta},1}^{\langle\rho,\alpha\rangle-\beta h(\alpha)} - e^{\langle\rho,\alpha\rangle-\beta h(\alpha)}\|_{C^0} < C_1 e^{-\sqrt{\beta}}$, we have $\|X_{\mathrm{Im}f} - \nabla \mathrm{Re}f\| = O(e^{-\sqrt{\beta}})$. From (4.6.2), we have

$$g(\nabla \mathrm{Re}f, \beta^{-1}\rho\partial_\rho) > C, \quad g(X_{\mathrm{Im}f}, \beta^{-1}\rho\partial_\rho) > C + O(e^{-\sqrt{\beta}}\|\beta^{-1}\rho\partial_\rho\|).$$

Since for $x \in \mathcal{H}_{\beta,1}^{\mathbb{R}}$, the vector $\|\rho\partial_\rho/\beta\|$ is bounded above and below by a constant independent of β. Thus using (4.6.3) with estimates on $g(\nabla \mathrm{Re}f, \rho\partial_\rho)$, $g(\rho\partial_\rho, X_{\mathrm{Im}f})$ and $g(\rho\partial_\rho, \rho\partial_\rho)$, we finish the proof. □

Proposition 4.6.4. *The Liouville vector field $X_{\lambda_{\mathcal{H}}}$ on the positive loci $\mathcal{H}_{\beta,1}^{\mathbb{R}}$ does not change the θ coordinate, i.e. under the projection map $\pi_T : M_{\mathbb{R}} \times T_M \to T_M$, $(\pi_T)_*(X_{\lambda_{\mathcal{H}}}(x)) = 0$ for all $(\rho, \theta) \in \mathcal{H}_{\beta,1}^{\mathbb{R}}$. In particular, the positive loci $\mathcal{H}_{\beta,1}^{\mathbb{R}}$ is preserved under the Liouville flow (for positive and negative time).*

Proof. From the following relation

$$X_{\lambda_{\mathcal{H}}} = X_\lambda^{\|} = X_\lambda - X_\lambda^{\perp},$$

suffice to check that X_λ and $X_\lambda^{\perp}$ does not change θ coordinates. Since $X_\lambda \propto \rho\partial_\rho$ everywhere, hence it does not change θ. At a point $x \in \mathcal{H}_{\beta,1}^{\mathbb{R}}$, $X_\lambda^{\perp}$ is proportional to the horizontal lift of $\frac{\partial}{\partial x} \in T_1\mathbb{C}$ by the local symplectic fibration $f_{\beta,1}$, so is $X_{\mathrm{Im}f}$, hence $X_\lambda^{\perp} = c(x)X_{\mathrm{Im}f}$. From the expression of $X_{\mathrm{Im}f}$ in (4.6.5), we see $X_{\mathrm{Im}f}$ does not change θ as well. □

Proposition 4.6.5. *For any face τ of P, the submanifold $\mathrm{Crit}_\tau = \{\beta\rho_{\beta,\tau}\} \times T_{\tau,\Theta}$ is a critical manifold of the flow $X_{\lambda_{\mathcal{H}}}$.*

Proof. Here we use $f = f_{\beta,1}$, $\varphi = \varphi_P$, $\mathcal{H} = \mathcal{H}_{\beta,1}$ for short. It is easy to check that Crit_τ is in the positive loci, and is contained in a neighborhood where the defining function $f_{\beta,1}$ is holomorphic. Thus, for any $x \in \mathrm{Crit}_\tau$, to check $X_{\lambda_{\mathcal{H}}}$ vanishes at x, suffice to check

$$\begin{aligned} X_{\lambda_{\mathcal{H}}}(x) = 0 \;\; &\Leftrightarrow \;\; \lambda_{\mathcal{H}}(x) = 0 \\ &\Leftrightarrow \;\; \lambda(v) = 0, \quad \forall v \in T_x\mathcal{H} \\ &\Leftrightarrow \;\; d^c\varphi(v) = 0, \quad \forall v \in T_x\mathcal{H} \\ &\Leftrightarrow \;\; d\varphi(v) = 0, \quad \forall v \in J(T_x\mathcal{H}) = T_x\mathcal{H} \\ &\Leftrightarrow \;\; d\varphi \in \mathrm{span}_{\mathbb{R}}\{d\mathrm{Re}f, d\mathrm{Im}f\}. \end{aligned}$$

Since x is in the positive loci, using (4.6.4) and

$$d\mathrm{Re}f = \sum_{\alpha\in\partial A} E_{\sqrt{\beta},1}^{\langle\rho,\alpha\rangle-\beta h(\alpha)} \cdot d\langle\rho,\alpha\rangle = d\left(\sum_{\alpha\in\partial A} e_{\sqrt{\beta},1}^{\langle\rho,\alpha\rangle-\beta h(\alpha)}\right) = dF_{\beta,1}.$$

Since $d\varphi|_{\rho_{\beta,\tau}} = cd\hat{F}_{\beta,1}|_{\rho_{\beta,\tau}}$ for some constant c, and $d\varphi|_{\beta\rho_{\beta,\tau}} = \beta d\varphi|_{\rho_{\beta,\tau}}$, $dF_{\beta,1}|_{\beta\rho_{\beta,\tau}} = \beta^{-1}d\hat{F}_{\beta,1}|_{\rho_{\beta,\tau}}$, hence

$$d\varphi \in \mathrm{span}_{\mathbb{R}}\, dF_{\beta,1} \subset \mathrm{span}_{\mathbb{R}}\{d\mathrm{Re}f, d\mathrm{Im}f\}.$$

This proves that any $x \in \mathrm{Crit}_\tau$ is a critical point for $X_{\lambda_{\mathcal{H}}}$. $\square$

Proposition 4.6.6. *There are no other critical point for the flow $X_{\lambda_{\mathcal{H}}}$, besides $\{\mathrm{Crit}_\tau\}_\tau$.*

Proof. Here we only sketch the proof. First, we show that there are no other critical point outside the positive loci $\mathcal{H}^{\mathbb{R}} = \mathcal{H}^{\mathbb{R}}_{\beta,1}$ by explicitly construct a tangent vector v for each point $x \in \mathcal{H}\backslash\mathcal{H}^{\mathbb{R}}$, such that $d^c\varphi(v) \neq 0$. Let $\pi_\beta : M_{\mathbb{C}^*} \to M_{\mathbb{R}}$ be the rescaled projection $\pi_\beta = \beta^{-1}\operatorname{Log}$. Let $\mathcal{A}$ be the tropical amoeba, and $\mathcal{A}_\beta$ be the actual image of $\pi_\beta(\mathcal{H})$. P is the compact complement of $\mathcal{A}$. For each α, let $l_\alpha(\rho) = \langle \rho, \alpha \rangle - h(\alpha)$. For $\alpha = 0$, let $l_0 = 0$. Define the piecewise linear convex function

$$\varphi_{\max}(\rho) := \max\{l_\alpha(x) : \alpha \in A\}$$

And also define the smooth version

$$\varphi_\beta(\rho) := \beta^{-1}\log(\sum_{\alpha\in A} \exp l_\alpha(\rho)).$$

We then have $0 < \sup_\rho \varphi_\beta(\rho) - \varphi_{\max}(\rho) < Ce^{-c\beta}$ for some constants c_1, c_2. Let $\rho = \pi_\beta(x)$. Define $\delta = \beta^{-1/2}$. Let

Let $\chi_\alpha(\rho) = \chi\left(\frac{\varphi_\beta(\rho) - l_\alpha(\rho)}{\delta}\right)$, where $\chi(x)$ is a cut-off function on $\mathbb{R}$ that smoothly drops from 1 to 0 smoothly as x increases from 1 to 2. And we modify the defining equation to

$$0 = \sum_\alpha \chi_\alpha(\rho)\, e^{i(\langle \theta, \alpha\rangle - \Theta(\alpha))}\, e^{\beta(\langle \rho, \alpha \rangle - h(\alpha))}.$$

For any $\rho \in M_{\mathbb{R}}$ define,

$$A_\rho := \{\alpha \in A \mid l_\alpha(\rho) - \varphi_{\max}(\rho) > -\delta\}.$$

For $\rho \in \mathcal{A}_\beta$, A_ρ contains at least two elements. We split the discussion into the following two cases.

(1) If $1 \notin A_\rho$, then ρ is close to a non-compact cell Π_{A_ρ} in the tropical amoeba $\mathcal{A}$. Let $\partial_0 \Pi_{A_\rho}$ be the compact component of the boundary of Π_{A_ρ}, then $\partial_0\Pi_{A_\rho}$ is also a face τ of P. Let ρ_τ be the minimum of φ on τ, we claim that ρ_τ is the global minimum on

Π_{A_ρ}, since the increasing level set of φ meets Π_{A_ρ} first at ρ_τ. Let ρ_Π be the point on Π_{A_ρ} closest to ρ, and consider the line segment γ from ρ_Π to ρ_τ, we then claim that φ restricted to γ is a strictly decreasing function from ρ_Π to ρ_τ. Let v_Π be the unit vector in the direction of $\dot{\gamma}$. Next, we view v_Π as a tangent vector at $T_x M_{\mathbb{C}^*}$, we claim that v_Π and Jv_Π are already in $T_x\mathcal{H}$. Hence $\langle d\varphi(\rho), J(J^{-1}v_\Pi)\rangle = \langle d\varphi(\rho_\Pi), J(J^{-1}v_\Pi)\rangle + O(1/\sqrt{\beta}) > 0$.

(2) If $1 \in A_\rho$, let $A_\rho = \{1, a_1, \cdots, a_k\}$, for some $k \leq n$. If $k < n$, we may look for vectors $v \in \ker(\alpha_1, \cdots, \alpha_k)$, such that $\langle d\varphi(x), v\rangle \neq 0$. If such vector exists, then $v, Jv \in T_x\mathcal{H}$, hence we have shown $d^c\varphi(x) \neq 0$. If such vector does not exist, then $d\varphi(x) \in \mathrm{span}_{\mathbb{R}}(\alpha_1, \cdots, \alpha_k)$, in fact $d\varphi(x) \in \mathrm{cone}(\alpha_1, \cdots, \alpha_k)$. Let $\theta_\alpha := \langle\theta, \alpha\rangle - \Theta(\alpha)$, if θ_{α_i} are all equal, then they have to be zero (modulo 2π), since the sum of the k terms equals to 1. If $\{\theta_\alpha\}$ are not all the same, say $\theta_{\alpha_1} \neq \theta_{\alpha_2}$. Here we assume that $\chi_{\alpha_i}(\rho)$ all equal to 1 in a neighborhood of ρ, the general case can be analyzed but is more complicted. Then we may reduce the modulus of the two complex numbers $z_{\alpha_1}, z_{\alpha_2}$

$$z_\alpha := \chi_\alpha(\rho)\, e^{i(\langle\theta,\alpha\rangle - \Theta(\alpha))}\, e^{\beta(\langle\rho,\alpha\rangle - h(\alpha))},$$

while keeping their sum invariant, and all other z_α fixed. This vector v satisfies $\langle\alpha_1, v\rangle < 0$, $\langle\alpha_2, v\rangle < 0$ and $\langle\alpha_i, v\rangle = 0$ for $i \neq 1, 2$, hence $\langle d\varphi, v\rangle < 0$, since $d\varphi$ is a strictly positive linear combination of α_i. $\square$

Proposition 4.6.7. *For any face τ of P, the unstable manifold for* Crit_τ *is* $W^-_{\beta,\tau} \times T_{\Theta,\tau}$.

Proof. We note that the critical manifold Crit_τ is in the positive loci, where the summands of f are positive, and the Liouville flow preserves the positivity of summand and does not change θ-coordinates.

On the positive loci, the contracting Liouville flow

$$-X_{\lambda_{\mathcal{H}}} = -X_\lambda + X_\lambda^{\perp_\omega} = -\nabla\varphi + (\nabla\varphi)^{\perp_\omega},$$

where $X_\lambda^{\perp_\omega}$ is the ω-orthogonal projection of X_λ with respect to $T\mathcal{H}$, and we know $X_\lambda^{\perp_\omega}$ is proportional to $X_{\mathrm{Im}f}$. On the other hand, the downward gradient flow $-\nabla(\varphi|_{\mathcal{H}})$ can be expressed as

$$-\nabla(\varphi|_{\mathcal{H}}) = -\nabla\varphi + (\nabla\varphi)^{\perp_g},$$

where this time one take g-orthogonal projection. If f is holomorphic, then $\mathcal{H}$ is Kähler , then $(\nabla\varphi)^{\perp_g} = (\nabla\varphi)^{\perp_\omega}$. However f is not holomorphic in the transition regions, hence $-X_{\lambda_{\mathcal{H}}}$ and $-\nabla(\varphi|_{\mathcal{H}})$ differs in the transition region, with difference bounded by $O(e^{-\sqrt{\beta}})$. Since $-\nabla(\varphi|_{\mathcal{H}})$ has no critical point in the transition region, hence $-X_{\lambda_{\mathcal{H}}}$ does not either. Since $(\nabla\varphi)^{\perp_\omega}$ is positively proportional to $X_{\mathrm{Im}f}$, by (4.6.5), hence by a similar argument as in Proposition 4.5.5, the unstable manifold from critical point ρ_τ along the flow $-X_{\lambda_{\mathcal{H}}}$ after projective Legendre transformation, is the interior of the cone $\mathrm{cone}(\tau^\vee)$. Hence, the unstable manifold for $-X_{\lambda_{\mathcal{H}}}$ and for $-\nabla(\varphi|_{\mathcal{H}})$ are the same. This finishes the proof of the proposition. $\square$

CHAPTER 5

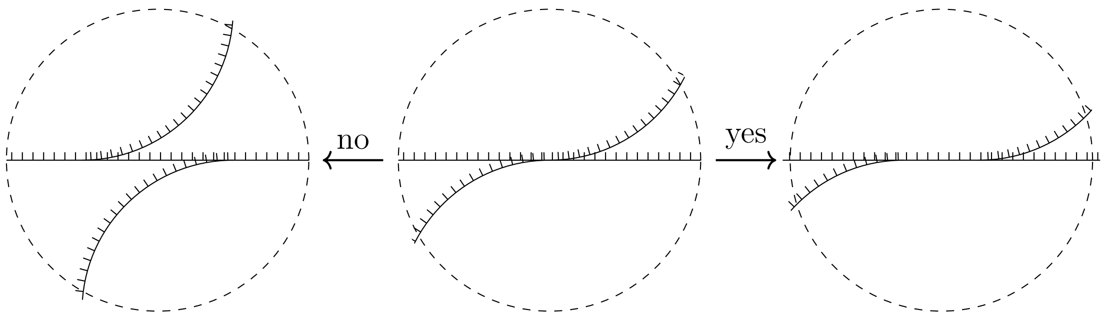
Variation of Constructible Sheaves: I

Let M be a smooth manifold, $\Lambda \subset T^*M$ a conical Lagrangian containing the zero section of T^*M and $\Lambda^\infty \subset T^\infty M$ the corresponding Legendrian. Let $Sh(M, \Lambda^\infty) = Sh(M, \Lambda)$ be the differential graded category of constructible sheaves with $SS^\infty(F) \subset \Lambda^\infty$. We are interested in the following question:

Given an initial Legendrian $\Lambda^\infty \subset T^\infty M$ and a constructible sheaf $F \in Sh(M, \Lambda^\infty)$, for what kinds of deformation of Λ^∞ can we find a corresponding deformation of F, such that $SS^\infty(F)$ remains in Λ^∞?

Constructible sheaves have both the simplicity of combinatorics and the flexibility of symplectic geometry, in that the data of a sheaf can be encoded as a representation of a quiver, and any Hamiltonian contactomorphism acting on $T^\infty M$ can be quantized to act on sheaves [**KS**, **GKS**]. Hence if Λ^∞ is a smooth Legendrian, and the deformation is an Legendrian isotopy, then the Legendrian isotopy can be embedded into a contactomorphism of $T^\infty M$, which can be quantized to give an equivalence of categories. However, if the Legendrian is not smooth, deformation of Legendrian may not come from contactomorphism, and not all Legendrian deformation results in equivalences of sheaf categories.

Example 5.0.1. Consider the following Legendrian deformation in $T^\infty\mathbb{R}^2$, represented as cooriented hypersurface (curve) on $\mathbb{R}^2$.



The deformation from the middle to the right keeps the category of sheaves invariant, and the one to the left does not. △

To setup the problem, we need to be more specific as to what singularity do we allow in the Legendrian, and how to describe the deformation. Here we list a few approaches, each having its own advantange and disadvantage.

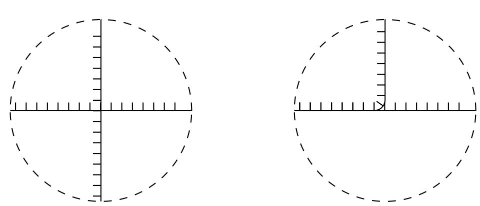
(1) One possible definition is this: A compact singular Legendrian is union of finitely many compact smooth Legendrians, $\mathcal{L} = \bigcup_{i=1}^N \mathcal{L}_i$, and deformation is realized by smooth Legendrian isotopy of each $\mathcal{L}_i$. The advantage is that only the gluing parameter is changing and the set of components is fixed. The disadvantage is that there is no canonical way to decompose $\mathcal{L}$ as a union of smooth components. In the above example for the picture in the middle, one write $\mathcal{L}$ as union of only two smooth Legendrians, or as many as four Legendrians.

(2) Another possible definition is: Let $\mathcal{S} = \{\mathcal{S}_\alpha : \alpha \in A\}$ be a Whitney stratification of M with finitely many strata, and $\Lambda_{\mathcal{S}}^\infty = \cup_\alpha T^*_{\mathcal{S}_\alpha} M$ is the union of the conormal

to the strata. The singular Legendrian $\mathcal{L}$ is defined as a closed subset in $\Lambda^\infty_{\mathcal{S}}$, and there is an induced stratification of $\mathcal{L} = \cup_\alpha \mathcal{L}_\alpha$ where $\mathcal{L}_\alpha \in T^*_{\mathcal{S}_\alpha} M$ which might be empty or of lower dimension. The advantage is that, it is closely related to the geometry on the base manifold; the disadvantage is that, even a smooth connected Legendrian $\mathcal{L} \subset T^\infty M$ may require a complicated stratification in M due to the singularity of $\pi_F(\mathcal{L})$ in the front projection $\pi_F : T^\infty M \to M$. To describe deformation of Legendrian, one can consider stratification in $M \times \mathbb{R}$ and Legendrian $T^\infty(M \times \mathbb{R})$, and consider the restriction to the slices $M \times \{t\}$.

Besides the above general cases, there are some interesting special cases as well. We give two examples below.

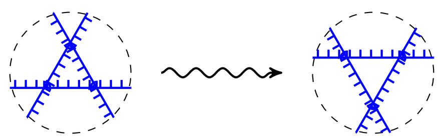
Example 5.0.2 (Slice of Arboreal Singularities)**.** Let Λ be a conical Lagrangian in $T^*\mathbb{R}^n$ with arboreal type singularity, studied by Nadler in [**N2**], given by a rooted tree $\mathcal{T}$ with $n+1$ nodes. The category of constructible sheaves are equivalent to the representation category of the quiver from the rooted tree, and is independent of the choice of the roots up to derived equivalence. We give two examples with $n = 2$.



$\bullet \leftarrow \bullet \rightarrow \bullet \qquad\qquad \bullet \rightarrow \bullet \rightarrow \bullet$

If we fix a linear function $f : \mathbb{R}^n \to \mathbb{R}$, we may consider sheaves on the slices $f^{-1}(t)$ for all $t \in \mathbb{R}$. It is interesting to ask whether categories of sheaves on different slices are equivalent. $\triangle$

Example 5.0.3 (Hyperplane Arrangements)**.** Let $\{H_i\}_{i=1}^N$ be a finite collection of affine co-oriented hyperplanes in $\mathbb{R}^n$, where each H_i is the zero locus of an affine linear equation $f_i = \langle c_i, x\rangle - b_i$ for some unit conormal vector c_i and offset parameter $b_i \in \mathbb{R}$, and the co-orientation is given by c_i.



For each subset $I \subset \{1, \cdots, N\}$ such that $\{c_i : i \in I\}$ is linearly independent, we define a conical Lagrangian

$$\Lambda_I^\infty = \bigcap_{i\in I} H_i \ \times \ \mathrm{cone}\{c_i : i \in I\} \subset T^*\mathbb{R}^n.$$

Let $\mathcal{I}$ be a collection of above subsets $I \subset \{1, \cdots, N\}$, then we may define a conical Lagrangian as $\Lambda_{\mathcal{I}}^\infty = \cup_{I\in\mathcal{I}}\Lambda_I^\infty$. It is interesting to study family of categories $Sh(\mathbb{R}^n, \Lambda_{\mathcal{I}}^\infty)$ where $\mathcal{I}$ is fixed and $\{c_i\}, \{b_i\}$ is changing. $\triangle$

In this chapter, we will study quantization for Legendrian deformation using stratification of M and $M \times \mathbb{R}$. In the next chapter, we will study the special case of hyperplane arrangements variation. The study for slices over arboreal singularities will be taken up in the future.

5.1. Definition of Variation of Legendrians and Sheaves

Let M be a smooth compact manifold. For any $t \in \mathbb{R}$, let $M_t = M \times \{t\} \subset M \times \mathbb{R}$ be the t-slice, and $j_t : M_t \hookrightarrow M \times \mathbb{R}$ be the inclusion. We will fix a Riemannian metric on M and use $T^\infty M$ and S^*M interchangeably. As before, our conical Lagrangian Λ includes the zero section in T^*M, and Λ^∞ is the corresponding Legendrian in $T^\infty M$.

Let $\Lambda^\infty_\mathbb{R} \subset T^\infty(M \times \mathbb{R})$ be a Legendrian in the product space. We say $\Lambda^\infty_\mathbb{R}$ is $\pi_\mathbb{R}$-*compatible for the projection* $\pi_\mathbb{R} : M \times \mathbb{R} \to \mathbb{R}$, if for any $(x, t; \xi, \tau) \in \Lambda^\infty_\mathbb{R}$, we have $\xi \neq 0$. For a $\pi_\mathbb{R}$-compatible $\Lambda^\infty_\mathbb{R}$, we denote its restriction at $t \in \mathbb{R}$ by

$$\Lambda^\infty_t = \{(x, \xi/|\xi|) \in T^\infty M \mid \exists \tau, \text{ such that } (x, t; \xi, \tau) \in \Lambda^\infty_\mathbb{R}\}.$$

We say a $\pi_\mathbb{R}$-compatible Legendrian $\Lambda^\infty_\mathbb{R}$ has *compactly supported deformation*, if there exists $[a, b] \subset \mathbb{R}$ such that Λ^∞_t is constant for $t < a$ and $t > b$.

Definition 5.1.1. A *deformation of Legendrian (over* $\mathbb{R}$*)* is a Legendrian $\Lambda^\infty_\mathbb{R} \subset T^\infty(M \times \mathbb{R})$, that is $\pi_\mathbb{R}$-compatible and has compactly supported deformation. Given two deformations of Legendrian, $\Lambda^\infty_{1,\mathbb{R}}$ and $\Lambda^\infty_{1,\mathbb{R}}$, we say they are *non-characteristic with respect to each other*, if $\Lambda^\infty_{1,t} \cap \Lambda^\infty_{1,t} = \emptyset$ for all $t \in \mathbb{R}$.

Suppose Λ^∞_t is constant outside $[a, b]$, we use Λ^∞_- (resp. Λ^∞_+) to denote the value of Λ^∞_t for $t < a$ (resp. $t > b$).

Definition 5.1.2. A *deformation of sheaf* is a sheaf $G_\mathbb{R} \in Sh(M \times \mathbb{R})$, such that the Legendrian $SS^\infty(G_\mathbb{R})$ is a deformation of Legendrian. For any $t \in \mathbb{R}$, let $G_t = G_\mathbb{R}|_{M_t}$ be the restriction of $G_\mathbb{R}$ over t. Let G_- (resp. G_+) be the value of G_t for $t \ll 0$ (resp. $t \gg 0$). Let $\Lambda^\infty_\mathbb{R}$ be a deformation of Legendrian $\Lambda^\infty_\mathbb{R} \subset T^\infty(M \times \mathbb{R})$, if $SS^\infty(G_\mathbb{R})$ and $\Lambda^\infty_\mathbb{R}$ are non-characteristic with respect to each other, then we say G_- and G_+ are $\Lambda^\infty_{\mathbb{R}}$-*isotopic*. If $\Lambda^\infty_{\mathbb{R}}$ has constant fiber $\Lambda^\infty_t \equiv \Lambda^\infty \subset T^\infty M$, we also say G_- and G_+ are Λ^∞-isotopic. If $G_+ = 0$, we say G_- is Λ^∞-*null-isotopic*. If $\Lambda^\infty_{\mathbb{R}} = \emptyset$, we may simply say G_- *and* G_+ *are isotopic*.

Proposition 5.1.3. *Let* $G_{\mathbb{R}}$ *and* $F_{\mathbb{R}}$ *be deformations of sheaves. If*

$$SS^\infty(G_t) \cap SS^\infty(F_t) = \emptyset \quad \textit{for all} \;\; t \in \mathbb{R},$$

then

$$hom(F_t, G_t) \cong hom(F_s, G_s) \quad \textit{for all } t, s \in \mathbb{R}$$

Proof. First, we claim that

$$hom_{M_t}(F_t, G_t) \cong hom_{M\times\mathbb{R}}(\mathbb{C}_{M_t}[-1], \underline{hom}(F_{\mathbb{R}}, G_{\mathbb{R}})).$$

Since $SS^\infty(j_{t*}\mathbb{C}_{M_t}) \cap SS^\infty(F_{\mathbb{R}}) = \emptyset$, by Petrowsky theorem for sheaves ([S], Cor. 4.6), we have isomorphism

$$\underline{hom}(\mathbb{C}_{M_t}, \mathbb{C}_{M\times I}) \otimes F_{\mathbb{R}} \xrightarrow{\sim} \underline{hom}(\mathbb{C}_{M_t}, F_{\mathbb{R}})$$

Since $\underline{hom}(\mathbb{C}_{M_t}, \mathbb{C}_{M\times I}) \cong \mathbb{C}_{M_t}[-1]$, we have

$$\Gamma_{M_t} F_{\mathbb{R}} \cong (F_{\mathbb{R}})_{M_t}[-1], \quad \Gamma_{M_t} G_{\mathbb{R}} \cong (G_{\mathbb{R}})_{M_t}[-1].$$

where we use the notation $F_Z = (j_Z)_*(j_Z)^*F \cong \mathbb{C}_Z \otimes F$ and $\Gamma_Z F = \underline{hom}(\mathbb{C}_Z, F)$. Thus

$$\begin{aligned} & hom_{M_t}(F_t, G_t) = hom_{M_t}({j_t}^* F_{\mathbb{R}}, {j_t}^* G_{\mathbb{R}}) \cong hom_{M\times\mathbb{R}}(F_{\mathbb{R}}, j_{t*}j_t^* G_{\mathbb{R}}) \\ \cong \;& hom_{M\times\mathbb{R}}(F_{\mathbb{R}}, \underline{hom}(\mathbb{C}_{M_t}, G_{\mathbb{R}})[1]) = hom_{M\times\mathbb{R}}(\mathbb{C}_{M_t}, \underline{hom}_{M\times\mathbb{R}}(F_{\mathbb{R}}, G_{\mathbb{R}}))[1] \\ = \;& hom_{M\times\mathbb{R}}(\mathbb{C}_{M_t}[-1], \underline{hom}_{M\times\mathbb{R}}(F_{\mathbb{R}}, G_{\mathbb{R}})) \end{aligned}$$

Next, we show that

$$hom_{M\times\mathbb{R}}(\mathbb{C}_{M_t}[-1], \underline{hom}_{M\times\mathbb{R}}(F_{\mathbb{R}}, G_{\mathbb{R}})) \cong [\pi_{\mathbb{R}*}\underline{hom}_{M\times\mathbb{R}}(F_{\mathbb{R}}, G_{\mathbb{R}})]_t$$

Indeed from [**KS**], Eq (2.3.10) and taking global sections, we have

$$hom_{\mathbb{R}}(\mathbb{C}_{\{t\}}, \pi_{\mathbb{R}*}\underline{hom}_{M\times\mathbb{R}}(F_{\mathbb{R}}, G_{\mathbb{R}})) \cong hom_{M\times\mathbb{R}}(\mathbb{C}_{M_t}, \underline{hom}_{M\times\mathbb{R}}(F_{\mathbb{R}}, G_{\mathbb{R}}))$$

Hence suffice to show that

$$hom_{\mathbb{R}}(\mathbb{C}_{\{t\}}, \pi_{\mathbb{R}*}\underline{hom}_{M\times\mathbb{R}}(F_{\mathbb{R}}, G_{\mathbb{R}})) \cong \pi_{\mathbb{R}*}\underline{hom}_{M\times\mathbb{R}}(F_{\mathbb{R}}, G_{\mathbb{R}}) \otimes \mathbb{C}_{\{t\}}[1]$$

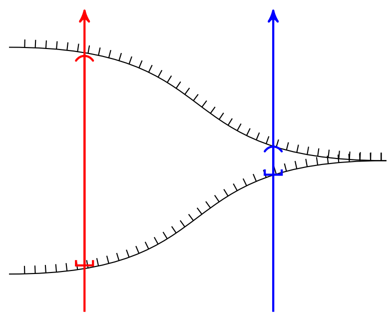
This follows from Petrowsky theorem if $\pi_{\mathbb{R}*}\underline{hom}_{M\times\mathbb{R}}(F_{\mathbb{R}}, G_{\mathbb{R}})$ is a local system, which we now prove. Since $\Lambda_t^G \cap \Lambda_t^F = \emptyset$, we have $\Lambda_F^\infty \cap \Lambda_G^\infty = \emptyset$, hence

$$SS^\infty(\underline{hom}(F_{\mathbb{R}}, G_{\mathbb{R}})) = SS^\infty(\underline{hom}(F_{\mathbb{R}}, \mathbb{C}_{M\times\mathbb{R}}) \otimes G_{\mathbb{R}}) = [SS(G_{\mathbb{R}}) - SS(F_{\mathbb{R}})]^\infty$$

however, $SS^\infty(F_t) \cap SS^\infty(G_t) = \emptyset$, that means if $(p, \tau) \in T^*_{x,t}(M \times \mathbb{R})$ is in $SS^\infty(G)$ for some $p \neq 0$, then there is no τ' such that $(p, \tau') \in SS^\infty(F)$, hence if $(p, \tau) \in SS(G) -$

$SS(F)$ then $p \neq 0$. In other words $SS^{\infty}(\underline{hom}(F_{\mathbb{R}}, G_{\mathbb{R}})) \cap (T^*_M M \times T^*I)^{\infty} = \emptyset$, hence $(\pi_{\mathbb{R}})_* \underline{hom}(F_{\mathbb{R}}, G_{\mathbb{R}})$ is a local system. This concludes the proof of the proposition. $\square$

Example 5.1.4. Let $M = \mathbb{R}$ and $G = \mathbb{C}_{[-1,1)}$, then G is null-isotopic via a right-cusp-like sheaf.



Indeed, let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth non-increasing function, such that $\chi(t) = 1$ for $t \leq 0$ and $\chi(t) = 0$ for $t \geq 1$, then the sheaf

$$G_{\mathbb{R}} := \mathrm{cone}(\mathbb{C}_{\{(x,t)|x \leq \chi(t)\}} \xrightarrow{\mathrm{res}} \mathbb{C}_{\{(x,t)|x \leq -\chi(t)\}})[-1] = \mathbb{C}_{\{(x,t)|-\chi(t) < x \leq \chi(t)\}}$$

is the desired deformation of sheaf. $\triangle$

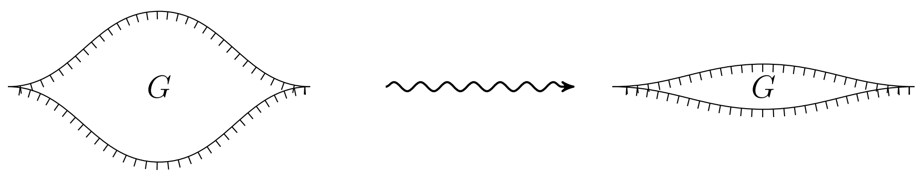
Example 5.1.5. Let $M = \mathbb{R}^2$, $j_B : B = \{x \in M \mid |x| < 1\} \hookrightarrow M$ be the inclusion of the open disk. Let $f : B \to \mathbb{R}$ be given by

$$f(x) = x_1 \exp\left(\frac{1}{1 - |x|^2}\right).$$

Let $G' \in Sh(B)$ be sheaf given by

$$G' := \mathrm{cone}(\mathbb{C}_{\{x: f(x) \leq 1\}} \xrightarrow{\mathrm{res}} \mathbb{C}_{\{x: f(x) \leq -1\}})[-1] = \mathbb{C}_{\{x: -1 < f(x) \leq 1\}}$$

and $G = j_{B*}G'$.



Then G is null-isotopic via a sheaf $G_{\mathbb{R}} = (j_B \times id_{\mathbb{R}})_* G'_{\mathbb{R}}$ for $G'_{\mathbb{R}} \in Sh(B \times \mathbb{R})$ defined by

$$G'_{\mathbb{R}} := \mathrm{cone}(\mathbb{C}_{\{(x,t)\in B\times\mathbb{R} | f(x)\leq \chi(t)\}} \xrightarrow{\mathrm{res}} \mathbb{C}_{\{(x,t)\in B\times\mathbb{R} | f(x)\leq -\chi(t)\}})[-1] = \mathbb{C}_{\{(x,t)\in B\times\mathbb{R} | -\chi(t) < f(x)\leq \chi(t)\}}.$$

This generalizes straightforwardly to $M = \mathbb{R}^n$ case. △

Proposition 5.1.6. *If G is Λ-null-isotopic, then for any $F \in Sh(M, \Lambda)$, $hom(G, F) = 0$ and $hom(F, G) = 0$.*

Proof. This is from Definition 5.1.2 of null-isotopic deformation of sheaves and Proposition 5.1.3 for the constancy of hom-complex. □

5.2. Constructible Sheaf as Yoneda Functor

In this section we explain the main idea of how to deform a constructible sheaf. First, we recall the definition of a presheaf: a presheaf F on M valued in chain complex takes in open set and output chain complexes, in a way consistent with the restriction map. If the sheaf $F \in Sh(M, \Lambda)$ is constructible, then we may deform the open set U in a Λ-non-characteritic way, while keeping $F(U)$ invariant up to quasi-isomorphism

$$SS^{\infty}(\mathbb{C}_{U_t}) \cap SS^{\infty}(F) = \emptyset \text{ for all } t \in \mathbb{R} \quad \Rightarrow \quad F(U_t) \cong F(U_s) \text{ for all } t, s \in \mathbb{R}. \tag{5.2.1}$$

More generally, we can view $F(U)$ as $hom(\mathbb{C}_U, F)$ and deform $\mathbb{C}_U$ as a sheaf, say by a sheaf-quantization $\widehat{\varphi}_t$ of Hamiltonian isotopy $\varphi_t : T^\infty M \to T^\infty M$. Let $P_t = \widehat{\varphi}_t \mathbb{C}_U$, we call P_t *probe sheaves*. Then we have

(5.2.2)

$$SS^\infty(P_t) \cap SS^\infty(F) = \emptyset \text{ for all } t \in \mathbb{R} \quad \Rightarrow \quad hom(P_t, F) \cong hom(P_s, F) \text{ for all } t, s \in \mathbb{R}.$$

Let Λ_t^∞ be a deformation of Legendrian, and assume $F_t \in Sh(M, \Lambda_t^\infty)$ is a deformation of constructible sheaves (see Definition 5.1.2), and P_t is a deformation of probe sheaves, then we have

(5.2.3)

$$SS^\infty(P_t) \cap SS^\infty(F_t) = \emptyset \text{ for all } t \in \mathbb{R} \quad \Rightarrow \quad hom(P_t, F_t) \cong hom(P_s, F_s) \text{ for all } t, s \in \mathbb{R}.$$

Hence, if we know F_0 and want to know the value of F_1 on certain probe sheaf P_1, $hom(P_1, F_1)$, we only need to deform P_1 back to P_0 avoiding collision with Λ_t^∞ along the way. And if we know the value of F_1 on sufficiently many such probe sheaves, we may reconstruct F_1.

Example 5.2.1. In this example, we illustrate how to deform a probe sheaf. Let $X = \mathbb{R}^2$, and $\Lambda_t \subset T^*X$ for $t \in [-1, 1]$ is given by

$$\begin{aligned} \Lambda_t &= T^*_X X \bigcup \left(\bigcup_{i=1}^{3} \{(x,\xi) \in T^*X \mid f_i(x) = t, \xi = \lambda df_i(x), \lambda > 0\} \right) \\ &\bigcup \left(\bigcup_{1 \le i < j \le 3} \{(x,\xi) \in T^*X \mid f_i(x) = f_j(x) = t, \xi = \lambda_i df_i(x) + \lambda_j df_j(x), \lambda_i, \lambda_j > 0\} \right) \end{aligned}$$

where $f_i(x) = x_1 \cos\theta_i + x_2 \sin\theta_i$, and $\theta_1 = \pi/2, \theta_2 = -5\pi/6, \theta_3 = -\pi/6$.

Let $F_{-1} \in Sh(X, \Lambda_{-1})$ be the standard sheaf supported in the closed set $\cup_i \{f_i(x) \geq -1\}$. We claim that as Λ_t changes from $t = -1$ to $t = 1$, F_{-1} changes to the costandard sheaf $F_1 \in Sh(X, \Lambda_1)$ supported on the open triangle $\cap_i \{f_i(x) < 1\}$ at degree -2.

To verify the claim of the stalk of F_1, we pick an open ball $B(0, 1/2)$ in the interior of the triangle, shown in the right of the picture Figure 6.1. Let $P = \mathbb{C}_{\{|x|<0.5\}}$ be a constant sheaf $\mathbb{C}$ supported on this open ball, then the stalk of F_{+1} at 0 can be computed as

$$(F_{+1})_0 \cong hom(P, F_{+1}).$$

Apply the unit speed geodesic flow R on the cosphere bundle $S^*\mathbb{R}^2$ with respect to the Euclidean metric on $\mathbb{R}^2$ for time -3, then by a result of [GKS] (Example 3.10), P changes to $\Phi_{-3}(P) = \mathbb{C}[-2]_{\{|x|\leq 2.5\}}$, where Φ_t quantizes the geodesic flow for time t. Since $SS^\infty(\Phi_t(P))$ remains disjoint from Λ_{+1}, as $t \in [-3, 0]$, we have

$$hom(P, F_{+1}) \cong hom(\Phi_{-3}(P), F_{+1}).$$

Finally, as Λ_{+1} varies back to Λ_{-1} and F_{+1} changes back to F_{-1}, Λ_s for $s \in [-1, 1]$ remains disjoint for $\Phi_{-3}(P)$. Hence, we get

$$hom(\Phi_{-3}(P), F') \cong hom(\Phi_{-3}(P), F) \cong hom(\mathbb{C}[-2]_{\{|x|\leq 2.5\}}, F) \cong \Gamma_c(B(0, 2.5), F)[2] \cong \mathbb{C}[2]$$

$\triangle$

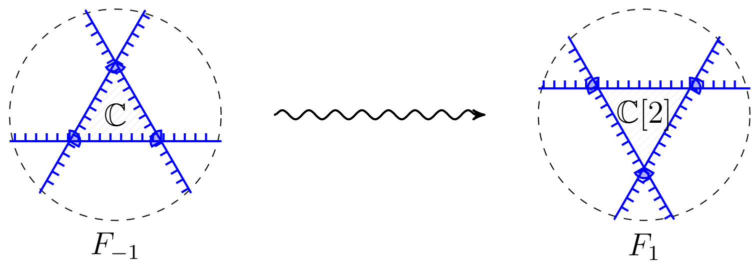


Figure 5.1. As the Legendrian moves, the sheaf F changes to F'.

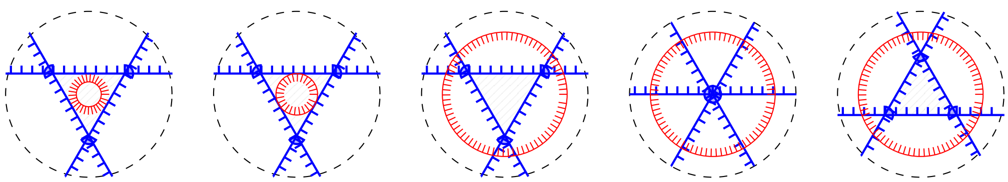


Figure 5.2. Invariance of hom-complex $hom(P, F)$ during deformation. The first three picture shows deformation of the probe sheaf P whose singular support is marked in red ,with F fixed as F_{+1}, then the last two picture deforms F keeping P fixed. Note that the $SS^\infty(P) \cap SS^\infty(F) = \emptyset$ throughout this deformation.

5.3. Family of Probe Sheaves and Reproducing Kernel

Let M be an n-dimensional manifold, $\Lambda^\infty \subset T^\infty M$ a Legendrian, and $\mathcal{S} = \{\mathcal{S}_\alpha\}_{\alpha \in A}$ a Whitney stratification of M compatible with Λ^∞. We will define a family of probe sheaves $\{P_p \mid p \in M\}$, such that their singular support at infinity $SS^\infty(P_p)$ is disjoint from Λ^∞ and $hom(P_p, F)$ compute the stalk of F at p for any $F \in Sh(M, \Lambda^\infty)$.

We then assemble the probe sheaves into a reproducing kernel $\Pi_\Lambda \in Sh(M \times M)$, which is supported on the graph of the almost retraction r. The main result in this section is Proposition 5.4.12, which says if a sheaf has singular support at infinity $SS^\infty(F)$ close enough to Λ^∞, then under the pushforward of the almost retraction r_*F has $SS^\infty(r^*F) = \Lambda^\infty$.

5.4. Whitney Stratification

We recall the definition of almost retraction, following ([N3], Section 2.2) closely.

A tubular neighborhood of a submanifold (may be not closed) $Y \subset M$ consists of an inner product on the normal bundle $p : N_Y \to Y$, and a smooth embedding

$$\varphi : N_Y[< \epsilon] = \{v \in N_Y | \langle v, v\rangle < \epsilon\} \hookrightarrow M \tag{5.4.1}$$

of the open ball bundle determeind by some $\epsilon > 0$. The image $T = \varphi(N_Y[< \epsilon])$ is requried to be an open neighborhood of $Y \subset M$, and the restriction of φ to the zero-section $Y \subset N_Y$ is required to be the identity map to $Y \subset M$. By rescaling the inner product, we can assume that $\epsilon = 1$.

By transport of structure, the neighborhood T comes with (smooth) tubular distance function $\rho : T \to \mathbb{R}_{\geq 0}$ and smooth tubular projection $\pi : T \to Y$ defined by

$$\rho(x) = \langle \varphi^{-1}(x), \varphi^{-1}(x)\rangle, \quad \pi(x) = p(\varphi^{-1}(x)) \tag{5.4.2}$$

We will write (T, ρ, π) to denote the tubular neighborhood, and remember that $\pi : T \to Y$ is an open unit ball bundle.

Given small $\epsilon \geq 0$, we have the inclusion

(5.4.3)

$$j[\epsilon] : T[\epsilon] = \{x \in T \mid \rho(x) = \epsilon\} \hookrightarrow T, \quad j[< \epsilon] : T[< \epsilon] = \{x \in T \mid \rho(x) < \epsilon\} \hookrightarrow T$$

and similarly with $< \epsilon$ replaced by $\leq \epsilon, \geq \epsilon$ or $> \epsilon$, or any subinterval of $[0, 1]$.

Any Whitney stratified subspace $X \subset M$ admits a compatible system of control data consisting of a tubular neighborhood $(T_\alpha, \rho_\alpha, \pi_\alpha)$ of each stratum $X_\alpha \subset X$. Whenever

$\alpha < \beta$, that is $S_\alpha \subset \overline{S_\beta}$, we require

$$\pi_\alpha(\pi_\beta(x)) = \pi_\alpha(x), \quad \rho_\alpha(\pi_\beta(x)) = \rho_\alpha(x) \tag{5.4.4}$$

on the common domain $x \in T_\alpha \cap T_\beta$, such that $\pi_\beta(x) \in T_\alpha$.

Fix ϵ to be small enough. Let $r_\alpha : T_\alpha[(0, 2\epsilon)] \to T_\alpha[2\epsilon]$ be a family of lines. We further require that ϵ is so small that different strata has disjoint tubes $T_\alpha[< 2\epsilon]$. If $\alpha < \beta$,

$$\rho_\beta = \rho_\beta r_\alpha, \quad \rho_\alpha = \rho_\alpha r_\beta, \quad \pi_\alpha = \pi_\alpha r_\alpha, \quad r_\alpha r_\beta = r_\beta r_\alpha \tag{5.4.5}$$

and further more, the restriction $r_\alpha|_{T_\alpha[(0,2\epsilon)]\cap S_\beta} : T_\alpha[(0, 2\epsilon)] \cap S_\beta \to T_\alpha[2\epsilon] \cap S_\beta$ is smooth ([**Go**], P194, property (7)).

For each chain of stratum $\alpha_1 < \alpha_2 < \cdots < \alpha_k$, and $B = \{\alpha_1, \cdots, \alpha_k\}$, we define a homeomorphism, (as a stand-in for polar coordinate)

$$h_B : \cap_{\alpha\in B}(T_\alpha[(0, 2\epsilon)]) \to (\cap_\alpha T_\alpha[2\epsilon]) \times \prod_{\alpha\in B}(0, 2\epsilon). \tag{5.4.6}$$

Fix a smooth non-decreasing function $q : \mathbb{R} \to \mathbb{R}$, such that $q(t) = 0$, for $t \le \epsilon$ and $q(t) = t$ for $t \ge 2\epsilon$, and $q'(t) > 0$ for $t \in (\epsilon, 2\epsilon)$. For each stratum S_α, we introduce

$$\Pi_\alpha : M \to M, \qquad \Pi_\alpha(x) = \begin{cases} x, & \text{when } x \notin T_\alpha[< 2\epsilon] \\ h_\alpha^{-1}(r_\alpha(x), q(\rho_\alpha(x))), & \text{when } x \in T_\alpha[< 2\epsilon]\backslash S_\alpha \end{cases} \tag{5.4.7}$$

One can show that Π_α commutes with each other. Define the ‘almost retraction’

$$r : M \to M, \quad r = \prod_{\alpha\in A} \Pi_\alpha \tag{5.4.8}$$

Thanks to the commutativity of Π_αs, the product is well-defined.

Proposition 5.4.1. *(1) If $p \in M$ is not in any S_α, then its preimage under r is a point in the same connected component of $|\mathcal{S}|$.*
(2) If $p \in S_\alpha$, then for any $\beta \neq \alpha$, $\Pi_\beta^{-1}(p) = \{p'\}$ for some $p' \in S_\alpha$, depending smoothly on p; and

$$\Pi_\alpha^{-1}(p) = \pi_\alpha^{-1}(p) \cap T_\alpha[\leq \epsilon]. \tag{5.4.9}$$

(3) If $p \in S_\alpha$, then there exists $p' \in S_\alpha$ depending smoothly on p, such that

$$r^{-1}(p) = \pi_\alpha^{-1}(p') \cap T_\alpha[\leq \epsilon] \tag{5.4.10}$$

Proof. A similar result is recorded in [GoMa] 6.13.4. □

For any family of lines r_α, and for any $0 < \delta < \epsilon$, we may define $r_\alpha^{(\delta)} : T_\alpha[(0, 2\delta)] \to T_\alpha[2\delta]$ induced by the family of lines. We may define the corresponding $\Pi_\alpha^{(\delta)}$ and almost retraction $r^{(\delta)}$.

5.4.1. Probe Sheaves

Recall in Section 5.4, for a stratification $\mathcal{S}$, we can define the control data $(T_\alpha, \rho_\alpha, \pi_\alpha)$ and a family of lines r_α, that is $\pi_\alpha : T_\alpha \to \mathcal{S}_\alpha$ is a tubular neighborhood of $\mathcal{S}_\alpha$, and ρ_α is a distance function to $\mathcal{S}_\alpha$, with certain compatibility conditions between strata. Let r denote the almost retraction

$$r : M \to M, \quad r = \prod_{\alpha \in A} \Pi_\alpha.$$

Definition 5.4.2. For each point $p \in M$, we define the *probe set for* p to be the closed set

$$A_p = r^{-1}(p), \tag{5.4.11}$$

and j_{A_p} be the inclusion map of A_p into M. We define the *probe sheaf* for p to be the costandard sheaf

$$P_p = j_{A_p!}\omega_{A_p} = \mathbb{D}(j_{A_p*}\mathbb{C}_{A_p}). \tag{5.4.12}$$

If p is in a stratum $\mathcal{S}_\alpha$ of dimension k for $0 \leq k \leq n$, then A_p is a fiber of $\pi_\alpha : T_\alpha[\leq \epsilon] \to \mathcal{S}_\alpha$ an embedded $(n-k)$ closed disk (Proposition 5.4.1). P_p has support in the relative interior of A_p, with stalk isomorphic to $\mathbb{C}[n-k]$. The singular support of P_p at infinity is the outward-conormal to A_p. The following proposition justifies the name for probe sheaf.

Proposition 5.4.3. *Fix a Riemannian metric g on M, and let $A_p[< \epsilon]$ be the set of points with distance to A_p less than ϵ. There exists a small enough $1 \gg \epsilon_0 > 0$ depending on $(M, g, \mathcal{S})$, such that for any constructible sheaf $F \in Sh_{\mathcal{S}}(M)$ adapted to the Whitney stratification S, we have*

$$F_p \cong F(A_p[< \epsilon]) \cong hom(P_p, F) \tag{5.4.13}$$

for all $0 < \epsilon < \epsilon_0$.

Proof. F_p can be computed by any small enough open ball around this point, thanks to the contructibility. We claim that the costandard sheaf on $A_p[<\epsilon]$ and the one on A_p are isotopic with respect to $\Lambda_{\mathcal{S}}^{\infty}$, and the isotopy is given by geodesic flow. $\square$

5.4.2. Reproducing Kernel

A classical reproducing kernel in functional analysis is the following, let X be a space, $H \subset L^2(X)$ is a Hilbert subspace, $\{K_x\}_{x\in X}$ is a family of functions in H, such that for any $f \in H$, $f(x) = \langle f, H_x\rangle$. We may equivalently define the reproducing kernel $K : X \times X \to \mathbb{R}$, by $K(x,y) = K_x(y)$, then we have

$$f(x) = \int_y f(y)K(x,y)dy, \qquad \text{for all } \ f \in H,$$

hence K is called the reproducing kernel for H.

Here we use the word *reproducing kernel* in a much looser sense: let $H \subset V$ be a subspace, then we say an operator $T : V \to V$ reproduces H, if $T|_H = id_H$. For example, if V the set of smooth complex valued function on $\mathbb{C}$, and $T_\epsilon(f)(z) = \oint_{|w-z|=\epsilon} \frac{f(w)}{w-z}\frac{dw}{2\pi i}$, then T_ϵ reproduces the subspace of holomorphic function (also harmonic functions).

In our case, we want to assemble these probe sheaves together into a reproducing kernel $\Pi_\Lambda \in Sh(M \times M)$, such that $(\Pi_\Lambda)_!$ reproduces the subcategory $Sh(M,\Lambda)$ in $Sh(M)$. Let π_1, π_2 be the projection from $M \times M$ to the first and second factor, and our convention is always to use the first factor as input and the second factor as output.

The definition of the kernel is just the constant sheaf over the graph of the almost retraction.

$$\Pi_\Lambda := \mathbb{C}_{\Gamma_r}, \quad \Gamma_r := \{(p, r(p)) \in M \times M \mid p \in M\} \tag{5.4.14}$$

Recall the definition of kernel operation $K_*, K_!$ in Eq. (3.4.2) and (3.4.4),

$$K_! : F \mapsto \pi_{2!}(K \otimes \pi_1^* F), \quad K_* : F \mapsto \pi_{2*}(\underline{hom}(K, \pi_1^! F))$$

Lemma 5.4.4.

$$\Pi_{\Lambda *} = r_* = r_! = \Pi_{\Lambda !} \tag{5.4.15}$$

Proof. $r_* = r_!$ since r is a proper map. The other two equal sign is by Proposition 3.4.1. □

Remark 5.4.5. We will use r_* and $r_!$ interchangeably from now on.

It is instructive to see the computation of a stalk of F at p to see the relation of the kernel and the probe sheaf.

Proposition 5.4.6. *Let $F \in Sh_{\mathcal{S}}(M)$, $p \in M$, then $hom(P_p, F) = (r_! F)_p = (r_* F)_p$.*

Proof. Let $F \in Sh_{\mathcal{S}}(M)$, let $p \in M$, and U_p be a small enough open ball around p, such that $F(U) = F_p$.Then

$$[\Pi_{\Lambda !} F]_p = [\pi_{2!}(\mathbb{C}_{\Gamma_r} \otimes \pi_1^* F)]_p = \Gamma_c(r^{-1}(p), F|_{r^{-1}(p)}) = \Gamma(r^{-1}(p), F|_{r^{-1}(p)})$$

where we have changed Γ_c to Γ since $r^{-1}(p)$ is compact.

$$[\Pi_{\Lambda *}F]_p = [\Pi_{\Lambda *}F](U) = hom(\Pi_\Lambda|_{\pi_2^{-1}(U)}, \pi_1^!(F)|_{\pi_2^{-1}(U)})$$

$$= hom(\pi_{1!}\Pi_\Lambda|_{\pi_2^{-1}(U)}, F) = hom(\mathbb{C}_{r^{-1}(U)}, F) = F(r^{-1}(U))$$

On the other hand

$$hom(P_p, F) = \Gamma \underline{hom}(P_p, F) = \Gamma(\mathbb{D}P_p \otimes F) = \Gamma(\mathbb{C}_{A_p} \otimes F) = \Gamma(r^{-1}(p), F|_{r^{-1}(p)})$$

Hence $hom(P_p, F)$, $(r_! F)_p$ and $(r_* F)_p$ are the same. $\square$

Proposition 5.4.7 ([N3], Lemma 6.3)**.** *r_* is canonically equivalent to the identity operator when restricted to S-constructible sheaves*

(5.4.16) $$r_* \cong id : Sh_{\mathcal{S}}(M) \xrightarrow{\sim} Sh_{\mathcal{S}}(M)$$

Corollary 5.4.8. *If $\Lambda \subset \Lambda^\infty_{\mathcal{S}}$ is a closed Legendrian, then r_* is canonically equivalent to the identity operator restricted to $Sh(M, \Lambda)$.*

We want to show that r_* has certain 'straightening effect', in that if a constructible sheaf F has its singular support at infinity $SS^\infty(F)$ in a small enough neighborhood of Λ^∞ in $T^\infty M$, then $r_* F$ would has its singular support in Λ^∞. We fix a Riemannian metric on M and induce a Riemannian metric on S^*M, we also identify $T^\infty M$ with S^*M. If Λ^∞ is a Legendrian in S^*M, we denote $\Lambda^\infty[< \epsilon]$ the set of points in S^*M whose distance to Λ^∞ are less than ϵ.

Proposition 5.4.9. *There exists ϵ_0 small enough, such that for any $p \in M$*

$$SS^\infty(P_p) \cap \Lambda_{\mathcal{S}}^\infty[< \epsilon_0] = \emptyset.$$

Proof. For each strata $\mathcal{S}_\alpha$, we claim there exists a $\epsilon_\alpha > 0$, such that $SS^\infty(SS^\infty(P_p)) \cap \Lambda_{\mathcal{S}}^\infty[< \epsilon_\alpha] = \emptyset$ for all $p \in \mathcal{S}_\alpha$, then $\epsilon_0 = \min_\alpha \epsilon_\alpha$ satisfy the condition. We define $r_\alpha = \prod_{\beta<\alpha} \Pi_\beta : \mathcal{S}_\alpha \to \overline{\mathcal{S}}_\alpha$, then $r_\alpha^{-1}(\mathcal{S}_\alpha)$ is a relative compact subset in $\mathcal{S}_\alpha$, denoted as B_α. For any $p \in \mathcal{S}_\alpha$, the probe set A_p is a fiber of $T_\alpha[< \epsilon] \to \mathcal{S}_\alpha$, where the base of the fiber is in B_α. Define

$$\epsilon_\alpha := \min_{q \in \overline{B}_\alpha} \operatorname{dist}(T^\infty_{\Pi_\alpha^{-1}(q)} M, \Lambda_{\mathcal{S}}^\infty),$$

where since $\overline{B}_\alpha$ is a compact subset in $\mathcal{S}_\alpha$, the minimum can be realized and is non-zero. Since $A_p = \Pi_\alpha^{-1}(r_\alpha^{-1}(p))$, and $SS^\infty(P_p) \subset T^\infty_{A_p} M$, we have

$$SS^\infty(P_p) \cap \Lambda_{\mathcal{S}}^\infty[< \epsilon_\alpha] = \emptyset.$$

This finishes the claim and proves the proposition. $\square$

Proposition 5.4.10. *Let ϵ_0 be as in Proposition 5.4.9. For any constructible sheaf $F \in Sh(M)$,*

$$SS^\infty(r^* F) \cap \Lambda_{\mathcal{S}}^\infty[< \epsilon_0] = \emptyset.$$

Proof. Let F be constructible with respect to a Whitney triangulation $\mathcal{T}$, which can be further refined such that any simplices in $\mathcal{T}$ is contained in some stratum in $\mathcal{S}$. Since F can be generated by the costandard sheaves in $Sh_{\mathcal{T}}(M)$, and that r^* is an exact functor, it suffices to prove that for any simplex τ in $\mathcal{T}$, the costandard sheaf $j_{\tau!}\omega_\tau$ satisfies the condition in the proposition. We note that

$$SS^\infty(r^* j_{\tau !}\omega_\tau) \subset \bigcup_{p\in\bar\tau} SS^\infty(P_p),$$

hence is disjoint from $\Lambda_{\mathcal{S}}^\infty[<\epsilon_0]$, hence proves the proposition. □

Proposition 5.4.11. *Let $\epsilon_0 > 0$ be small enough as in Proposition 5.4.9. Then for any $F \in Sh(M)$ such that $SS^\infty(F) \in \Lambda_{\mathcal{S}}^\infty[<\epsilon_0])$, we have $SS^\infty(r_* F) \subset \Lambda_{\mathcal{S}}^\infty$, i.e.*

$$r_* = r_! : Sh(M, \Lambda_{\mathcal{S}}^\infty[<\epsilon_0]) \to Sh(M, \Lambda_{\mathcal{S}}^\infty). \tag{5.4.17}$$

Proof. Fix any $F \in Sh(M, \Lambda[<\epsilon_0])$, suffice to show that $r_! F = r_* F$ is locally constant on each stratum of $\mathcal{S}$. Let $p_0, p_1 \in \mathcal{S}_\alpha$, then there is a path $\{p_t\}_{t\in[0,1]}$ in $\mathcal{S}_\alpha$ connecting p_0, p_1. Consider the set of probe sheaves for point in this path, we have

$$SS^\infty(P_{p_t}) \cap SS^\infty(F) \subset SS^\infty(P_{p_t}) \cap \Lambda_{\mathcal{S}}^\infty[<\epsilon_0] = \emptyset, \quad \text{for all } \ t \in [0,1].$$

Hence by the non-characteristic deformation proposition (Proposition 5.1.3),

$$hom(P_{p_t}, F) \cong hom(r^*(\mathbb{C}_{p_t}[n-k]), F) \cong hom(\mathbb{C}_{p_t}[n-k], r_* F)$$

is constant for all $t \in [0,1]$, hence $r_* F$ has constant (co)stalk along $\mathcal{S}_\alpha$ for all α, thus is $\mathcal{S}$ constructible. □

Next, we prove the more refined version.

Proposition 5.4.12. *Let $\epsilon_0 > 0$ be small enough as in Proposition 5.4.9. Then for any $F \in Sh(M)$ such that $SS^\infty(F) \in \Lambda^\infty[< \epsilon_0])$, we have $SS^\infty(r_*F) \subset \Lambda^\infty$, i.e.*

$$r_* = r_! : Sh(M, \Lambda^\infty[< \epsilon_0]) \to Sh(M, \Lambda^\infty). \tag{5.4.18}$$

Proof. This proof follows that of Theorem 6.7 in [N3]. To show that r_*F actually lands in $Sh(M, \Lambda)$, one need to show that r_*F has no non-trivial microlocal stalk on $\Lambda^\infty_{\mathcal{S}} \backslash \Lambda$.

For each strata $\mathcal{S}_\alpha$, we define $\Lambda^{\infty,sm}_{\mathcal{S}_\alpha} = \Lambda^\infty_{\mathcal{S}_\alpha} \setminus \overline{\cup_{\beta > \alpha} \Lambda^\infty_{\mathcal{S}_\alpha}}$, and $\Lambda^{\infty,sm}_{\mathcal{S}} = \cup_\alpha \Lambda^{\infty,sm}_{\mathcal{S}_\alpha}$, then we have

$$\overline{SS^\infty(r_*F) \cap \Lambda^{\infty,sm}_{\mathcal{S}}} = SS^\infty(r_*F), \quad \overline{\Lambda^\infty \cap \Lambda^{\infty,sm}_{\mathcal{S}}} = \Lambda^\infty.$$

Hence $SS^\infty(r_*F) \subset \Lambda^\infty$ is equivalent of

$$SS^\infty(r_*F) \cap \Lambda^{\infty,sm}_{\mathcal{S}} \subset \Lambda^\infty \cap \Lambda^{\infty,sm}_{\mathcal{S}}$$

which in turn is equivalent of

$$\Lambda^{\infty,sm}_{\mathcal{S}} \backslash SS^\infty(r_*F) \supset \Lambda^{\infty,sm}_{\mathcal{S}} \backslash \Lambda.$$

That is, for any $u \in \Lambda^{\infty,sm}_{\mathcal{S}} \backslash \Lambda$, we need to show that $u \notin SS^\infty(r_*F)$.

We induct on the number of strata of $\mathcal{S} \subset M$. The base case $\mathcal{S} = \emptyset$ is clear, r is the identity map on M.

Suppose given a closed stratum $i_0 : \mathcal{S}_0 \hookrightarrow M$. Let $M[> \epsilon] = M \backslash T_{\mathcal{S}_0}[\leq \epsilon]$, $\mathcal{S}[> \epsilon] = \mathcal{S} \cap M[> \epsilon]$. Then the Whitney stratification, system of control data and family of lines are the same for $\mathcal{S}[> \epsilon]$. Denoting the resulting almost retraction by $r[> \epsilon] : M[> \epsilon] \to M[> \epsilon]$.

Since $\mathcal{S}[>\epsilon]$ has fewer strata than $\mathcal{S}$, by induction, the push-forward induces a functor

$$r[>\epsilon]_* : Sh(M[>\epsilon], \Lambda^\infty[<\epsilon_0]|_{M[>\epsilon]}) \to Sh(M[>\epsilon], \Lambda^\infty|_{M[>\epsilon]})$$

Since $(r_*F) \in Sh(M, \Lambda^\infty_{\mathcal{S}})$ and

$$(r_*F)|_{M\backslash\mathcal{S}_0} = (\Pi_0)_* r[>\epsilon]_*(F|_{M[>\epsilon]}),$$

hence

$$(r_*F)|_{M\backslash\mathcal{S}_0} \in Sh(M\backslash\mathcal{S}_0, \Lambda^\infty|_{M\backslash\mathcal{S}_0}).$$

Thus, it suffices to show that $SS^\infty(r_*F)|_{\mathcal{S}_0} \subset \Lambda^\infty|_{\mathcal{S}_0}$, or as remarked in the beginning

$$\Lambda^{\infty,sm}_{\mathcal{S}_0} \setminus (SS^\infty(r_*F)|_{\mathcal{S}_0}) \supset \Lambda^{\infty,sm}_{\mathcal{S}_0} \setminus (\Lambda^\infty|_{\mathcal{S}_0}).$$

By taking the normal slice to $\mathcal{S}_0$ at p, we may assume $\mathcal{S}_0$ is zero dimensional.

Let $u \in \Lambda^{\infty,sm}_{\mathcal{S}_0} \backslash (\Lambda^\infty|_{\mathcal{S}_0})$, a covector over $p \in \mathcal{S}_0$, we shall prove u is not in $SS^\infty(r_*F)|_{\mathcal{S}_0}$. Since T_0 is the image of a smooth embedding of an open ball in $\mathbb{R}^n$, hence by a diffeomorphism, we may assume $T_0 = B(0,1) \subset \mathbb{R}^n$, $p = 0$, and ρ_0 is the standard Euclidean inner product. Let $\mathcal{S}' = \mathcal{S} \cap T_0$ be the induced strata in T_0. We may choose a linear function u that realize the covector, and a small enough $\delta > 0$, such that u has no critical points on $\mathcal{S}'_\beta \cap B(0,\delta)$ for all positive dimensional strata $\mathcal{S}'_\beta \in \mathcal{S}'$. We want to prove that

$$\mathrm{cone}(r_*F(B(\delta)) \to r_*F(B(\delta) \cap \{u < 0\})) \cong 0,$$

or equivalently,

$$\text{cone}(F(U_+) \to F(U_-)) \cong 0 \tag{5.4.19}$$

where $U_+ = r^{-1}(B(\delta))$ and $U_- = U_+ \cap \{u \circ r < 0\}$. Note $r^{-1}(B(\delta)) = B(q^{-1}(\delta))$ by definition of the almost retraction (5.4.7). By Corollary 5.4.8, $\mathbb{C}_{U_-}$ and $\mathbb{C}_{U_+}$ are Λ^∞-isotopic. By Proposition 5.4.10, we have $SS^\infty(\mathbb{C}_{U_-})$ and $SS^\infty(\mathbb{C}_{U_+})$ are disjoint from $\Lambda[< \epsilon_0]$, hence $\mathbb{C}_{U_-}$ and $\mathbb{C}_{U_+}$ are $\Lambda^\infty[< \epsilon]$-isotopic, in particular $SS^\infty(F)$-isotopic. Thus, by Proposition 5.1.3, we have

$$hom(\mathbb{C}_{U_+}, F) \cong hom(\mathbb{C}_{U_-}, F).$$

This proves Eq.(5.4.19) and finishes the proof of this proposition. $\square$

Corollary 5.4.13. *Let $\epsilon_0 > 0$ be small enough as in Proposition 5.4.9. Let $G \in Sh(M, \Lambda^\infty[> \epsilon_0])$ and $F \in Sh(M, \Lambda^\infty[< \epsilon_0])$, then we have*

$$hom(G, F) \cong hom(G, r_* F)$$

and similarly

$$hom(F, G) \cong hom(r_* F, G)$$

Proof. Let ϵ be the parameter that controls the family of lines. Given a family of lines with parameter ϵ, it naturally induces the same family of lines for smaller ϵ. let $r^{(t\epsilon)}$ be the almost contraction with the parameter $0 \leq t \leq 1$, we claim that $\{r_*^{(t\epsilon)} F\}_{t \in [0,1]}$, is a variation of sheaves, non-characteristic with respect to $\Lambda^\infty[> \epsilon']$, hence to $SS^\infty(G)$. Applying Proposition 5.1.3, we get the desired result. $\square$

5.5. Quantization of Contactomorphism of the Legendrian Complement

In many cases studying the dual object is easier than the original object. For example, to define the weak derivative on distribution one use integration by part and let the derivative acts on the smooth test function. Here the idea is similar, to deform a sheaf such that its singular support at infinity adhere to a prescribed singular Legendrian is hard, but it is much easier to deform a probe sheaf whose singular support at infinity *avoid* the prescribed Legendrian. Just like the GordonLuecke theorem about knot complement, which says if K and K are two knots with homeomorphic complements in three-sphere then there is a homeomorphism of the three-sphere taking one knot to the other, here we shall prove that categories $Sh(M, \Lambda_t^\infty)$ are equivalent to each other as long as the complement $T^\infty M \backslash \Lambda_t^\infty$ are contactomorphic to each other.

Theorem 7. *Let $\Lambda_{\mathbb{R}}^\infty$ be a variation of Legendrian in $T^\infty(M \times \mathbb{R})$. If for any $t, s \in \mathbb{R}$, we have contactomorphism*

$$\varphi_{t\to s} : T^\infty M \backslash \Lambda_t^\infty \to T^\infty M \backslash \Lambda_s^\infty.$$

such that for any $t_1, t_2, t_3 \in \mathbb{R}$,

$$\varphi_{t_2\to t_3} \cdot \varphi_{t_1\to t_2} = \varphi_{t_1\to t_3}, \quad \varphi_{t_1\to t_1} = id,$$

and φ_t change smoothly with t. Then, there are equivalences of categories

$$\widehat{\varphi}_{t\to s} : Sh(M, \Lambda_t^\infty) \to Sh(M, \Lambda_s^\infty),$$

such that for any $t_1, t_2, t_3 \in \mathbb{R}$,

$$\widehat{\varphi}_{t_2 \to t_3} \cdot \widehat{\varphi}_{t_1 \to t_2} \cong \widehat{\varphi}_{t_1 \to t_3}, \quad \widehat{\varphi}_{t_1 \to t_1} \cong id.$$

Remark 5.5.1. This may seems a useless theorem in practice, since the geometric question of finding contactomorphism between non-compact contact spaces might be harder than the algebraic question of finding equivalences of categories, however it maybe useful to prove non-existence result on contactomorphism. Later, we will give some easy to check sufficient conditions that implies the existence of contactomorphisms.

Let $f_t : T^\infty M \backslash \Lambda_t^\infty \to \mathbb{R}$ be a family of smooth functions, such that $f_t \circ \varphi_{s \to t} = f_s$ and $f_t(x) \to \infty$ as $\mathrm{dist}(x, \Lambda_t^\infty) \to 0$ uniformly in x. For example, we may fix $t_0 \in \mathbb{R}$, construct f_{t_0} first, then define $f_t = f_{t_0} \circ \varphi_{t \to t_0}$.

For each $t \in \mathbb{R}$, let X_t be the contact vector field on $T^\infty M \backslash \Lambda_t^\infty$ given by $X_t(x) = \frac{d\varphi_{t,t+\epsilon}(x)}{d\epsilon}$. And let $H_t = \langle \alpha, X_t \rangle$ be the Hamiltonian function generating X_t. Fix a standard smooth cut-off function $\chi : \mathbb{R} \to \mathbb{R}$, i.e. $\chi(x) = 1$ for all $x \leq 1$ and $\chi(x) = 0$ for $x \geq 2$, and $\chi'(x) \leq 0$. For any $R > 0$, we define the truncated Hamiltonian function

$$H_{t,R}(x) := H_t(x) \cdot \chi\left(\frac{f_t(x)}{R}\right),$$

and let $X_{t,R}$ be the corresponding contact vector field on the entire T^*M.

For any $t \in \mathbb{R}$, we let $\epsilon(t)$ be small enough, as in Proposition 5.4.3, such that

$$SS^\infty(P_{p,t}) \cap \Lambda_t^\infty[< \epsilon(t)] = \emptyset, \quad \text{for all } p \in M.$$

We let $R(t)$ be large enough, such that

$$\Lambda_t^\infty[< \epsilon(t)] \subset \{f_t(x) < R(t)\}.$$

This can always be achieved, since $f_t(x) \to \infty$ as $\mathrm{dist}(x, \Lambda_t^\infty) \to 0$ uniformly in x.

From the GKS construction of the quantization for Hamiltonian contactomorphism, for each $R > 0$ and $t, s \in \mathbb{R}$, we have a family of kernels

$$K_{t\to s}^{(R)} \in Sh(M \times M)$$

such that

$$K_{t\to t}^{(R)} = \mathbb{C}_\Delta, \quad K_{t\to s}^{R} = (K_{s\to t}^{R})^t \text{ and } K_{t_2\to t_3}^{(R)} \circ K_{t_1\to t_2}^{(R)} = K_{t_1\to t_3}^{(R)}.$$

However, they only perform the desired quantization for contactomorphism on $\{f_t < R\}$. We will use almost retraction to finish the construction.

Definition 5.5.2. For any $t, s \in \mathbb{R}$, and any $R \in \mathbb{R}$ such that $R > R(t)$ and $R > R(s)$, we define functor

$$\widehat{\varphi}_{t\to s}^{R} : Sh(M) \to Sh(M), \quad F \mapsto (\Pi_{\Lambda_s})_!(K_{t\to s}^{R})_!(\Pi_{\Lambda_t})_!F.$$

Proposition 5.5.3. *If $F \in Sh(M, \Lambda_t^\infty)$, then $SS^\infty(\widehat{\varphi}_{t\to s}^{R}(F)) \in \Lambda_s^\infty$.*

Proof. The Hamiltonian flow $\varphi_{t,s}^{R}$ of $X_{t,R}$ from t to s gives a diffeomorphism $\{f_t < R\}$ to $\{f_s < R\}$, also give a diffeomorphism of the complement $\{f_t > R\}$ and $\{f_s > R\}$.

Hence

$$SS^\infty((K^R_{t\to s})_! F) = \varphi^R_{t\to s}(SS^\infty(F)) \subset \{f_s > R\} \subset \Lambda^\infty_s[<\epsilon(s)].$$

From Proposition 5.4.12, and $(\Pi_{\Lambda_r})_! = (r_s)_!$, we have

$$(\Pi_{\Lambda_s})_!(K^R_{t\to s})_!(\Pi_{\Lambda_t})_! F = (\Pi_{\Lambda_s})_!(K^R_{t\to s})_! F \in Sh(M, \Lambda^\infty_s).$$

□

Proposition 5.5.4. *For any $t, s \in \mathbb{R}$, and any $R \in \mathbb{R}$ such that $R > R(t)$ and $R > R(s)$,*

$$\widehat{\varphi}^R_{t\to s} : Sh(M, \Lambda^\infty_t) \to Sh(M, \Lambda^\infty_s)$$

is independent of R.

Proof. Consider any sheaf in $G \in Sh(M, T^\infty M)$, and any sheaf $F \in Sh(M, \Lambda^\infty_{t_1})$. Then we have

$$\begin{aligned} hom(G, \widehat{\varphi}^R_{t\to s} F) &= hom(G, (\Pi_s) K^R_{t\to s*} F) \\ &= hom(\Pi^*_s G, K^R_{t\to s*} F) \end{aligned}$$

Since as we vary R, $K^R_{t\to s*} F$ will vary such that $SS^\infty(K^R_{t\to s*} F)$ remains in $\Lambda_s[<\epsilon(s)]$. On the other hand, $SS^\infty(\Pi^*_s G)$ has singular support disjoint from $\Lambda_s[<\epsilon(s)]$ by Proposition 5.4.10. Hence, by non-characteristic deformation result in Proposition 5.1.3, the hom complex is invariant as R varies. By faithfulness of Yoneda embedding functor, we see $\widehat{\varphi}^R_{t\to s} F$ is independent of R. □

Remark 5.5.5. We will drop the R supscript from $\widehat{\varphi}^R_{t\to s}$ and simply write $\widehat{\varphi}_{t\to s}$. We will also write Π_t for $\Pi_{\Lambda^\infty_t}$.

Now that we have constructed the functors $\varphi_{t\to s}$, we proceed to finish the proof of the theorem.

Proof of Theorem 7. We only need to verify the compositions of $\widehat{\varphi}_{t\to s}$, since there are only finitely many t_i involved, we can fix an R large enough for all $R(t_i)$. Consider any sheaf in $G \in Sh(M, T^\infty M)$, and any sheaf $F \in Sh(M, \Lambda^\infty_{t_1})$ then we have

$$\begin{aligned} hom(G, \widehat{\varphi}_{t_2\to t_3} \circ \widehat{\varphi}_{t_1\to t_2} F) &= hom(G, \Pi_{t_3!} K_{t_2\to t_3!} \Pi_{t_2!} \Pi_{t_2!} K_{t_1\to t_2!} \Pi_{t_1!} F) \\ &= hom(G, \Pi_{t_3!} K_{t_2\to t_3!} \Pi_{t_2!} K_{t_1\to t_2!} \Pi_{t_1!} F) \\ &= hom(K_{t_3\to t_2!} \Pi^*_{t_3} G, \Pi_{t_2!} K_{t_1\to t_2!} F) \end{aligned}$$

Since $SS^\infty(K_{t_3\to t_2!}\Pi^*_{t_3} G)$ and $SS^\infty(K_{t_1\to t_2!} F)$ satisfies the hypothesis of Corollary 5.4.13, we may take out the almost retraction $\Pi_{t_2!}$ in the last line above on the right slot. Thus, we have

$$\begin{aligned} hom(G, \widehat{\varphi}_{t_2\to t_3} \circ \widehat{\varphi}_{t_1\to t_2} F) &\cong hom(K_{t_3\to t_2!} \Pi^*_{t_3} G, K_{t_1\to t_2!} F) \\ &\cong hom(G, \Pi_{t_3!} K_{t_2\to t_3!} K_{t_1\to t_2!} F) \\ &= hom(G, \widehat{\varphi}_{t_1\to t_3} F). \end{aligned}$$

By faithfulness of Yoneda embedding functor, we have $\widehat{\varphi}_{t_2\to t_3} \circ \widehat{\varphi}_{t_1\to t_2} F \cong \widehat{\varphi}_{t_1\to t_3} F$. $\square$

5.6. Thickening of Legendrian: Definition and Existence

The sufficient condition in Theorem 7 is quite general, though hard to check in practice. We now give an easier to check sufficient condition based on the singular Legendrian itself (or rather, its tubular neighborhood) rather than its complement.

The intuitive idea is that, there should be no *new* self-intersection of the Legendrian during the deformation. Since the initial singular Legendrian itself can be viewed as several component of smooth Legendrians glued together, there are intersections (non-transversal even) between the smooth components to begin with. There are several possible geometric conditions, which we now discuss below. As always, we fix a Riemannian metric on M and hence on S^*M, and identify $T^\infty M$ with S^*M. Let R_t denote the Reeb flow by time t. We also use $\mathcal{L} \subset S^*M$ to denote a Legendrian, $\Lambda \subset T^*M$ a conical Lagrangian, and $\Lambda^\infty \subset T^\infty M$ the corresponding Legendrian at infinity.

(1) The first possible condition is that there is no short Reeb chord emerging or disappearing during the deformation. More precisely, for a deformation of Legendrian $\{\mathcal{L}_s\}$, there exists a small $\epsilon > 0$ so that

$$\mathcal{L}_s \cap R_t(\mathcal{L}_s) = \emptyset, \quad \text{for all } 0 < |t| < \epsilon, \quad \text{uniformly in } s. \tag{5.6.1}$$

This condition prevent self-collision along the Reeb direction, but not when two local pieces of the Legendrian approaches each other within the contact distribution $\ker(\alpha)$. For example, consider two Legendrians in $J^1(\mathbb{R})$ with $\alpha = dz - ydx$, one with the front of $z = 0$, the other with front $z = x^3 + tx$, with parameter $t \in [0, 1]$. As t tends to 0 from above, the second Legendrian approaches the first one, at meets when $t = 0$, and there is no short Reeb chord with non-zero length between them for all $t \in [0, 1]$. The resulting sheaf categories are different for $t = 0$ and $t > 0$. Hence, this condition alone is not enough to guarantee the sheaf categories to be invariant.

(2) The second possible condition is to strenghen the first one, by requiring the singular Legendrian $\mathcal{L}_s$ to be contained in a hypersurface $\mathcal{H}_s$ transverse to the Reeb flow, such that $\mathcal{H}_s$ deformation retracts to $\mathcal{L}_s$ and

$$\mathcal{H}_s \cap R_t(\mathcal{H}_s) = \emptyset, \quad \text{for all } 0 < |t| < \epsilon, \quad \text{uniformly in } s. \tag{5.6.2}$$

We call such hypersurface a *thickening* of $\mathcal{L}$. Of course the condition of a hypersurface transverse to Reeb flow is not an intrinsic notion on a contact manifold, since for any point p in a contact manifold C, any vector R not in the contact distribution can be made into a Reeb vector by choosing some contact one-form.

We will show that the second condition above (with some Liouville flow condition on $\mathcal{H}_s$) is enough to construct a family tubular neighborhoods U_s for $\mathcal{L}_s$, by thicken $\mathcal{H}_s$ in the (positive and negative) Reeb flow direction, such that there is a contact vector field flowing across the tubular boundary ∂U_s. This condition is enough to show that the complement of the tubular neighborhoods $\{U_s\}$ are contactomorphic to each other, hence allows one to show equivalence of sheaf categories just as the Theorem 7.

In the remaining part of this section, we will first review some elementary properties about hypersurface in contact manifold and tubular neighborhood around singular Legendrian. Then, we construct a tubular neighborhood for any singular Legendrian. The result is summarized in Theorem 8.

5.6.1. Contact Hamiltonian vector field

Let M be a smooth manifold, T^*M its cotangent bundle with the canonical one-form $\alpha = pdq$, two form $\omega = d\alpha = dp \wedge dq$, and the outward Liouville vector field (or the Euler vector field) $V = p\partial_p$ such that $\iota_V \omega = \alpha$. Given a Hamiltonian function H, the Hamiltonian flow ξ_H is defined by $\iota_{\xi_H}\omega = -dH$. We will use ξ_H for sympletic Hamiltonian vector field, and X_H for contact Hamiltonian vector fields.

Let g be any Riemaninan metric on M, then T^*M has induced norm. Let $\dot{T}^*M = T^*M \backslash M$, where M is identified with the zero section in T^*M. Let $S^*M = \{(q,p) \in T^*M \mid |p| = 1\}$ be the unit cosphere bundle, the one-form α restrict to S^*M to be a contact form, and the contact distribution $\xi = \ker(\alpha)$. In fact, any smooth function $f : S^*M \to \mathbb{R}$ defines another hypersurface H_f, by flowing every point in $p \in S^*M$ along the expanding Liouville vector field V by time $f(p)$, and let $P_f : S^*M \to H_f$ denote this diffeomorphism. Then $(H_f, \alpha|_{H_f})$ is also a contact manifold, contactomorphic to $(S^*M, \alpha|_{S^*M})$ via P_f, and $P_f^*(\alpha|_{H_f}) = e^f \cdot \alpha|_{S^*M}$.

The Reeb vector field R corresponding for a contact one-form α, is such that

$$\iota_R \alpha = 1, \quad \iota_R d\alpha = 0. \tag{5.6.3}$$

Its flowline is the characteristic foliation of the hypersurface S^*M with respect to ω. And R is also the restriction of the symplectic Hamiltonian flow for function $|p|$ on its level set $|p| = 1$.

Given a smooth function $H : S^*M \to \mathbb{R}$, the contact Hamiltonian vector field X_H is defined by

$$X_H = H \cdot R + X_H^{\parallel} \in \mathbb{R}R \oplus \xi, \quad \iota_{X_H} d\alpha = \langle H, R \rangle \alpha - dH \tag{5.6.4}$$

Proposition 5.6.1 ([Ge] Theorem 2.3.1)**.** *With a fixed choice of contact form α there is a one-to-one correspondence between infinitesimal automorphisms X of $\xi = \ker \alpha$ and smooth functions $HM \to \mathbb{R}$. The correspondence is given by*

$$X \mapsto H = \langle \alpha, X \rangle, \quad H \mapsto X_H.$$

Alternatively, one can think in terms of homoegenous Hamiltonian vector field, extending H on S^*M to $\dot{T}^*M$ as a homogeneous degree-one function

$$H|p| : \quad \dot{T}^*M \to \mathbb{R} \tag{5.6.5}$$

then take the usual symplectic Hamiltonian vector field $\xi_{H|p|}$. This flow $\xi_{H|p|}$ is conic, in that it commutes with the fiberwise scaling action by $\mathbb{R}_{>0}$. However it does not perserve the hypersurface S^*M but only level sets of $H|p|$. Let π_{S^*M} denote the projection of $T(T^*M)|_{S^*M} \cong T(S^*M) \oplus \mathbb{R}V$ onto the factor $T(S^*M)$, then we can recover the contact Hamiltnonian flow X_H on S^*M by restricting $\xi_{H|p|}$ on S^*M then projection away the radial component $\mathbb{R}V$. We have proved the following lemma:

Lemma 5.6.2. *Let $H : S^*M \to \mathbb{R}$ be any smooth function, with X_H the contact Hamiltonian vector field for H. Let $H|p|$ be the homogeneous degree-one function on*

$\dot{T}^*M$ *with* $\xi_{H|p|}$ *the symplectic contact vector field, then*

$$X_H = \pi_{S^*M}(\xi_{H|p|}|_{S^*M}).$$

Lemma 5.6.3.

$$\langle X_H, dH\rangle = H\langle R, dH\rangle \tag{5.6.6}$$

Proof. Since $X_H = HR + X_H^{\parallel}$, where $X_H^{\parallel} \in \ker(\alpha)$, we have

(5.6.7)

$$\langle X_H - HR, dH\rangle = \langle X_H - HR, dH - \alpha\rangle = \langle X_H - HR, -\iota_{X_H}(d\alpha)\rangle = d\alpha(X_H - HR, X_H) = 0$$

where we have used $R \in \ker(d\alpha)$. □

5.6.2. Hypersurface Thickening of a Legendrian

Let $(C, \xi = \ker(\alpha))$ be a co-oriented contact manifold with fixed contact form α and Reeb vector field R, $\mathcal{L}$ a Legendrian in C, we will construct hypersurface $\mathcal{H}$ containing $\mathcal{L}$ such that $\mathcal{H}$ is transverse to R.

Let $\mathcal{H}$ be any smooth hypersurface transverse to R, we identify $U_{\mathcal{H},\epsilon:} = \mathcal{H} \times (-\epsilon, +\epsilon)$ with a neighborhood of $\mathcal{H}$ via Reeb flow for small enough ϵ, and smooth function

$$H : U_{\mathcal{H},\epsilon} \to \mathbb{R}, \quad (x,t) \mapsto t.$$

In particular, we have $R(H) = 1$, and $\mathcal{H} = \{H = 0\}$. We have the following property

Proposition 5.6.4. *We use the above notation.*

(1) The contact flow X_H generated by H preserves $\mathcal{H}$.

(2) $\mathcal{H}$ is an exact symplectic manifold with the one-form $\alpha|_{\mathcal{H}}$, with the Liouville flow equal to the restriction of the contact flow $X_H|_{\mathcal{H}}$. If $\mathcal{H}$ is tangential to $\xi = \ker(\alpha)$ at $p \in \mathcal{H}$, then the Liouville flow on $\mathcal{H}$ vanishes at p.

(3) If $\mathcal{L}$ is a Legendrian contained in $\mathcal{H}$, then $\mathcal{L}$ is an exact Lagrangian in $\mathcal{H}$ with $\alpha|_{T\mathcal{L}} = 0$, and is invariant under the Liouville flow.

(4) Let $\pi_{\mathcal{H}} : U_{\mathcal{H},\epsilon} \to \mathcal{H}$ be the projection along the Reeb trajectory, then the contact form α is

$$\alpha = dH + \pi_{\mathcal{H}}^*(\alpha|_{\mathcal{H}}).$$

Proof. (1) This follows from Lemma 5.6.3 and $\langle R, H\rangle = 1$. On $\mathcal{H}$, we have $\langle X_H, dH\rangle = H\langle H, R\rangle = 0$, hence X_H preserves the zero set of H.

(2) Since R is transversal to $\mathcal{H}$ and R span $\ker(d\alpha)$, we have $d\alpha|_{\mathcal{H}} = d(\alpha|_{\mathcal{H}})$ is non-degenerate. To compute the Liouville field, we notice that

$$\iota_{X_H}(d\alpha)|_{\mathcal{H}} = (\langle H, R\rangle\alpha - dH)|_{\mathcal{H}} = \alpha \tag{5.6.8}$$

where we have used $\langle H, R\rangle = 1$ and $dH|_{\{H=0\}} = 0$.

(3) From the definition of Legendrian, we have $T\mathcal{L} \subset \ker(\alpha)$, hence $\alpha|_{T\mathcal{L}} = 0$. For any $p \in \mathcal{L}$ and $Y \in T_p\mathcal{L}$, we have

$$d\alpha_p(X_H, Y) = \alpha_p(Y) = 0 \tag{5.6.9}$$

hence $(X_H)_p \in (T_p\mathcal{L})^\perp = T_p\mathcal{L}$, hence X_H is tangential to $\mathcal{L}$.

(4) Let $(x, t) \in \mathcal{H} \times (-\epsilon, +\epsilon)$, we may decompose the tangent space at this point as $\langle \partial_t \rangle \oplus T_x\mathcal{H}$. When tested against ∂_t, we use $\partial_t = R$ to get equality. When tested against $Y \in T_x\mathcal{H}$, we may apply $\exp(-tR)$ to go from (x, t) to $(x, 0)$. Since $\mathcal{L}_R\alpha = 0$, $\exp(-tR)^*\alpha = \alpha$, we have

$$\alpha_{(x,t)}(Y) = [\exp(-tR)^*\alpha]_{(x,t)}(Y) = \alpha_{(x,0)}(\exp(-tR)_*Y) = \alpha|_{\mathcal{H}}(Y). \tag{5.6.10}$$

□

Remark 5.6.5. The above statement still holds if H and $\mathcal{H}$ are only C^1.

Next, we construct hypersurface containing singular Legendrian. To fix idea, we consider the special case when the Legendrian $\mathcal{L}$ is smooth. Locally, there is a neighborhood U of $\mathcal{L}$, and a embedding $\iota : U \hookrightarrow J^1(\mathcal{L})$, sending $\mathcal{L}$ to the zero section, such that $\iota^*(\alpha_{J^1\mathcal{L}}) = \alpha$, where $J^1(\mathcal{L}) \cong \mathbb{R} \times T^*\mathcal{L}$ is equipped with contact form $\alpha_{J^1\mathcal{L}} = dz - pdq$ (see e.g. Theorem 6.2.2 in [Ge]). Thus, we may work in a tubular neighborhood of $\mathcal{L}$ in $J^1\mathcal{L}$. The local hypersurface can be taken as $\{z = 0\}$ in U, we see it is indeed transverse to $R = \partial_z$.

In the above construction, the hypersurface $\mathcal{H}$ is smooth and contains $\mathcal{L}$ as the fixed point for the contracting Liouville flow: $-X_H = -p\partial_p - z\partial_z$ in local coordinate of $J^1\mathcal{L} = \mathbb{R}_z \times T^*\mathcal{L}_{(q,p)}$. If $\mathcal{L}$ is singular, in general we cannot achieve both conditions that $\mathcal{H}$ is smooth and $\mathcal{H}$ tangent to the contact distribution ξ along $\mathcal{L}$. There are two options, we either give up smoothness of $\mathcal{H}$, or we allow the Liouville flow along $\mathcal{L}$ to be non-zero. We only give results in the first direction.

Definition 5.6.6. Let $\mathcal{L}$ be a singular Legendrian in a contact manifold $(C, \alpha, \xi = \ker(\alpha))$. We say $(\mathcal{H}, U)$ is a *local C^1-hypersurface thickening of $\mathcal{L}$ in U* if

(1) U is a tubular neighborhood of $\mathcal{L}$, $\mathcal{H}$ is a C^1 hypersurface in U, and $\mathcal{L} \subset \mathcal{H}$,

(2) $\mathcal{H}$ is transverse to the Reeb vector field R,

(3) $\alpha_{\mathcal{H}} := \alpha|_{T\mathcal{H}}$ vanishes exactly on $\mathcal{L}$.

Proposition 5.6.7. *Let $\mathcal{L}$ be a compact singular Legendrian in S^*M, then there exists a C^1-hypersurface thickening of $\mathcal{L}$.*

Proof. We first prove a local version. For any $p \in \mathcal{L}$, we identity a neighborhood U of p with a neighborhood of origin $J^1\mathbb{R}^n$ with contact form $\alpha = dz - \sum_i y_i dx_i$, hence suffice to construct a hypersurface containing a singular Legendrian in $J^1\mathbb{R}^n$. We write $\mathcal{L} \cap U$ as $\mathcal{L}$ as a local singular Legendrian in $J^1\mathbb{R}^n$. Take the Lagrangian projection $\pi : J^1\mathbb{R}^n \to T^*\mathbb{R}^n$, then $\pi(\mathcal{L})$ is a singular exact Lagrangian L in $T^*\mathbb{R}^n$ near the origin. If $\pi : \mathcal{L} \to L$ is not bijective, then there is a Reeb chord corresponding to the self-intersection of L, since the Reeb chords has lower bound on length, we may shrink the neighborhood around the origin so that there is no Reeb chord. Note that since $0 \in \mathcal{L}$, we have $0 \in L$.

We now define a function $f : L \to \mathbb{R}$, such that

$$\begin{cases} f(0) = 0 \\ df|_{TL} = (ydx)|_{TL}. \end{cases}$$

In fact, for any point $(x, y) \in L$, $f(x, y)$ is the z-value of the corresponding point in $\mathcal{L}$ under the Lagrangian projection h. Our goal here is to extend the definition of L to a neighborhood of the origin in $T^*\mathbb{R}^n$, with additional condition that for any $(x, y) \in L$ and any $(v_x, v_y) \in T_{(x,y)}(T^*\mathbb{R}^n)$, we have

$$\langle df(x,y), (v_x, v_y)\rangle = \langle \alpha, (v_x, v_y)\rangle = \sum_i y_i (v_x)_i.$$

In other words, we prescribe the first derivatives of f along L. This extension problem can be achieved by the Whitney extension theorem (see e.g. [**?**], Theorem 2.3.6), which gives a C^1 function $F(x,y)$ with prescribed first derivative on L, and smooth away from L. Then the local hypersurface $\mathcal{H}$ is defined by $\{(x, y, F(x,y)) \mid (x,y) \in W\}$ for sufficiently small neighborhood W of the origin.

One still need to glue the locally constructed the hypersurfaces together. This can be done by standard partition of unity and we omit the detail here. $\square$

5.6.3. Convex Tubular Neighborhood of Singular Legendrian

Given a C^1-hypersurface $\mathcal{H}$ transverse to the Reeb flow, we may construct a C^1 Hamiltonian function H locally near $\mathcal{H}$, such that $\{H = 0\}$ and $R(H) = 1$. The Hamiltonian vector field X_H is well-defined, but will be only be C^0. The C^0 vector field $-X_H$ vanishes on $\mathcal{L}$ and is a local attracting basin.

Proposition 5.6.8. *There exists a function $\rho : U \to [0,1)$ (possibly for smaller U), such that*

(1) $\rho|_{\mathcal{L}} = 0$, $\rho|_{\partial U} = 1$ and $\rho|_{U \setminus \mathcal{L}}$ is positive and smooth with no critical point

(2) X_H is gradient-like for ρ, i.e. $\langle X_H, d\rho\rangle > \delta(|X_H|^2 + |d\rho|^2)$ for some positive δ.

Proof. Suffice to construct nested family of smooth hypersurfaces transverse to X_H converging to $\mathcal{L}$, then let ρ be defined with these as level sets. $\square$

The next proposition says we can smooth the construction while keeping the attracting property of vector field $-X_H$.

Proposition 5.6.9. *Let (C, α) be a contact manifold with fixed contact one-form α. Let $\mathcal{L}$ be any singular Legendrian in C, and $(\mathcal{H}, U)$ be a C^1 hypersurface thickening of $\mathcal{L}$, and $\rho : U \to \mathbb{R}$ a defining function for $\mathcal{L}$ satisfying properties in Proposition 5.6.8. Then for any $\epsilon > 0$, we may find a smooth Hamiltonian function $\widetilde{H}$, such that $X_{\widetilde{H}}$ is gradient like for ρ on $\{x \in U : \rho(x) \geq \epsilon\}$.*

Proof. We take a smoothing $\widetilde{\mathcal{H}}$ of $\mathcal{H}$, that is C^1-close to the original $\mathcal{H}$ but may no longer contain $\mathcal{L}$, and define a Hamiltonian function $\widetilde{H}$ and vector field $X_{\widetilde{H}}$. Then $X_{\widetilde{H}}$ is a vector field C^0 close to X_H, hence the transversality of $X_{\widetilde{H}}$ to level sets of ρ with $\rho \geq \epsilon$ can be perserved if $\widetilde{H}$ is close enough to H, since $\{\rho \geq \epsilon\}$ is a compact set. $\square$

We collect the above construction into the following definition and Theorem.

Definition 5.6.10. Let (C, α) be a contact manifold with fixed contact one-form α. A *convex tubular neighborhood* of a Legendrian $\mathcal{L}$ is the following data $(U, \rho, H, \mathcal{H})$:

(1) U is an open neighborhood of $\mathcal{L}$, that admit a deformation retract to $\mathcal{L}$.

(2) $\rho : U \to [0, 1)$ with continuous extension to $\overline{U}$, such that $\rho|_{\mathcal{L}} = 0$, $\rho|_{\partial U} = 1$ and $\rho|_{U \setminus \mathcal{L}}$ is smooth and with $d\rho \neq 0$.

(3) $H : U \to \mathbb{R}$ is a C^1-function, such that X_H is gradient-like for ρ,

(4) $\mathcal{H} = \{x \in U : H(x) = 0\}$ is a C^1-hypersurface thickening of $\mathcal{L}$. (c.f. Definition 5.6.6)

Theorem 8. *For any singular Legendrian $\mathcal{L}$, there exists a convex tubular neighborhood $(U, \rho, H, \mathcal{H})$ for $\mathcal{L}$. For any non-zero level set $\rho^{-1}(c)$ of ρ, there is a smooth contact vector field $X_{\widetilde{H}}$ transverse to it.*

Proof. This follows from Proposition 5.6.7 and 5.6.8, and the existence of smooth transverse contact vector field is due to Proposition 5.6.9. □

5.7. Quantization of Variation of Thickened Legendrian

In previous subsections, we saw that any singular Legendrian there exists a convex tubular neighborhood in the sense of Definition 5.6.10. However, as the singular Legendrian varies the neighborhood may not be able to vary continuously. We show that if there exists a continuous deformation of the convex tubular neighborhoods as well, then there exists a contactomorphism of the complement of the convex tubular neighborhoods, and we can quantize the variation of Legendrians.

First we define what is a variation of a convex tubular neighborhood for a variation of Legendrian. Here it is more convenient to work in $T^*M \times \mathbb{R}$ instead of $T^*(M \times \mathbb{R})$, and similarly $T^\infty M \times \mathbb{R}$ instead of $T^\infty(M \times \mathbb{R})$. Given a variation of Legendrian $\Lambda^\infty_\mathbb{R}$ in $T^\infty(M \times \mathbb{R})$, we define $\mathcal{L}(\Lambda^\infty_\mathbb{R}) \subset T^\infty M \times \mathbb{R}$ by

$$\mathcal{L}(\Lambda^\infty_\mathbb{R}) := \cup_{t \in \mathbb{R}} \Lambda^\infty_t \times \{t\} \subset S^*M \times \mathbb{R}$$

Definition 5.7.1. Let $\Lambda^\infty_\mathbb{R}$ be a variation of Legendrian in $T^\infty(M \times \mathbb{R})$, and $\mathcal{L}_\mathbb{R} = \mathcal{L}(\Lambda^\infty_\mathbb{R}) \subset S^*M \times \mathbb{R}$. A *variation of convex tubular neighborhood* for $\Lambda^\infty_\mathbb{R}$ is the following data $(U_\mathbb{R}, \rho_\mathbb{R}, H_\mathbb{R}, \mathcal{H}_\mathbb{R})$, where $U_\mathbb{R}$ is a tubular neighborhood of $\mathcal{L}_\mathbb{R}$, $\mathcal{H}_\mathbb{R} \subset U_\mathbb{R}$ is a hypersurface, and $\rho_\mathbb{R}, H_\mathbb{R}$ are functions on $U_\mathbb{R}$. Let $U_t = U_\mathbb{R}|_{S^*M \times \{t\}}$, and $\rho_t, H_t, \mathcal{H}_t$ be the restriction of $\rho_{\mathbb{R}}, H_{\mathbb{R}}, \mathcal{H}_{\mathbb{R}}$ on U_t, then we require $(U_t, \rho_t, H_t, \mathcal{H}_t)$ to be a convex tubular neighborhood of $\mathcal{L}_{\mathbb{R}}$ and is constant for $t \ll 0$ and $t \gg 0$.

Theorem 9. *Let $\Lambda^{\infty}_{\mathbb{R}}$ be a variation of Legendrian in $T^{\infty}(M \times \mathbb{R})$. If $\Lambda^{\infty}_{\mathbb{R}}$ admits variation of convex tubular neighborhood $(U_{\mathbb{R}}, \rho_{\mathbb{R}}, H_{\mathbb{R}}, \mathcal{H}_{\mathbb{R}})$, then there exists equivalence of categories*

$$\widehat{\varphi}_{t \to s} : Sh(M, \Lambda^{\infty}_t) \to Sh(M, \Lambda^{\infty}_s)$$

that is identity when $t = s$ and compatible with composition:

$$\widehat{\varphi}_{t_1 \to t_2} \circ \widehat{\varphi}_{t_0 \leftarrow t_1} \cong \widehat{\varphi}_{t_0 \to t_2}, \quad \widehat{\varphi}_{t \to t} \cong Id. \tag{5.7.1}$$

We prove the above theorem analogously as in Theorem 7. The only difference is one need to construct contactomorphism across different t.

5.7.1. Construction of the slice reproducing kernel

Let M be a smooth compact manifold. For any $t \in \mathbb{R}$, let $M_t = M \times \{t\} \subset M \times \mathbb{R}$ be the t-slice, and $j_t : M_t \hookrightarrow M \times \mathbb{R}$ be the inclusion. Let $\Lambda^{\infty}_{\mathbb{R}} \subset S^*(M \times \mathbb{R})$ be a variation of Legendrian, with slice Legendrian $\Lambda^{\infty}_t \in S^* M_t$. Let $\mathcal{S}_{\mathbb{R}}$ be projection image of $\Lambda^{\infty}_{\mathbb{R}}$ in $M \times \mathbb{R}$ and $\mathcal{S}_t$ the image for Λ^{∞}_t.

Let $\mathcal{S}_{\mathbb{R}}$ be equipped with a minimal Whitney stratification

$$\mathcal{S}_{\mathbb{R}} = \bigcup_{\alpha \in A_{\mathbb{R}}} \mathcal{S}_{\alpha},$$

and $\mathcal{S}_t$ be equipped with the induced stratification $\mathcal{S}_{t,\alpha} = \mathcal{S}_{\alpha} \cap M_t$, and let $A_t \subset A_{\mathbb{R}}$ consists of α such that $\mathcal{S}_{t,\alpha} \neq \emptyset$. We assume that the restriction of the projection map

$\pi_{\mathbb{R}} : M \times \mathbb{R} \to \mathbb{R}$ to each stratum $\mathcal{S}_\alpha$ with positive dimension is non-singular. Let $\{t_1, \cdots, t_N\}$, $t_1 < t_2 \cdots < t_N$, be the projection images of the zero-dimensional strata, which we assume to be distinct, and let $t_0 = 0, t_{N+1} = 1$. Then the stratification $\mathcal{S}_t$ is topologically constant over intervals (t_i, t_{i+1}). (cf [N1], §3.7).

Fix a $t \in \mathbb{R}$, let $(T_\alpha, \rho_\alpha, \pi_\alpha, r_\alpha)_{\alpha \in A_t}$ be a control system with a family of lines, for the Whitney stratification $\mathcal{S}_t$, as in §5.4. Let r_t be the almost retraction for M_t, and let $\Pi_{\Lambda_t^\infty} = \mathbb{C}_{\Gamma(r_t)}$ be the reproducing kernel for $Sh(M_t, \Lambda_t^\infty)$ as in §5.3.

Let ϵ_t' denote a lower bound between the distance of the singular support of the probe sheaves in M_t defined by r_t and the Legendrian Λ_t^∞, as given in Proposition 5.4.9. We note that ϵ_t' does not have a uniform lower bound over $t \in \mathbb{R}$, it may tends to zero as $t \to t_i$ for $i = 1, \cdots N$.

For each Λ_t^∞, let $(U_t, \rho_t, H_t, \mathcal{H}_t)$ be a convex tubular neighborhood of $\mathcal{L}_t$, as defined in Section 5.6.3, varying smoothly and compactly in t. Let $\widetilde{H}_t(q,p)$ be the homogeneous degree-one extension on $\dot{T}^*M_t$, and $\widehat{H}(q,p,t) = \widehat{H}_t(q,p)$ the function on $\dot{T}^*M \times \mathbb{R}$. We may extend φ to be 1 outside U.

5.7.2. Symplectic fibration over $\mathbb{R}_t \times \mathbb{R}_s$

Let $E = \dot{T}^*M \times \mathbb{R}_t \times \mathbb{R}_s$ be the total space, $B = \mathbb{R}_t \times \mathbb{R}_s$ be the base. Here $\mathbb{R}_t$ is the deformation direction, and $\mathbb{R}_s$ is the direction that generates the Hamiltonian flow.

Let $T^*M = \{(q,p) \mid q \in M, p \in T_q^*M\}$. Let $\omega_{T^*M} = dp \wedge dq$ be the standard symplectic form (up to sign), and let

$$\Omega_E = \pi_{T^*M}^* \omega_{T^*M} - d\widehat{H}(q,p,t) \wedge ds. \tag{5.7.2}$$

be a two-form that restricts to each fiber of $E_{t,s}$ of $(t,s) \in B$ is non-degenerate. This gives $\pi_B : E \to B$ a sympletic bundle structure.

The Ω_E orthogonal complement for each fiber of E defines a horizontal distribution in TE, called the symplectic connection. The horizontal lift is given by

$$\partial_t \mapsto \partial_t \in TE, \quad \partial_s \mapsto \partial_s + \xi_{\widehat{H}} \in TE$$

where our sign convention is $\iota_{\xi_H}(\omega) = -dH$. Indeed for any vertical tangent vector $Y \in TE|_v$, we have

$$\Omega_E(Y, \partial_s + \xi_{\widehat{H}}) = \omega_{T^*M}(Y, \xi_H) - d\widehat{H}(q,p,t)(Y) = 0 \tag{5.7.3}$$

For each $r \in (0,1)$, let

$$C_2(r) = \inf\{\langle X_{H_t}, \rho(q,p,t)\rangle \mid (q,p,t) \in U, \rho(q,p,t) = r\} > 0 \tag{5.7.4}$$

$$C_3(r) = \sup\{|\langle \partial_t, \rho(q,p,t)\rangle| \mid (q,p,t) \in U, \rho(q,p,t) = r\} \geq 0 \tag{5.7.5}$$

and let $C_4(r) > 0$ be large enough, such that $C_4(r)C_2(r) - C_3(r) > 1$.

Definition 5.7.2 (Admissable for level r)**.** We say a tangent vector $a\partial_t + b\partial_s \in TB$ is admissible for level r, if $b > C_4(r)|a|$. If a smooth path on B has all its tangent vectors admissible for level r, we say the path is admissible for level r. A piecewise smooth path is admissible for level r if its each smooth component is.

Proposition 5.7.3. *For any piecewise smooth path $\gamma : [0,1] \to I \times \mathbb{R}$, the symplectic parallel transport from $\dot{T}^*M_{\gamma[0]}$ to $\dot{T}^*M_{\gamma[1]}$ is conic, and induces a contactomorphism (isotopic to identity) from $S^*M_{\gamma[0]}$ to $S^*M_{\gamma[1]}$. Furthermore, if γ is admissible for level r, then it will send $S^*M_{\gamma[0]} \backslash \Lambda^\infty_{\gamma[0]}[< r]$ into $S^*M_{\gamma[1]} \backslash \Lambda^\infty_{\gamma[1]}[< r]$.*

Proof. For any (q,p,t) that $\rho(q,p,t) = r$, we have

$$\langle a\partial_t + bX_H, \rho(q,p,t)\rangle \geq |a|C_3(r) + bC_2(r) > |a|(C_3(r) + C_4(r)C_2(r)) > |a| \geq 0 \tag{5.7.6}$$

Hence for a point in U with initial value ρ above r, its value will never get below r along the parallel transport trajectory. □

Paths in B can be translated in the s variable, hence an admissible path of level r from t_0 to t_1 means a level r path from (t_0, s_0) to (t_1, s_1) for some $s_1 > s_0$.

Paths can be concatenated in the obvious way. By definition, piecewise smooth level r path will concatenate into piecewise smooth level r path.

The space of all level r path with the same endpoints (t_0, s_0) and (t_1, s_1) are contractible.

Fix any $r \in (0,1)$. We may shrink ϵ'_t, such that $\epsilon'_t < r$ for all $t \in \mathbb{R}$.

For any $t \in \mathbb{R}$, there is a large enough $T_t > 0$, such that the a straightline path γ_t from $(t,0)$ to (t,T_t) sends $\Lambda^\infty_t[> \epsilon'_t]$ into $\Lambda^\infty_t[> r]$. We may choose T_t as

$$T_t = \frac{r - \epsilon'_t}{\inf\{\langle X_{H_t}, \rho_t(q,p)\rangle \mid (q,p) \in U_t, \rho_t(q,p) \in [\epsilon'_t, r]\}} \tag{5.7.7}$$

For any $t_0, t_1 \in \mathbb{R}$, we may build a path $\gamma_{t_1 \leftarrow t_0}$ from t_0 to t_1 admissible for level r. For example, we can take the straightline from $(t_0, 0)$ to $(t_1, |t_1 - t_0|C_4(r))$.

To summarize the above construction, we make the following definitions:

Definition 5.7.4. (1) For any $t \in \mathbb{R}$, let γ_t be the straightline path from $(t, 0)$ to (t, T_t); ϕ_t be the contactomorphism induced on S^*M that sends $\Lambda_t^\infty[> \epsilon_t']$ into $\Lambda_t^\infty[> r]$; and $K_t \in Sh(M \times M)$ be the GKS kernel for ϕ_t.

(2) For any $t, s \in \mathbb{R}$, let $\gamma_{t\to s}$ be the straightline path from $(t_0, 0)$ to $(t_1, |t_1 - t_0|C_4(r))$; $\phi_{t\to s}$ be the contactomorphism induced on S^*M that sends $\Lambda_t^\infty[> r]$ into $\Lambda_s^\infty[> r]$; and $K_{t\to s} \in Sh(M \times M)$ be the GKS kernel for $\phi_{t\to s}$.

Proposition 5.7.5. *Fix $t_0, t_1 \in \mathbb{R}$. Let γ_a, γ_b be two level-r path from t_0 to t_1. Fix an isotopy in level r path $h_u(l) : [0,1]\times[0,1] \to B$, where $u \in [0,1]$, such that $h_0 = \gamma_a, h_1 = \gamma_b$. Let K_u be the 1-parameter family of kernels in $Sh(M_{t_1} \times M_{t_0})$ with parameter in u, such that K_u is the GKS quantization of the sympletic parallel transport over the path $h_u \circ \gamma_{t_0}$. Then, for any $F \in Sh(M, \Lambda_{t_1}^\infty[< r])$, $p \in M$, $u_1, u_2 \in [0,1])$, we have*

$$hom(K_{u_0!}P_{p,t_0}, F) \cong hom(K_{u_1!}P_{p,t_0}, F)$$

Proof. The one-parameter family of probe sheaves $K_{u!}P_{p,t_0}$ over parameter u defined a variation of sheaves in $\Lambda_{t_1}^\infty[< r])$, hence is $SS^\infty(F)$-non-characteristic. □

5.7.3. Constructing the parallel transport kernel

As before, we denote the reproducing kernel for slice t by Π_t, and K_t and $K_{t\to s}$ are the GKS kernels defined in Definition 5.7.4. We use subscript $t \to s$ and $s \leftarrow t$ interchangeably. We define kernel $\Phi_{t_1\leftarrow t_0}$, by

$$\Phi_{t_1\leftarrow t_0} = \Pi_{t_1} \circ K_{t_1}^t \circ K_{t_0\leftarrow t_1}^t, \tag{5.7.8}$$

recall K^t represent the transpose of the kernel (cf. Section 3.6)

Proposition 5.7.6. *If* $F \in Sh(M_{t_0}, \Lambda^\infty_{t_0}[< r])$, *then* $\Phi_{t_1 \leftarrow t_0 *} F \in Sh(M_{t_1}, \Lambda^\infty_{t_1})$.

Proof. Since $(K^t)_* = K^!$, we have

$$\Phi_{t_1 \leftarrow t_0 *} F = \Pi_{\Lambda^\infty_{t_1} *} \circ K^!_{t_1} \circ K^!_{t_0 \leftarrow t_1} F. \tag{5.7.9}$$

Then since $K_!$ is adjoint to $K^!$, and $SS^\infty((K_\varphi)_! F) = \varphi SS^\infty(F)$, hence $SS^\infty((K_\varphi)^! F) = \varphi^{-1} SS^\infty(F)$, and we have

$$SS^\infty(K^!_{t_1} \circ K^!_{t_0 \leftarrow t_1} F) = \phi^{-1}_{t_1} \circ \phi^{-1}_{t_0 \leftarrow t_1} SS^\infty(F). \tag{5.7.10}$$

Since $\phi_{t_0 \leftarrow t_1}$ sends $\Lambda^\infty_{t_1}[> r]$ into $\Lambda^\infty_{t_0}[> r]$, then $\phi^{-1}_{t_0 \leftarrow t_1}$ pullback the complement $\Lambda^\infty_{t_0}[\leq r]$ into $\Lambda^\infty_{t_1}[\leq r]$. Similarly, $\phi^{-1}_{t_1}(\Lambda^\infty_{t_1}[< r]) \subset \Lambda^\infty_{t_1}[< \epsilon'_{t_1}]$. Hence, we have

$$K^!_{t_1} \circ K^!_{t_0 \leftarrow t_1} F \in Sh(M_{t_1}, \Lambda^\infty_{t_1}[< \epsilon'_{t_1}]) \tag{5.7.11}$$

Apply Proposition 5.4.12 to $(r_{t_1})_*$, we get

$$r_{t_1 *} K^!_{t_1} \circ K^!_{t_0 \leftarrow t_1} F \in Sh(M_{t_1}, \Lambda^\infty_{t_1}) \tag{5.7.12}$$

$\square$

Next, we prove Eq (5.7.1).

Proposition 5.7.7. *Given an isotopy of level* r *paths*

$$h : \gamma_{t_2 \leftarrow t_1} \circ \gamma_{t_1} \circ \gamma_{t_1 \leftarrow t_0} \circ \gamma_{t_0} \rightsquigarrow \gamma_{t_2 \leftarrow t_0} \circ \gamma_{t_0}$$

and given any $G \in Sh(M)$, we have

$$hom(G, \Phi_{t_2 \leftarrow t_1 *} \circ \Phi_{t_1 \leftarrow t_0 *} F) \cong hom(G, \Phi_{t_2 \leftarrow t_0 *} F)$$

natural in G. Hence there is an isomorphism $\Phi_{t_2 \leftarrow t_1 *} \circ \Phi_{t_1 \leftarrow t_0 *} F \to \Phi_{t_2 \leftarrow t_0 *} F$.

Proof. The proof is analogous to the proof of Theorem 7. We will be brief here.

$$\begin{aligned}
& hom(G, \Phi_{t_2 \leftarrow t_1 *} \circ \Phi_{t_1 \leftarrow t_0 *} F) \\
\cong \ & hom(K_{t_1 \leftarrow t_2 !} \circ K_{t_2 !} \circ \Pi^*_{t_2}(G),\ \Pi_{t_1 *} \circ K^!_{t_1} \circ K^!_{t_0 \leftarrow t_1} F) \\
\cong \ & hom(K_{t_1 \leftarrow t_2 !} \circ K_{t_2 !} \circ \Pi^*_{t_2}(G),\ K^!_{t_1} \circ K^!_{t_0 \leftarrow t_1} F) \\
\cong \ & hom(K_{t_0 \leftarrow t_1 !} \circ K_{t_1 !} \circ K_{t_1 \leftarrow t_2 !} \circ K_{t_2 !} \circ \Pi^*_{t_2}(G),\ F) \\
\cong \ & hom(K_{t_0 \leftarrow t_2 !} \circ K_{t_2 !} \circ \Pi^*_{t_2}(G),\ F) \\
= \ & hom(G, \Phi_{t_2 \leftarrow t_0 *} F)
\end{aligned}$$

where in the third line above, we applied Corollary 5.4.13, to drop $\Pi_{t_1 *}$ on the right slot; and in the fifth line above, we apply the Proposition 5.7.5 to quantize the isotopy that changes the composition $\phi_{t_0 \leftarrow t_1} \circ \phi_{t_1} \circ \phi_{t_1 \leftarrow t_2}$ to $\phi_{t_0 \leftarrow t_2}$. This proves the first statement, and a standard Yoneda faithfulness argument gives the second statement. □

This also conclucdes the proof of the Theorem 9.

CHAPTER 6

Variation of Constructible Sheaves: II

In the last chapter, we studied two sufficient conditions on deformation of Legendrians in $T^{\infty}M$, such that the corresponding sheaf categories are invariant. While being quite general, those conditions are still hard to verify in practice. Here we consider a special case of Legendrian deformation, Legendrians supported on affine hyperplanes on $\mathbb{R}^n$ (c.f. Example 5.0.3).

Results from this section will be used in the proof of non-equivariant coherent-constructible correpondence, and in the proof of Fukaya-Seidel category equivalent to constructible sheaf category on a torus, since in both cases we need to vary the singular support on the torus $\Lambda_{\mathcal{T},\Theta}$ by changing the function Θ (see the notation section in the introduction for the definition of $\Lambda_{\mathcal{T},\Theta}$).

The main idea is illustrated in the following example.

Example 6.0.1. We still consider the following deformation of sheaves on $\mathbb{R}^2$.

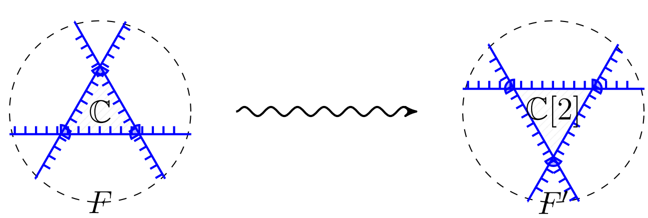


Figure 6.1. As the Legendrian moves, the sheaf F changes to F'.

The generators for $Sh(X, \Lambda_{-1})$ and $Sh(X, \Lambda_{+1})$ are shown in Figure 6.2. It is clear that $hom(P_i, P_j) \cong hom(P_i', P_j')$ for all i, j, hence there is an equivalence of categories. F is quasi-isomorphic to the following chain complex

$$F \cong (P_0 \to P_1 \oplus P_2 \oplus P_3 \to P_4 \oplus P_5 \oplus P_6)$$

with P_0 at degree -2 and maps given by the obvious restriction maps, and F' is quasi-isomorphic to a similar complex with P_i replaced by P_i'. Hence F is sent to F' under the equivalence.

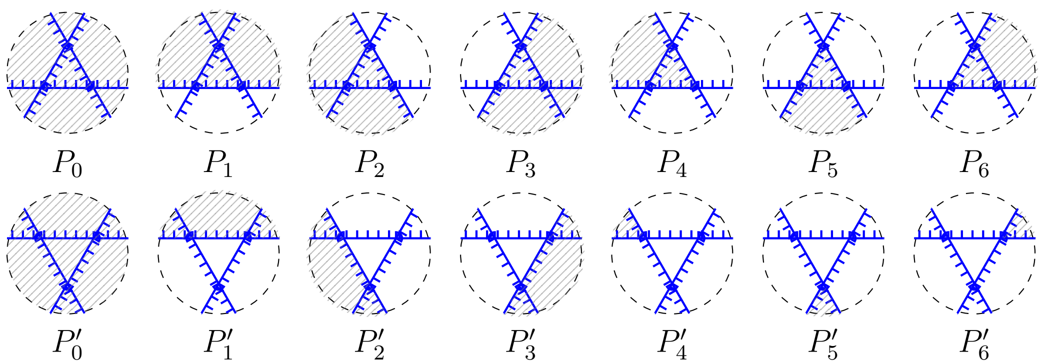


Figure 6.2. The generators for $Sh(X, \Lambda_{-1})$ (first row) and $Sh(X, \Lambda_{+1})$ (second row), as standard sheaves supported on closed set marked by the shaded regions. Under the deformation of Legendrian, P_i changes to P_i'.

$\triangle$

Our main theorem in this chapter is the following

Theorem 10. *Let M be a smooth compact manifold, $\Lambda_{\mathbb{R}}^{\infty} \in T^{\infty}(M \times \mathbb{R})$ a variation of Legendrian. If $M \times \mathbb{R}$ admits a Cech covering with product open sets $\{U_i \times I_i\}_i$, such* *that on each patch the Legendrian deformation is diffeomorphic to admissible deformation of hyperplane arrangements on $\mathbb{R}^n$, then the sheaf categories are invariant under deformation.*

We will first study the deformation of affine hyperplanes (by translation only) as the local picture. Then we study how to glue the local sheaves (of complexes) into a global sheaf. Finally, we prove the theorem by patching the local sheaf deformations.

6.1. Affine Hyperplanes on Vector Space

6.1.1. Legendrian supported on affine hyperplanes.

Let $V \cong \mathbb{R}^n$ be an n-dimensional real vector space, V^* be the dual space. Let $\langle , \rangle : V \times V^* \to \mathbb{R}$ denote the canonical pairing.

Let $v_1, \cdots, v_m$ be nonzero covectors in V^*, and $\Omega = \{1, \cdots, m\}$. Let S be a collection of subsects of Ω, such that

(1) $S = \sqcup_{k=0}^n S_k$, where S_k is a (possibly empty) collection of size k subsets of Ω.
(2) $S_0 = \{\emptyset\}$ and $S_1 = \{\{1\}, \cdots, \{m\}\}$.
(3) If $\sigma_k \in S_k$, then $\{v_i : i \in \sigma_k\}$ are linearly independent.
(4) If $\sigma_k \in S_k$, then any non-empty subsets of σ_k is also in S.

Then S is a partially ordered set, with $\sigma_1 \leq \sigma_2$ if and only if $\sigma_1 \subset \sigma_2$.

Pick m real numbers $b_1, \cdots, b_m \in \mathbb{R}$. Using the co-vectors v_i and the 'offsets' b_i, we may define the closed half-spaces Q_i and their boundaries H_i for for all $1 \leq i \leq m$

$$Q_i = \{x \in \mathbb{R}^n : \langle x, v_i \rangle \geq b_i\}, \quad H_i = \partial Q_i = \{x \in \mathbb{R}^n : \langle x, v_i \rangle = b_i\}.$$

For each $\sigma \in S$, we define the closed 'corners' Q_σ, and their 'spine' H_σ

$$Q_\sigma = \bigcap_{i \in \sigma} Q_i, \quad H_\sigma = \bigcap_{i \in \sigma} H_i.$$

Note that $H_\sigma \neq \partial Q_\sigma$. We define $Q_\emptyset = V$ and $H_\emptyset = V$.

For any $\sigma \in S$, let

$$v_\sigma = \text{cone}\{v_i \mid i \in \sigma\} = \{\sum_{i \in \sigma} a_i v_i \mid a_i \geq 0\} \subset V^*$$

be the closed cone, with $v_\emptyset = 0$. And we define the conical Lagrangian

$$\Lambda = \bigcup_{\sigma \in S} \Lambda_\sigma \subset T^*V \quad \text{where} \quad \Lambda_\sigma = H_\sigma \times v_\sigma \subset V \times V^* \cong T^*V.$$

Finally, let

$$\mathcal{L}_\sigma = \text{Leg}(\Lambda_\sigma) \quad \text{and} \quad \mathcal{L} = \text{Leg}(\Lambda),$$

be the corresponding Legendrian for the conical Lagrangians Λ_σ and Λ. For example, if $\sigma = \{i\}$, then $\mathcal{L}_\sigma$ consists of unit covectors with foots on H_i and pointing in the v_i direction. And if $\sigma = \{i, j\}$, then $\mathcal{L}_\sigma$ consists of unit covectors with foots on the codimension-2 affine linear subspace $H_i \cap H_j$, and pointing in the direction within the cone spanned by v_i and v_j.

Let $b = (b_1, \cdots, b_m) \in \mathbb{R}^m$, and we will write the subscript b explicitly, as in $H_{\sigma,b}$ and $\mathcal{L}_b$, when we want to emphasize the dependence on b. We call the above data, $\{(v_i, b_i)\}_{i=1,\cdots,m}, S$, a *hyperplane arrangement*, and denoted by $\mathcal{H}_{v,b,S}$, or sometimes only $\mathcal{H}_b$ if v, S are clear from the context.

We will consider the category of constructible sheaves on V, with singular support contained in Λ_b, denoted as $Sh(V, \Lambda_b)$, or $Sh(V, \mathcal{L}_b)$. As the offset parameter b changes, the Legendrian $\mathcal{L}_b$ also changes, and we are interested in how a sheaf $F \in Sh(V, \mathcal{L}_b)$ changes along with b.

For a generic choice of b, the hyperplanes have transversal intersection, then as b undergoes a small perturbation, the stratification induced by $\mathcal{L}_b$ has the same structure, hence the constructible sheaf F has a natural continuation. As b changes along a path through a critical moment b_0, when a non-generic intersection occurs, then some strata will disappear as b approaches b_0 from one side, and some new strata will apear as b leaves b_0 from the other side, the question then is how to define F on the new strata.

The idea is to resolve F using sheaves that each admits an obvious deformation as b changes. One choice of such sheaves are standard sheaves supported on the closed sets Q_σ for $\sigma \in S$. More precisely, let $j_\sigma : Q_\sigma \hookrightarrow V$ be the closed embedding, and $\mathbb{C}_{\mathbb{Q}_\sigma}$ be the constant sheaf with stalk $\mathbb{C}$ on Q_σ, then we define

$$P_\sigma := (j_\sigma)_* \mathbb{C}_{Q_\sigma}, \quad \text{with stalks at } x \quad (P_\sigma)_x = \begin{cases} \mathbb{C} & \text{if } x \in Q_\sigma \\ 0 & \text{otherwise} \end{cases}.$$

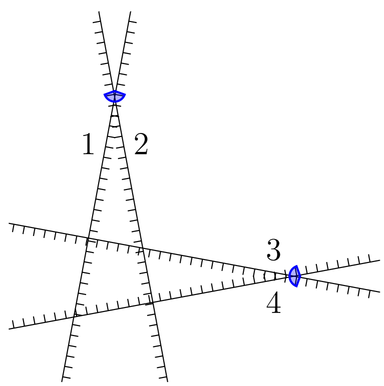
The problem then reduces to the two following steps:

(1) Show that all the sheaves can be generated using $\{P_\sigma \mid \sigma \in S\}$, and

(2) Show that as b chanages, the full subcategory with objects $\{P_\sigma \mid \sigma \in S\}$ is invariant.

If $\sigma \subset \tau$, then $Q_\sigma \supset Q_\tau$, then there is a restriction morphism between the standard sheaves $P_\sigma \to P_\tau$. These morphism are induced by the poset relation of S, and is stable under changes of b. However, there may be other morphisms, which is sensitve to b, as the following examples shows. In this case, the step (2) above would fail.

Example 6.1.1. Consider the following Legendrian, given by four co-oriented hyperplanes H_i for $i = 1, \cdots, 4$ and the corner at $1, 2$ and $3, 4$.

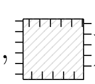


$$\begin{tikzcd}
 & P_1 \arrow[dr] & \\
 & & P_{12} \arrow[dddl, dotted] \arrow[ddd, dotted, bend left] \\
 & P_2 \arrow[ur] & \\
P_0 \arrow[uuur] \arrow[ur] \arrow[dr] \arrow[dddr] & & \\
 & P_3 \arrow[dr] & \\
 & & P_{34} \arrow[uuul, dotted] \arrow[uuu, dotted, bend left] \\
 & P_4 \arrow[ur] &
\end{tikzcd}$$

The standard sheaves are

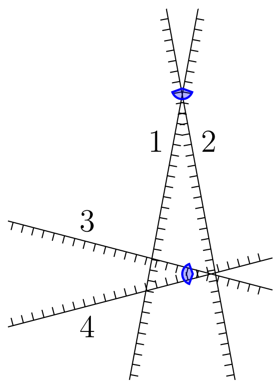
$$P_0, \quad P_1, \cdots, P_4, \quad P_{12}, P_{34}$$

where $P_0 = \mathbb{C}_{\mathbb{R}^2}$, P_i supported on the half-space Q_i for $i = 1, \cdots, 4$, and P_{ij} supported on $Q_i \cap Q_j$ for $ij = 12, 34$. The solid arrows in the diagram (and their compositions which are not drawn) represent the homs induced by restriction morphism, e.g. $\mathrm{Hom}(P_0, P_1) \cong \mathbb{C}$ is generated by the restriction morphism $P_0 \to P_1$. These morphisms are stable under changes of the offset parameter b. The dotted arrows represent other homs

$$hom(P_{12}, P_{34}) = \Gamma(\mathbb{R}^2, \underline{hom}(P_{12}, P_{34})) = \Gamma(\mathbb{R}^2,$$  $)$

$$\cong \quad C^\bullet([0,1] \times [0,1], \{0,1\} \times [0,1]; \mathbb{C}) \cong \mathbb{C}[-1].$$

The dashed arrows are not stable under changes of b. For example, if we shift the horiztonal wedge to the left, then we get



$$
\begin{tikzcd}
 & P_1 \arrow[r] & P_{12} \arrow[ddl, dotted] \\
 & P_2 \arrow[ur] \arrow[dr, dotted] & \\
P_0 \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] & P_3 \arrow[r] & P_{34} \\
 & P_4 \arrow[ur] &
\end{tikzcd}
$$

Here $P_2 \to P_{34}$ is given by restriction, since $Q_{34} \subset Q_2$. $\triangle$

Here we do not know if the $\{P_\sigma\}$ generates the sheaf categories associated to the two Legendrians; even if they do, by matching the generators will not induce an equivalence of categories, since the morphisms between the generators changed as b changed. This motivates the following conditions on the hyperplane arrangments.

Definition 6.1.2. We say a hyperplane arrangment $\mathcal{H}_{v,b,S}$ is admissible, if there exists a fan Σ in V^*, such that the set of cones in Σ has a one-to-one correspondence with $\{\sigma \in S\}$, by $\sigma \mapsto v_\sigma$.

We will sometimes denote the cone v_σ by σ, to be consistent with usual convention.

Remark 6.1.3. In the example above, the cones $v_{\{1,2\}}$ and $v_{\{3,4\}}$ have intersecting interiors, hence cannot be fit into a fan, thus the hyperplane arrangment is not admissible. As pointed out in [**N3**] by Nadler, the change of the hom space is related with the change of the short Reeb trajectories starting and ending on $\mathcal{L}_b$. In this case, as the Legendrian moves from the first picture to the second one, the Reeb chord from $\mathcal{L}_{\{3,4\}}$ to $\mathcal{L}_{\{2\}}$, the interval from $H_3 \cap H_4$ and ending perpendicularly on H_2, disappears and is replaced by a Reeb chord from $\mathcal{L}_{\{2\}}$ to $\mathcal{L}_{\{3,4\}}$.

The above condition is first studied by [**FLTZ1**] as condition (Z1) in Theorem 5.2, the (Z2) condition there is automatically satisfied by condition (4) in the definition of S. They considered more general situations, which allow for multiple hyperplanes with the same co-vectors. The simpler case we study here will serve as the local model, e.g. in a small open ball, hence we do not allow for multiple occurence of a cone in Σ.

The following two results from [**FLTZ1**] will be used.

Proposition 6.1.4. *If $\mathcal{H}_{v,b,S}$ is an admissible hyperplane arrangment with fan Σ. Then for any cones $\sigma, \tau \in \Sigma$, we have the following hom-complex*

$$hom(P_\sigma, P_\tau) = \begin{cases} \mathbb{C} \cdot \rho_{\sigma \to \tau} & \text{if } \sigma \subset \tau \\ 0 & \text{otherwise.} \end{cases}$$

where $\rho_{\sigma\to\tau} : P_\sigma \to P_\tau$ is the canonical restriction morphism. In particular, the hom-space is independent of the offset parameter b.

Proof. This follows from [**FLTZ1**] Proposition 3.3. Here we note that, $\sigma \subset \tau$ if and only if $Q_\sigma \supset Q_\tau$. ☐

Proposition 6.1.5. *The category $Sh(V, \Lambda_b)$ is generated by $\{P_{\sigma,b} \mid \sigma \in \Sigma\}$.*

Proof. This is proved in [**FLTZ1**], Theorem 5.2. ☐

Explicitly, any sheaf $F \in Sh(V, \Lambda_b)$ admits a resolution using P_σ's:

(6.1.1)
$$[F] := \left(\cdots \to \bigoplus_{\sigma_1 < \sigma_2 \in S} P_{\sigma_1} \otimes hom(P_{\sigma_1}, P_{\sigma_2}) \otimes hom(P_{\sigma_2}, F) \to \bigoplus_{\sigma_1 \in S} P_{\sigma_1} \otimes hom(P_{\sigma_1}, F) \right)$$

6.1.2. Universal deformation space for hyperplane arrangements

Let $\mathcal{H}_{v,b,S}$ be an admissible hyperplane arrangement with fan Σ, where $v = (v_1, \cdots, v_m) \in (V^*)^m$, and $b = (b_1, \cdots, b_m) \in \mathbb{R}^m$.

Let $\widetilde{V} = V \times \mathbb{R}^m$ and $V_b = V \times \{b\} \cong V$ for each $b \in \mathbb{R}^m$. Let $\iota_b : V \cong V_b \hookrightarrow \widetilde{V}$ be the inclusion, and $\pi_V : \widetilde{V} \to V$ be the projection. We define the closed half-spaces and their intersections

$$\widetilde{Q}_i = \{(x, b) \in V \times \mathbb{R}^m : \langle x, v_i \rangle \geq b_i\}, \quad \widetilde{Q}_\sigma = \bigcap_{i \in \sigma} \widetilde{Q}_i$$

and similarly

$$\widetilde{H}_i = \{(x, b) \in V \times \mathbb{R}^m : \langle x, v_i \rangle = b_i\} = \partial \widetilde{Q}_i, \quad \widetilde{H}_\sigma = \bigcap_{i \in \sigma} \widetilde{H}_i.$$

Let $\widetilde{V}^*$ be the dual space of $\widetilde{V}$ and let $\pi_{V^*} : \widetilde{V}^* \to V^*$ be the projection. We lift $v_i \in V^*$ as

$$\widetilde{v}_i = (v_i, -e_i) \in V^* \times (\mathbb{R}^m)^* = \widetilde{V}^*.$$

Then as v_i spans cones in Σ, we may define the lifted fan $\widetilde{\Sigma} \in \widetilde{V}^*$ as

$$\widetilde{\Sigma} = \{\widetilde{v}_\sigma : \sigma \in S\} \quad \text{where} \quad \widetilde{v}_\sigma = \text{cone}\{\widetilde{v}_i : i \in \sigma\}.$$

Then we have an admissible hyperplane arrangments $\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}_{\widetilde{v},\widetilde{b}=0,S}$ with fan $\widetilde{\Sigma}$. Let the conical Lagrangian $\widetilde{\Lambda}$ and the associated Legendrian $\widetilde{\mathcal{L}}$ be defined as

$$\widetilde{\Lambda} = \bigcup_{\sigma \in S} \widetilde{H}_\sigma \times \widetilde{v}_\sigma \subset T^*\widetilde{V}, \quad \widetilde{\mathcal{L}} = \mathrm{Leg}(\widetilde{\Lambda}).$$

Let $\widetilde{P}_\sigma$ be the standard sheaf on the closed set $\widetilde{Q}_\sigma$. By Proposition 6.1.4, we have

$$\mathrm{Hom}(\widetilde{P}_\sigma, \widetilde{P}_\tau) \cong \begin{cases} \mathbb{C} & \text{if } \sigma \subset \tau \\ 0 & \text{otherwise} \end{cases}, \quad \text{for any } \sigma, \tau \in S. \tag{6.1.2}$$

And by Proposition 6.1.5, $Sh(\widetilde{V}, \widetilde{\Lambda})$ is generated by $\{\widetilde{P}_\sigma \mid \sigma \in S\}$.

Proposition 6.1.6. *With the above notation, the restriction functor*

$$\iota_b^* : Sh(\widetilde{V}, \widetilde{\Lambda}) \to Sh(V_b, \Lambda_b)$$

is an quasi-equivalence of dg derived category, where $\iota_b : V_b \hookrightarrow \widetilde{V}$ in the inclusion.

Proof. ι_b^* sends the generators $\widetilde{P}_\tau$ to $P_{\tau,b}$, while preserving the morphisms and their compositions, hence induces an quasi-equivalence of dg derived category. $\square$

We denote the inverse of ι_b^* by

$$\epsilon_b = (\iota_b^*)^{-1} : Sh(V_b, \Lambda_b) \to Sh(\widetilde{V}, \widetilde{\Lambda}),$$

that sends generators $\{P_{\tau,b}\}$ to generators $\{\widetilde{P}_\tau\}$.

Corollary 6.1.7. *For any $b_1, b_2 \in \mathbb{R}^m$, we have an equivalence of category*

$$\Phi_{b_1\to b_2} = \epsilon_{b_1} \circ \iota_{b_2}^* : Sh(V, \Lambda_{b_1}) \to Sh(V, \Lambda_{b_2})$$

such that

$$\Phi_{b_1\to b_1} \cong \mathrm{id}, \quad \Phi_{b_1\to b_2} \circ \Phi_{b_2\to b_3} \cong \Phi_{b_1\to b_3},$$

where compositions are from left to right, and $\cong$ means natural isomorphism of functors.

Proof. $\Phi_{b_1\to b_2}$ is a composition of equivalence of categories. In particular, it sends generators P_{τ,b_1} to P_{τ,b_2}, and induces isomorphism

$$hom(P_{\sigma,b_1}, P_{\tau,b_1}) \cong hom(P_{\sigma,b_2}, P_{\tau,b_2}) \cong \begin{cases} \mathbb{C} & \text{if } \sigma \subset \tau \\ 0 & \text{otherwise} \end{cases},$$

such that the restriction morphism $P_{\sigma,b_1} \to P_{\tau,b_1}$ for b_1 goes to the corresponding restriction morphism for b_2, if $\sigma \subset \tau$.

To be more concrete about the natural equivalence, for $F \in Sh(V, \Lambda_b)$, we consider the resolution $\eta_F : [F] \xrightarrow{\sim} F$ as in Eq. (6.1.1). Then $\Phi_{b_1\to b_2}(F)$ is defined by first build the resolution $[F]$, then replace $P_{\sigma,b_1} \otimes \cdots$ by $P_{\sigma,b_2} \otimes \cdots$ for all $\sigma \in \Sigma$ in the above resolution. Thus we have the natural equivalence $\eta : \Phi_{b_1\to b_1} = [\cdot] \cong \mathrm{id}$:

$$\eta_F : \Phi_{b_1\to b_1}(F) = [F] \to F = \mathrm{id}(F)$$

We note that $M = \iota_b^* \circ \epsilon_b = [\cdot] : Sh(\widetilde{V}, \widetilde{\Lambda}) \to Sh(\widetilde{V}, \widetilde{\Lambda})$ is naturally equivalent to id, with $\eta : M \to \mathrm{id}$. Then

$$\Phi_{b_1 \to b_2} \circ \Phi_{b_2 \to b_3} = \epsilon_{b_1} \circ \iota_{b_2}^* \circ \epsilon_{b_2} \circ \iota_{b_3}^* = \epsilon_{b_1} \circ M \circ \iota_{b_3}^* \xrightarrow{\epsilon_{b_1} \circ \eta \circ \iota_{b_3}^*} \epsilon_{b_1} \circ \mathrm{id} \circ \iota_{b_3}^* = \Phi_{b_1 \to b_3}.$$

□

6.1.3. Cut-off and Extension functor

So far we have discussed sheaves on a linear space V. This can be used as a local model for the Legendrian deformation over the torus T_M under study here. Thus, we need to be able to go back and forth between sheaves defined on the entire linear space V, and sheaves defined on a 'extendable' open set U of V, where extendable is going to be defined shortly.

Let $\mathcal{H}_{v,b,S}$ be an admissible hyperplane arrangement on V with fan Σ. Let $i_U : U \hookrightarrow V$ be an open inclusion of an open subset U of V. The cut-off functor is the restriction

$$i_U^* : Sh(V, \Lambda) \to Sh(U, \Lambda|_U).$$

However, if U is too small, say contained in an open stratum of the stratification given by Λ, then $Sh(U, \Lambda|_U) \cong Loc(U)$ is a local system on U, and i_U^* would fail to be an equivalence of category. Hence we are interested in open sets U such that i_U^* admits an inverse ϵ_U,

$$\epsilon_U : Sh(U, \Lambda|_U) \xrightarrow{\sim} Sh(V, \Lambda),$$

and we call these open sets U *extendable* (for $\mathcal{H}_{v,b,S}$).

One special case is easy to consider, that is when the hyperplanes H_i in $\mathcal{H}_{v,b,S}$ all pass through the origin of V. In this case, sheaves in $Sh(V,\Lambda)$ is invariant under the $\mathbb{R}_{>0}$ dilation action on V, and any convex open set U containing the origin is extendable.

Let $\mathcal{H}_b = \mathcal{H}_{v,b,S}$ be an admissible hyperplane arrangement, $\mathcal{S}_b = \mathcal{S}(\Lambda_{v,b,S})$ be the induced Whitney stratification with $\mathcal{S}^k$ the set of dimension k strata. Then all the strata are convex polytopes, some possibly non-compact.

Proposition 6.1.8. *If U is a convex open set that intersects with all the strata of $\mathcal{S}$, then U is extendable for $\mathcal{H}$.*

Proof. Let $\mathcal{S}|_U$ denote the stratification of U by $\{\mathcal{S}_\alpha \cap U \mid \alpha \in A\}$. By the proposition hypothesis, for any $\alpha \in A$, $\mathcal{S}_\alpha \cap U$ is convex and non-empty. Let $P_{\alpha,U}$ denote standard sheaves with stalk $\mathbb{C}$ support on $\mathcal{S}_\alpha \cap U$, then

$$hom(P_{\alpha,U}, P_{\beta,U}) = \begin{cases} \mathbb{C} & \text{if } \overline{\mathcal{S}_\alpha \cap U} \supset (\mathcal{S}_\beta \cap U) \\ 0 & \text{otherwise.} \end{cases}$$

If $\overline{\mathcal{S}_\alpha} \supset \mathcal{S}_\beta$, then $\overline{\mathcal{S}_\alpha \cap U} \supset \overline{\mathcal{S}_\alpha} \cap U \supset \mathcal{S}_\beta \cap U$. Conversely, if $\overline{\mathcal{S}_\alpha \cap U} \supset (\mathcal{S}_\beta \cap U)$, then $\overline{\mathcal{S}_\alpha} \cap \mathcal{S}_\beta \neq \emptyset$, hence $\overline{\mathcal{S}_\alpha} \supset \mathcal{S}_\beta$. Hence we have

$$hom(P_{\alpha,U}, P_{\beta,U}) \cong hom(P_\alpha, P_\beta).$$

Let $Sh(V,\mathcal{S})$ be the category of constructible sheaves on V with stratification $\mathcal{S}$, and $Sh(U,\mathcal{S}|_U)$ the corresponding restriction. We first claim that

$$\iota_U^* : Sh(V,\mathcal{S}) \xrightarrow{\sim} Sh(U,\mathcal{S}|_U)$$

is an equivalence of category. Since both categories are generated by standard sheaves supported on convex strata, and restriction matches the generators and preserves the homs and compositions, hence $\iota_{\mathcal{S}}^*$ is an equivalence. Let ϵ_U be the inverse functor of $\iota_{\mathcal{S}}^*$,

$$\epsilon_U : Sh(U, \mathcal{S}|_U) \xrightarrow{\sim} Sh(V, \mathcal{S}),$$

and be called the *extension functor*.

Since $Sh(V, \Lambda)$ is a full subcategory of $Sh(V, \mathcal{S})$, we have a fully-faithful functor

$$\iota_U^* : Sh(V, \Lambda) \hookrightarrow Sh(U, \Lambda|_U).$$

To show it is essentially surjective, we want to show that ϵ_U applied to sheaves in $Sh(U, \Lambda|_U)$ will land in the subcategory $Sh(V, \Lambda)$.

For any stratum $\mathcal{S}_\alpha$, let $x \in \mathcal{S}_\alpha$ and $\xi \in T^*_{\mathcal{S}_\alpha} V|_x \backslash \Lambda|_x$. Let $y \in \mathcal{S}_\alpha \cap U$. Since $\mathcal{S}_\alpha$ is a convex polytope, hence $\xi \in T^*_{\mathcal{S}_\alpha} V|_y \cong T^*_{\mathcal{S}_\alpha} V|_x$, and $\gamma(t) = ((1-t)x + ty, \xi)$ for $t \in [0,1]$ is a path from (x, ξ) to (y, ξ) in the smooth part of $T^*_{\mathcal{S}_\alpha} V$. Let $F \in Sh(U, \Lambda|_U)$, and $F' = \epsilon_{\mathcal{S}}(F) \in Sh(V, \mathcal{S})$. Since the cohomology of the microlocal stalk of F' is locally constant along the path γ, and vanishing at $\gamma(1) = (y, \xi)$, hence it vanishes at $\gamma(0) = (x, \xi)$. Thus $F' \in Sh(V, \Lambda)$. $\square$

Proposition 6.1.9. *Let U be a convex neighborhood of $0 \in V$. Then there is a neighborhood W of $0 \in \mathbb{R}^m$, such that for any $b \in W$, U is extendable for $\mathcal{H}_{v,b,S}$.*

Proof. For any $b \in \mathbb{R}^m$, and any subset $I \subset \{1, \cdots, m\}$ such that $\{v_i \mid i \in I\}$ are linearly independent, we define the intersection of hyperplanes $H_{I,b} = \cap_{i \in I} H_{i,b}$. Let $x_{I,b} \in H_{I,b}$ be the points that is closest to 0. For example, if $b = 0$, then $x_{I,0} = 0 \in V$ for all $H_{I,0}$. Let $r(b) := \max_I |x_{I,b}|$, then $r(b)$ depends on b continuously, $r(b) = 0 \iff b = 0$ and $r(\lambda b) = \lambda r(b)$.

Claim: for any $r > r(b)$, $B(0,r)$ intersects all the strata of $\mathcal{S}_{v,b,S}$.

Indeed, if $\mathcal{S}_\alpha$ is a closed strata (minimal under inclusion relation), then $\mathcal{S}_\alpha = H_{I,b}$ for some intersection of hyperplane, and $B(0,r) \cap \mathcal{S}_\alpha \ni x_{I,\alpha}$ hence is not empty. For any stratum $\mathcal{S}_\beta$, there exists a minimal strata $\mathcal{S}_\alpha \subset \overline{\mathcal{S}_\beta}$, hence $B(0,r) \cap \overline{\mathcal{S}_\beta} \neq \emptyset$, hence $B(0,r) \cap \mathcal{S}_\beta \neq \emptyset$. This finishes the proof of the claim.

Let r be small enough, such that $B(0,r) \subset U$. Take W open neighborhood of b small enough, such that for all $b \in W$, $r(b) < r$. Thus for all $b \in W$, U intersects all the strata of $\mathcal{S}_{v,b,S}$, hence is extendable for $\mathcal{H}_{v,b,S}$ by Proposition 6.1.8. This finishes the proof of the proposition. □

Corollary 6.1.10. . *Let U be an open set in V and W be a contractible open set in $\mathbb{R}^m$, such that for any $b \in W$, U is extendable for $\mathcal{H}_{v,b,S}$. Then for any $b_1, b_2 \in W$, there is a canonical equivalence of category*

$$\Phi_{b_1 \to b_2, U} = \epsilon_U \circ \epsilon_{b_1} \circ \iota_{b_2}^* \circ \iota_U^* : Sh(U, \Lambda_{b_1}|_U) \to Sh(U, \Lambda_{b_2}|_U)$$

such that

$$\Phi_{b_1 \to b_1, U} \cong \mathrm{id}, \quad \Phi_{b_1 \to b_2, U} \circ \Phi_{b_2 \to b_3, U} \cong \Phi_{b_1 \to b_3, U},$$

where compositions are from left to right, and $\cong$ means natural isomorphism of functors.

Proof. This follows from the definition of 'extendable' open set, and the Corollary 6.1.7. □

6.2. Gluing Sheaves of Complexes

Let X be a topological space and let $\mathscr{U} = \{U_i\}_{i \in A}$ be an open cover of X index by a finite set A. In this section, we want to build a sheaf (of complexes) F on X from sheaves F_i on U_i, together with some gluing data.

The complexity arises when the restrictions of sheaves on different patches, $F_i|_{U_i \cap U_j}$ and $F_j|_{U_i \cap U_j}$, are not isomorphic as sheaf of set, but only isomorphic in the dg derived category. For example, if $x \in U_i \cap U_j$, then on the stalk level, one only has zig-zag of quasi-isomorphism of chain complex of $\mathbb{C}$-vector spaces $(F_i)_x \xrightarrow{\text{q-iso}} (F_j)_x$ in $Sh(\{x\})$ means $(F_i)_x \xleftarrow{\text{q-iso}} B_1 \xrightarrow{\text{q-iso}} B_2 \xleftarrow{\text{q-iso}} \cdots B_k \xrightarrow{\text{q-iso}} (F_j)_x$ in $Sh_{\text{naive}}(\{x\})$, instead of an honest (bijection) isomorphism of chain complexes $(F_i)_x \xrightarrow{\text{iso}} (F_j)_x$.

In the remaining part of the paper, we will always work with dg derived category of sheaves $Sh(X)$, with quasi-isomorphism denoted as $F \xrightarrow{\text{q-iso}} G$ or $F \xrightarrow{\sim} G$, meaning isomorphism in $H(Sh(X))$.

6.2.1. Gluing Sheaf of Sets

First we review the simpler case of gluing sheaf of sets, where the gluing data are honest isomorphism with tricycle condition on the nose. We specify the following local data: for each $i \in A$, let F_i be a sheaf on U_i, and for each $i, j \in A$, let

$$\varphi_{i \to j} : F_i|_{U_i \cap U_j} \xrightarrow{\text{iso}} F_j|_{U_i \cap U_j}$$

be an isomorphism, such that for distinct triples $i, j, k \in A$, such that $U_i \cap U_j \cap U_k \neq \emptyset$, we have tricycle condition

$$\varphi_{i\to j} \circ \varphi_{j\to k} \circ \varphi_{k\to i} = \mathrm{id} : (F_i)_x \xrightarrow{\text{iso}} (F_i)_x, \quad \forall x \in U_i \cap U_j \cap U_k.$$

Then, we may define the global sheaf F as the equalizer, that is for any open set $U \subset X$, we define

$$F(U) = \mathrm{eq}\left(\prod_i F_i(U \cap U_i) \Longrightarrow \prod_{i \neq j} F_i(U \cap U_i \cap U_j) \right).$$

6.2.2. Cech Resolution

Let F be a sheaf over X, valued in sets or chain complexes. We have the following Cech resolution of F (e.g. see Kashiwara-Schapira [KS], §2.8). Fix a total ordering of A (for the purpose of having correct signs), and for each subset $I \subset J$, we let $U_I = \cap_{i \in I} U_i$, then $U_I \subset U_J$ when $I \supset J$. Let $j_U : U \hookrightarrow X$ be the inclusion, and let $j_{U!}\mathbb{C}_U$ be the constant sheaf with stalk $\mathbb{C}$ supported on the open set U, and for $U \subset V$ let $\rho_{U\hookrightarrow V} : j_{U!}\mathbb{C}_U \to j_{V!}\mathbb{C}_V$ denote the canonical morphism. Then we have a resolution of $\mathbb{C}_X$:

$$P_\bullet := \left(\cdots \xrightarrow{d} \bigoplus_{I_1} j_{U_{I_1}!}\mathbb{C}_{U_{I_2}} \xrightarrow{d} \bigoplus_{I_0} j_{U_{I_0}!}\mathbb{C}_{U_{I_0}} \to 0 \right), \qquad I_k \subset A, \quad |I_k| = k+1$$

where P_k corresponds to the direct sum over I_k and the differential on direct summand P_{I_k} is given by

$$d|_{P_{I_k}} = \sum_{j=0}^{k} (-1)^j \rho_{U_{I_k} \hookrightarrow U_{I_k - v_j}}, \quad \text{where } I_k = \{v_0, v_1, \cdots, v_k\}, v_0 < v_1 < \cdots < v_k.$$

Using $F \cong F \otimes \mathbb{C}_X$ or $F \cong \underline{hom}(\mathbb{C}_X, F)$, we get resolution of F by replacing $\mathbb{C}_X$ with $P_\bullet$:

$$\mathscr{C}_\bullet(\mathscr{U}; F) := F \otimes P_\bullet = \left(\cdots \xrightarrow{d} \bigoplus_{I_2} j_{U_{I_2}!} F|_{U_{I_2}} \xrightarrow{d} \bigoplus_{I_1} j_{U_{I_1}!} F|_{U_{I_1}} \to 0 \right),$$

and

$$\mathscr{C}^\bullet(\mathscr{U}; F) := \underline{hom}(P_\bullet, F) = \left(0 \to \bigoplus_{I_1} j_{U_{I_1}*} F|_{U_{I_1}} \xrightarrow{d} \bigoplus_{I_2} j_{U_{I_2}*} F|_{U_{I_2}} \xrightarrow{d} \cdots \right),$$

for all $I_k \subset A$, and $|I_k| = k + 1$.

6.2.3. Local Data on Cech Cover

Here we define the necessary gluing data for sheaf of complexes.

Let $P(A)$ be the partially ordered set

$$P(A) = \{ I \mid \emptyset \neq I \subset A, U_I \neq \emptyset \}.$$

with ordering $I \leq J$ if $I \subset J$. Let $P(A)$ also denote the category with objects being elements in $P(A)$ and morphism being $I \to J$ if $I \leq J$ and the obvious composition.

Definition 6.2.1. A path γ in $P(A)$ is a sequence of composable morphisms $I_0 \to I_1 \to \cdots \to I_k$. We say a path γ is *non-degenerate* if none of the morphisms in it is an identity-morphism. The length of a path is the number of arrows.

Definition 6.2.2. The *Cech data* of sheaves of complexes with respect to a finite open cover $\{U_i\}_{i \in A}$ is the following data $(F = \{F_I\}, h = \{h(\gamma)\})$,

(1) For each $I \in P(A)$, let F_I be a complex of sheaves over U_I.

(2) For each non-degenerate path $\gamma = (I \to J)$ of length 1, there is a isomorphism in the dg derived category of chain complexes

$$h(I \to J) : F_I|_{U_J} \xrightarrow{\sim} F_J$$

(3) For each non-degenerate path $\gamma = (I = I_0 \to \cdots \to I_k = J)$ of length $k \geq 2$, there is an element $h(\gamma)$ in $\mathrm{Hom}^{1-k}(F_I|_{U_J}, F_J)$, such that

$$\begin{aligned} & dh(I_0 \to \cdots \to I_k) \\ &= \sum_{i=1}^{k-1} (-1)^i \left(-h(I_0 \to \cdots \widehat{I_i} \cdots \to I_k) + h(I_0 \to \cdots \to I_i) \circ h(I_i \to \cdots \to I_k) \right), \end{aligned}$$

where $h(I_0 \to \cdots \to I_i)$ is an abuse of notation for $h(I_0 \to \cdots \to I_i)|_{U_J} \in \mathrm{Hom}^{1-k}_{U_J}(F_{I_0}|_{U_J}, F_{I_j}|_{U_J})$

Example 6.2.3. If $\mathcal{F}$ is a sheaf of complex over X, then it induces the following canonical Cech data:

(1) For each $I \in P(A)$, $F_I := \mathcal{F}|_{U_I}$.

(2) For each non-degenerate path $\gamma = (I \to J)$ of length 1, let $h(I \to J) : F_I|_{U_J} \to F_J$ be the identify morphism, since $F_I|_{U_J} = (\mathcal{F}|_{U_I})|_{U_J} = \mathcal{F}|_{U_J} = F_J$.

(3) For each non-degenerate path $\gamma = (I = I_0 \to \cdots \to I_k = J)$ of length $k \geq 2$, let $h(\gamma) = 0$.

$\triangle$

The following proposition show that given a Cech data of sheaves of complexes, we may construct a global sheaf of complexes (or complex of sheaves).

Proposition 6.2.4. *Let X be a topological space with a finite open cover $\{U_i\}_{i\in A}$, and let (F,h) be a Cech data of sheaves with respect to this cover. Then there exists a complex of sheaves $\mathcal{F} := \mathcal{F}_{F,h}$, such that for each $i \in A$, there is a quasi-isomorphism*

$$\psi_i : \mathcal{F}|_{U_i} \xrightarrow{\sim} F_i.$$

Proof. We give the construction of the global sheaf first. We will write $\mathcal{F}$ for $\mathcal{F}^\bullet$, and understand sheaf as sheaf of complexes. The hom-complex is in the category of dg derived category of sheaves over X. Let

$$\mathcal{F} = \left(\bigoplus_I \mathcal{F}_I, d = \sum_{I\subset J} d_{I\to J}\right), \quad \mathcal{F}_I := j_{I*}F_I[1-|I|], \quad d_{I\to J} \in \mathrm{Hom}^1(\mathcal{F}_I, \mathcal{F}_J)$$

where the sums are over $I \in P(A)$ and over $I, J \in P(A)$. The restriction isomorphism $\psi_i : \mathcal{F}|_{U_i} \to F_i$ is given by its restriction on the direct summands:

$$\psi_i|_{\mathcal{F}_I|_{U_i}} = \begin{cases} id & \text{if } I = \{i\} \\ 0 & \text{else} \end{cases}.$$

(Step 1: sign conventions). To specify d, such that $d^2 = 0$, we need to treat the signs carefully. Fix any linear ordering of A, and let any subset of A be equipped with the induced linear order. If $I = \{v_1, \cdots, v_k\}$ is a set with linear ordering, with $v_1 \leq \cdots \leq v_k$, we identify I with the sequence (or k-tuple) $(v_1 \cdots v_k)$. If I_1 and I_2 are two sequences that are disjoint as sets, then let $I_1 \sqcup I_2$ denote the concatenation of the two. If J is a linear ordered set, and $I \subset J$, then let $J - I$ be the complement of I in J equipped with the induced linear ordering. For an ordered set J, let $\Sigma(J)$ be the permutation group acting on J.

Let $\emptyset \neq I \subset J$ be a nested pair of linearly ordered sets. Let $\sigma(I,J) \in \Sigma(J)$ be the permutation, that 'pushes I to the left inside J',

$$\sigma(I,J) : J \mapsto I \sqcup (J - I).$$

Let $\neq I \subset J \subset K$ be a nested triple of linearly ordered sets. Let $\sigma(I,J,K) \in \Sigma(K)$ be the permutation that

$$\sigma(I,J,K) : K \mapsto I \sqcup (J - I) \sqcup (K - J).$$

Then we have relation

$$\sigma(I,J,K) = \sigma(I,J) \circ \sigma(J,K) = \sigma(J - I, K - I) \circ \sigma(I,K).$$

For any permuation σ, let $(-1)^{\sigma} = \pm 1$ be the signature of σ.

(Step 2: define d). Now we define $d_{I \to J}$. For $I = J$, we have

$$d_{I \to I} = (-1)^{|I|-1} d_{F_I}.$$

For $I \subset J$ with $|J - I| = 1$, we have

$$d_{I \to J} = (-1)^{|I|-1} (-1)^{\sigma(I,J)} h(I \to J).$$

where $h(I \to J) \in \mathrm{Hom}^0(F_I|_{U_J}, F_J)$ is the quasi-isomorphism specified in the Cech data.

For $I \to J$ with $|J - I| = k \geq 2$, we have

$$d_{I\to J} = (-1)^{|I|-1}(-1)^{\sigma(I,J)} \sum_{\sigma\in\Sigma(J-I)} (-1)^{\sigma} h(\gamma_\sigma)$$

where $\gamma_\sigma = (I_0^\sigma \to I_1^\sigma \to \cdots \to I_k^\sigma)$ is a non-degenerate path in $P(A)$ from I to J and

$$I_i^\sigma = I \sqcup (\text{first } i \text{ terms in } \sigma(J-I)).$$

(Step 3: checking $d^2 = 0$). We claim that, for any $I \to J$,

$$\sum_{K, I\subset K\subset J} d_{IK} \circ d_{KJ} = 0$$

where the composition is from left to right.

$$\begin{aligned}
& d_{II}d_{I\to J} + d_{I\to J}d_{JJ} \\
= & \sum_{\sigma\in\Sigma(J-I)} (-1)^{2(|I|-1)+\sigma(I,J)}(-1)^{\sigma}(d_{F_I} \circ h(\gamma_\sigma) - (-1)^{1-k}h(\gamma_\sigma) \circ d_{F_J}) \\
= & \sum_{\sigma\in\Sigma(J-I)} (-1)^{\sigma(I,J)}(-1)^{\sigma} d(h(\gamma_\sigma)) \\
= & \sum_{\sigma\in\Sigma(J-I)} \sum_{i=1}^{k-1} (-1)^{\sigma(I,J)}(-1)^{\sigma}(-1)^{i-1} \Big(h(I_0^\sigma \to \cdots \widehat{I_i^\sigma} \cdots \to I_k^\sigma) + h(I_0^\sigma \to \cdots \to I_i^\sigma) \\
& \circ h(I_i^\sigma \to \cdots \to I_k^\sigma)) \\
= & \sum_{\sigma\in\Sigma(J-I)} \sum_{i=1}^{k-1} (-1)^{\sigma(I,J)}(-1)^{\sigma}(-1)^{i-1} h(I_0^\sigma \to \cdots \to I_i^\sigma) \circ h(I_i^\sigma \to \cdots \to I_k^\sigma)
\end{aligned}$$

where the double sum of terms $\sum_{i=1}^{k-1}\sum_{\sigma}(-1)^i(-1)^{\sigma}h(I_0^{\sigma} \to \cdots \widehat{I_i^{\sigma}} \cdots \to I_k^{\sigma})$ is zero, as it is the differential of a top-dimensional cycle in the relative simplicial chain of the k-cube $C_k([0,1]^k, \partial[0,1]^k)$, where $[0,1]^k$ is equipped with the canonical triangulation into $k!$ simplices.

Next we rearrange the sum of the permutations σ, by summing over σ with fixed I_i^{σ} first.

$$\begin{aligned}
& d_{II}d_{I\to J} + d_{I\to J}d_{JJ} \\
= \; & \sum_{\sigma\in\Sigma(J-I)}\sum_{i=1}^{k-1}\sum_{K\in P(A)}(-1)^{\sigma(I,J)}(-1)^{\sigma}(-1)^{i-1}\cdot \mathbb{I}_{K=I_i^{\sigma}}\cdot h(I_0^{\sigma}\to\cdots\to I_{i-1}^{\sigma}\to K) \\
& \circ h(K\to I_{i+1}^{\sigma}\to\cdots\to I_k^{\sigma}) \\
= \; & \sum_{K:I\subsetneq K\subsetneq J}\;\sum_{\sigma_L\in\Sigma(K-I)}\;\sum_{\sigma_R\in\Sigma(J-K)}(-1)^{|K-I|-1+\sigma_R+\sigma_L+\sigma(K-I,J-I)+\sigma(I,J)}h(\gamma_{\sigma_L})\circ h(\gamma_{\sigma_R}) \\
= \; & \sum_{K:I\subsetneq K\subsetneq J} - \left(\sum_{\sigma_L\in\Sigma(K-I)}(-1)^{|I|-1}(-1)^{\sigma(I,K)}(-1)^{\sigma_R}h(\gamma_{\sigma_L})\right)\circ \\
& \left(\sum_{\sigma_R\in\Sigma(J-K)}(-1)^{|K|-1}(-1)^{\sigma(K,J)}(-1)^{\sigma_R}h(\gamma_{\sigma_R})\right) \\
= \; & -\sum_{K:I\subsetneq K\subsetneq J} d_{IK}\circ d_{KJ}.
\end{aligned}$$

This finishes the proof of the claim that $d^2 = 0$.

Given the claim, we have $d^2 = 0$. Indeed, let $(d^2)_{I\to J}$ denote the component that goes from $\mathcal{F}_I$ to $\mathcal{F}_J$, then

$$(d^2)_{I\to J} = \sum_{K:I\subset K\subset J} d_{IK}d_{KJ} = \begin{cases} d^2_{I\to I} = 0 & \text{if } I = J, \\ d_{I\to I}d_{I\to J} + d_{I\to J}d_{J\to J} = 0 & \text{if } |J-I| = 1, \\ \sum_{K:I\subset K\subset J} d_{I\to K}d_{K\to J} = 0 & \text{if } |J-I| \geq 2 \end{cases}$$

where the case $I = J$ follows since $F_I^\bullet$ are chain complexes, $|J-I| = 1$ since $d_{I\to J}$ are quasi-isomorphism, and $|J-I| \geq 2$ follows from the claim.

(Step 4): Finally, we verify that ψ_i is indeed a quasi-isomorphism. First, we note that ψ_i is a closed degree 0 morphism, that is $\psi_i \circ d_{\mathcal{F}} = d_{F_i} \circ \psi_i$. Then, suffice to the check the claim at the stalk level. Fix any $x \in U_i$, then suffice to prove that the cone of $\psi_{i,x} : \mathcal{F}_x \to F_{i,x}$

$$K^\bullet := \operatorname{cone}(\psi_{i,x})[-1] = \left(\mathcal{F}_x \oplus F_{i,x}[-1], \begin{bmatrix} -d_{F_{i,x}} & \pi_{\{i\}} \\ 0 & d_{\mathcal{F}} \end{bmatrix} \right).$$

is acyclic. We will use the spectral sequence of a filter complex to show this ([GeMa] §III.7.5). Consider a finite decreasing filtration $F^pK^\bullet$, $p \geq 0$

$$F^pK^\bullet = \begin{cases} \bigoplus_{I\in P(A)} \mathcal{F}_{I,x} \oplus F_{i,x}[-1] & \text{if } p = 0 \\ \bigoplus_{I\in P(A),|I|\geq p+1} \mathcal{F}_{I,x} & \text{if } p \geq 1 \end{cases}$$

At E_0 page, we have $d_0 = \begin{bmatrix} -d_{F_{i,x}} & \pi_{\{i\}} \\ 0 & \sum_{I\in P(A)} d_{II} \end{bmatrix}$, the cohomology of E_0 is

$$E_1^{p,q} \cong H^{p,q}(E_0, d_0) = \bigoplus_{I\in P(A), I\neq\{i\}, |I|=p+1} H^q(\mathcal{F}^\bullet_{I,x}, d_{II}).$$

At E_1 page, we consider the horizontal differential $d_1 = \sum_{I\subset J, |J-I|=1} d_{I\to J}$. Let $E_1^{I,q} = H^q(\mathcal{F}^\bullet_{I,x}, d_{II})$, for $I \neq \{i\}$. Let $P(A)_x$ consists of I such that $x \in \overline{U_I}$. Then $P(A)_x$ can be decomposed as

$$P(A)_x = P(A)'_x \sqcup P(A)''_x \sqcup P(A)'''_x := \{\{i\}\} \sqcup \{I \mid i \notin I\} \sqcup \{I \mid i \in I, I \neq \{i\}\}.$$

There is a bijection from $P(A)''_x$ to $P(A)'''_x$, sending $I \mapsto I \sqcup \{i\}$. On the otherhand, for $|J-I| = 1$, $d_{I\to J} = h(I \to J)$ are isomorphism on the cohomology. Hence, for each fixed q, the chain complex $(E_1^{\bullet,q}, d_1)$ admits a contraction homotopy

$$g : E_1^{p,q} \to E_1^{p-1,q}, \quad g|_{E_1^{I,q}} = \begin{cases} d^{-1}_{I-i,I} & \text{if } i \in I \\ 0 & \text{else,} \end{cases}$$

Then $d_1 \circ g + g \circ d = id$. Hence, for each q, $E^{\bullet,q}$ is acyclic, and

$$E_2^{p,q} \cong H^{p,q}(E_1, d_1) = 0.$$

Thus, $K^\bullet$ is acyclic, and $\psi_{i,x}$ is a quasi-isomorphism. This finishes the proof of the proposition. $\square$

Example 6.2.5. Let $\mathcal{F}_0$ be a sheaf of complex over X and (F,h) be the canonical Cech data induced by $\mathcal{F}_0$. Then the reconstruction in the above proposition reduces to the Cech resolution $\underline{hom}(P_\bullet, \mathcal{F}_0)$. △

6.2.4. Equivalence of Cech data

Let (F,h) be a Cech data associated with the open cover $\mathscr{U}$ of X. Sometimes we want to replace the local sheaves F_I by certain nice resolutions $\widetilde{F}_I$, the next proposition gives a recipe for changing the gluing data 'by conjugation'.

Proposition 6.2.6. *Let (F,h) be a Cech data associated with the open cover $\mathscr{U}$ of X. Suppose for each $I \in P(A)$, we have the following quasi-isomorphisms,*

$$f_I : F_I \rightleftarrows \widetilde{F}_I : g_I \ , \quad g_I \circ f_I = \mathrm{id}_{G_I} + d\alpha_I, \quad f_I \circ g_I = \mathrm{id}_{F_I} + d\beta_I$$

where f_I, g_I are degree-zero closed morphism, and α_I, β_I are degree-(-1) morphism, and composition is from left to right.

Then for any path $\gamma = (I = I_0 \to \cdots \to I_k = J)$, we may define $\widetilde{h}(I_0 \to \cdots \to I_k) = g_{I_0} \circ \widehat{h}(I_0 \to \cdots \to I_k) \circ f_{I_k}$, where

$$\widehat{h}(I_0 \to \cdots \to I_k) = \sum_{m=0}^{k-1} \sum_{0<r_1<\cdots<r_m<k} h(I_0 \cdots I_{r_1}) \circ \beta_{I_{r_1}} \circ h(I_{r_1} \cdots I_{r_2}) \circ \cdots \circ \beta_{I_{r_m}} \circ h(I_{r_m} \cdots I_k).$$

Then $(\widetilde{F}, \widetilde{h})$ is a Cech data. And there is a quasi-isomorphism $\psi : \mathcal{F}_{\widetilde{F},\widetilde{h}} \xrightarrow{\sim} \mathcal{F}_{F,h}$.

Proof. First, we verify that $\widetilde{h}$ satisfies the condition (2) and (3) in the definition of Cech data. The condition of quasi-isomorphism in (2) and the condition of degree in (3) are easy to check. We now check the differential condition in (3).

$$
\begin{aligned}
& d(\widetilde{h}(I_0 \to \cdots \to I_k)) \\
= & \sum_{m=0}^{k-1} \sum_{0<r_1<\cdots<r_m<k} g_{I_0} \circ d[h(I_0 \to \cdots I_{r_1}) \circ \beta_{I_{r_1}} \circ h(I_{r_1} \cdots I_{r_2}) \circ \cdots \\
& \cdots \circ \beta_{I_{r_m}} \circ h(I_{r_m} \cdots I_k)] \circ f_{I_k}.
\end{aligned}
$$

If d hit β, then we have $d\beta_{I_i} = (f_{I_i} \circ g_{I_i} - \mathrm{id}_{F_{I_i}})$, and denote the corresponding terms as '$f \circ g$'-type or 'id'-type terms. If d hit $h(\cdots)$, then we have '$h(\cdots \widehat{I} \cdots)$'-type and '$h \circ h$'-type terms. By a straightforward yet tedious calculation, we find that 'id'-type term cancels with $h \circ h$'-type terms, and '$h(\cdots \widehat{I} \cdots)$'-type term become $\widetilde{h}(\cdots \widehat{I} \cdots)$-type term, and '$f \circ g$'-type term become $\widetilde{h} \circ \widetilde{h}$ type term. This finishes the proof that $(\widetilde{F}, \widetilde{h})$ is a Cech data.

Next we define $\psi : \widetilde{\mathcal{F}} = \mathcal{F}_{\widetilde{F},\widetilde{h}} \xrightarrow{\sim} \mathcal{F}_{F,h}$. Recall that as sheaf of graded vector space, we have

$$
\widetilde{\mathcal{F}} = \bigoplus_{I \in P(A)} \widetilde{\mathcal{F}}_I, \quad \mathcal{F} = \bigoplus_{I \in P(A)} \mathcal{F}_I.
$$

Then $\psi = \sum_{I \subset J} \psi_{I \to J}$ is given schematically as

$$
\psi = g \circ (1 + q \circ \beta + q \circ \beta \circ q \circ \beta + \cdots) = g \circ \frac{1}{1 - q \circ \beta}
$$

where the composition is from left to right, and

(1) $g : \widetilde{\mathcal{F}} \to \mathcal{F}$ component-wise by $g_I : \widetilde{\mathcal{F}}_I \to \mathcal{F}_I$,

(2) $q : \mathcal{F} \to \mathcal{F}$ is given by $d_{\mathcal{F}} = d_{\mathcal{F},0} + q$, or $q = \sum_{I \subsetneq J} d_{\mathcal{F}, I \to J}$,

(3) $\beta : \mathcal{F} \to \mathcal{F}$ is given component-wise by $\beta_I : \mathcal{F}_I \to \mathcal{F}_I$, a degree-$(-1)$ morphism.

In the above notation, we have

$$d_{\widetilde{\mathcal{F}}} = d_{\widetilde{\mathcal{F}},0} + \widetilde{q} = d_{\widetilde{\mathcal{F}},0} + g \circ \frac{1}{1 - q \circ \beta} \circ q \circ f$$

where $f : \mathcal{F} \to \widetilde{\mathcal{F}}$ is given componentwise by $f_I : \mathcal{F}_I \to \widetilde{\mathcal{F}}_I$.

To show that ψ indeed is a chain map, we need to show

$$d_{\widetilde{\mathcal{F}}} \circ \psi = \psi \circ d_{\mathcal{F}}$$

The left hand side is (suppressing $\circ$ sign, and let $\widetilde{d}_0 = d_{\widetilde{\mathcal{F}},0}, d_0 = d_{\mathcal{F},0}$)

$$\left(\widetilde{d}_0 + g\frac{1}{1-q\beta}qf\right) g\frac{1}{1-q\beta} = gd_0\frac{1}{1-q\beta} + g\frac{1}{1-q\beta}qfg\frac{1}{1-q\beta}$$

Since $(d_0 + q)^2 = 0$ and $d_0^2 = 0$, we have

$$d_0 q\beta = -(qd_0 + q^2)\beta = -q^2\beta - q(d_0\beta + \beta d_0 - \beta d_0) = -q^2\beta - q(fg - 1) + q\beta d_0.$$

Hence $[d_0, q\beta] = q(1 - fg - q\beta)$, and we have

$$[d_0, \frac{1}{1-q\beta}] = \frac{1}{1-q\beta}q(1 - fg - q\beta)\frac{1}{1-q\beta} = \frac{1}{1-q\beta}q - \frac{1}{1-q\beta}qfg\frac{1}{1-q\beta}$$

Thus we have

$$\begin{aligned} d_{\widetilde{\mathcal{F}}} \circ \psi &= g\frac{1}{1-q\beta}d_0 + g\frac{1}{1-q\beta}q - g\frac{1}{1-q\beta}qfg\frac{1}{1-q\beta} + g\frac{1}{1-q\beta}qfg\frac{1}{1-q\beta} \\ &= g\frac{1}{1-q\beta}(d_0 + q) = \psi \circ d_{\mathcal{F}} \end{aligned}$$

Thus ψ is indeed a chain map.

Finally, we want to show that ψ is a quasi-isomorphism, or $K = \text{cone}(\psi)$ is acyclic. Let K be equipped with a decreasing filtration $K = F^0K \supset F^1K \supset \cdots$, where F^pK contains component of $\widetilde{\mathcal{F}}_I$ and $\mathcal{F}_I$ with $|I| \geq p+1$. The spectral sequence sequence has $E_1^{p,q} = 0$, hence K is acyclic. $\square$

6.3. Deformation of Constructible Sheaves: Existence and Uniqueness

We have seen in the first section how to deform constructible sheaves whose singular supports lies in the conormal of co-oriented affine hyperplanes and their intersections. In this section, we show that if a deformation on a manifold is locally of the above type, then local deformations can be glued together uniquely to give a global deformation.

First, we state a uniqueness result about sheaf extension.

Proposition 6.3.1. *Let $\Lambda_{\mathbb{R}}^{\infty}$ be a variation of Legendrian in $T^{\infty}(M \times \mathbb{R})$. Suppose $\Lambda_{\mathbb{R}} \widehat{+} \Lambda_{\mathbb{R}}^a$ is disjoint from $T^*_{M\times\{t\}}(M \times \mathbb{R})$ away from the zero section for all t, then the restriction functor*

$$\iota_t^* : Sh(M \times \mathbb{R}, \Lambda_{\mathbb{R}}) \to Sh(M \times \{t\}, \Lambda_t)$$

is fully faithful, where $\iota_t : M \times \{t\} \hookrightarrow M \times \mathbb{R}$ is the inclusion.

Proof. Suffice to check on the hom space between objects. Let $F, G \in Sh(M\times\mathbb{R}, \Lambda_{\mathbb{R}})$, hence the hom-sheaf has the following bound on singular support

$$SS(\underline{hom}(F,G)) \subset SS(G) \widehat{+} SS(F)^a \subset \Lambda_{\mathbb{R}} \widehat{+} \Lambda_{\mathbb{R}}^a.$$

By the assumption of the proposition, we have

$$SS(\underline{hom}(F,G)) \cap T^*_{M\times\{t\}}(M \times \mathbb{R}) \subset T^*_{M\times\mathbb{R}}(M \times \mathbb{R}).$$

Hence section of $\underline{hom}(F,G)$ can propagate from a thin strip $M \times (t-\epsilon, t+\epsilon)$ around any slice $M \times \{t\}$ to the entire space $M \times \mathbb{R}$, thus we have

$$hom(F,G) \cong \iota_t^* \underline{hom}(F,G) \cong \underline{hom}(\iota_t^* F, \iota_t^* G),$$

where the last step follows from a similar argument in Proposition 5.1.3. $\square$

Corollary 6.3.2. *With the same setup as Proposition 6.3.1. Fix any $t_0 \in \mathbb{R}$, if $F_{t_0} \in Sh(M, \Lambda_{t_0}^\infty)$ and $G_\mathbb{R}, H_\mathbb{R}$ are two sheaves in $Sh(M, \Lambda_\mathbb{R})$ such that we have isomorphism*

$$\varphi : G_{t_0} \xrightarrow{\sim} F_{t_0}, \quad \psi : H_{t_0} \xrightarrow{\sim} F_{t_0},$$

then there is unique isomorphism $\Phi : G_\mathbb{R} \cong H_\mathbb{R}$.

Proof. Since $hom(G_\mathbb{R}, H_\mathbb{R}) \cong hom(G_{t_0}, H_{t_0})$, we can extend the isomorphism $\psi^{-1} \circ \varphi : G_{t_0} \xrightarrow{\sim} H_{t_0}$ to $\Phi : G_\mathbb{R} \to H_\mathbb{R}$ uniquely. The cone cone(Φ) restricts to the slice t_0 is trivial, hence the entire cone is trivial, thus Φ is a quasi-isomorphism. $\square$

Next, we prove that one can glue the sheaf deformation over local patches, hence proving the theorem 10.

Proof of Theorem 10. Given any slice $M \times \{t\} \subset M \times \mathbb{R}$, and any sheaf $F_t \in Sh(M \times \{t\}, \Lambda_t)$, we may cover $M \times \{t\}$ by finitely many open patches $\mathscr{U}(t) := \{U_i \mid t \in I_i\}$, and fix $\epsilon(t) > 0$ small enough, such that $I(t) = (t - \epsilon(t), t + \epsilon(t)) \subset I_i$ for all i with $t \in I_i$. Since locally $F_t|_{U_i}$ can be represented by chain complexes of standard sheaves in U_i with singular support in $\Lambda_t|_{U_i}$, as in Proposition 6.1.5, and these standard sheaves in U_i can be extended to the open patch $U_i \times I_i$, hence we can extend $F_t|_{U_i}$ to

$U_i \times I(t)$. We also do so for all the intersection of patches in $\mathscr{U}(t)$. Then for the sheaf F_t on initial slice $M \times \{t\}$, with respect to cover $\mathscr{U}(t)$, we may build the set of local Cech data (Definition 6.2.2) as in Example 6.2.3. In other words, we resolve F_t as a chain complex of sheaves supported on Cech covers and their intersection, such that each building block can be extended from t to the open neighborhood $I(t)$ of t and their hom's are preserved during the extension, hence we can extend the chain-complex of sheaves itself, which is a resolution of F_t. This way, we extended F_t from $M \times \{t\}$ to $M \times I(t)$. Since $\cup_t I(t)$ covers $\mathbb{R}$, and the variation of Legendrian is compactly supported in $\mathbb{R}$, hence we may extend F_t to the entire $M \times \mathbb{R}$. By Corollary 6.3.2, we know such extension is unique. $\square$

CHAPTER 7

Twisted Polytope Sheaves and Coherent-Constructible Correspondence for Toric Variety

7.1. Introduction

Toric varieties are certain compactifications of the complex torus $(\mathbb{C}^*)^n$. They provide many interesting examples, and can be studied in various ways, using algebraic geometry, symplectic geometry or combinatorics.

For example, let X_Σ be a smooth projective toric vareity corresponding to a fan Σ, and L an ample line bundle with a lifting of the $(\mathbb{C}^*)^n$-action. Then there is a convex polytope Δ_L in $\mathbb{R}^n$, where $\mathbb{R}^n$ is identified with the dual Lie algebra $\mathrm{Lie}(T^n)^\vee$ and $T^n = (U(1))^n$ is the maximal compact real subgroup of $(\mathbb{C}^*)^n$. The convex polytope Δ_L can be understood in the following ways,

(1) Algebraically, Δ_L is the convex hull of the characters appearing in the weight decomposition of $H^0(X, L)$ under the $(\mathbb{C}^*)^n$-action.
(2) Symplectically, Δ_L is the moment polytope of the Hamiltonian action T^n on (X, ω), where ω is a symplectic 2-form with $[\omega] = c_1(L)$.
(3) Combinatorially, Δ_L is the intersection of half-spaces $Q_\rho = \{x \in \mathbb{R}^n \mid \langle x, v_\rho \rangle \leq a_\rho\}$, one for each compactifying divisor D_ρ of X given by a vector $v_\rho \in \mathbb{Z}^n$, and a_ρ is the vanishing order of the invariant (meromorphic) section along the divisor D_ρ.

In the case where L is not ample, the corresponding polytope becomes a 'twisted polytope', as explained in Figure 1.2. The name originates from the paper of Karshon and Tolman [KT], where they generalized the moment map to the case where ω is degenerate.

The above correspondence between equivariant line bundles and twisted polytopes enjoys a categorification under the name of (equivariant) *Coherent Constructible Correspondence (CCC).*

Theorem 11 ([FLTZ1]). *If X is a proper toric variety, there is a corresponding conical Lagrangian $\Lambda_\Sigma \subset T^*\mathbb{R}^n$ and an equivalence of derived (or rather, triangulated dg categories)*

$$\kappa : Perf_T(X_\Sigma) \xrightarrow{\sim} Sh_{cc}(\mathbb{R}^n, \Lambda_\Sigma)$$

where

- *$Perf_T(X_\Sigma)$ is the triangulated dg category of perfect complexes of torus-equivariant coherent sheaves on X_Σ.*
- *$Sh_{cc}(\mathbb{R}^n, \Lambda_\Sigma)$ is the triangulated dg category of constructible sheaves on $\mathbb{R}^n$ which are compactly supported, whose singular supports lie in Λ_Σ.*

The equivariant CCC implies that there is a quasi-embedding for the non-equivariant case:

Proposition 7.1.1 ([Tr],Proposition 2.4, 2.7). *Let $\pi : \mathbb{R}^n \to T^n \cong \mathbb{R}^n/\mathbb{Z}^n$ be the projection. Then there exists a functor $\overline{\kappa}$ and commutative diagrams*

$$\begin{tikzcd} Perf_T(X_\Sigma) \arrow[r, "\sim \kappa"] \arrow[d, "forget"] & Sh_{cc}(\mathbb{R}^n, \Lambda_\Sigma) \arrow[d, "\pi_!"] \\ Perf(X_\Sigma) \arrow[r, hook, "\overline{\kappa}"] & Sh(T^n, \overline{\Lambda}_\Sigma), \end{tikzcd}$$

Remark 7.1.2. When X_Σ is smooth, the homotopy category of $Perf_T(X_\Sigma)$ (resp. $Perf(X_\Sigma)$) coincide with the usual $D^bCoh_T(X_\Sigma)$ (resp. $D^bCoh(X_\Sigma)$).

Remark 7.1.3. Under the quotient map $\pi : \mathbb{R}^n \to T^n$, all the upstairs objects in $\mathbb{R}^n$ are unadorned, and downstairs objects in T^n have overlines.

And it is conjectured that this quasi-embedding is a quasi-equivalence. The conjecture has been verified in certain cases by Treumann [**Tr**], Scherotzke-Sibilia [**SS**] and Kuwagaki [**Ku1**]. Recently, it has been fully proven by Kuwagaki [**Ku2**] in the generality of toric stacks, using gluing descriptions of ∞-categories on both sides.

In this paper, we prove the non-equivariant CCC for smooth projective toric varieties, by showing the $\overline{\kappa}$-images of line bundles generate the constructible sheaf category.

Theorem. *Let X_Σ be a smooth projective toric variety of complex dimension n, then there is an quasi-equivalence of category*

$$\overline{\kappa} : Coh(X_\Sigma) \xrightarrow{\sim} Sh(T^n, \overline{\Lambda}_\Sigma)$$

where $\overline{\Lambda}_\Sigma$ is a conical Lagrangian in T^*T^n.

The key part of the proof is as following. For any point $\theta \in T^n$, there is a constructible sheaf $\overline{P}_{[\theta]}$ on T^n as the $\overline{\kappa}$-image of a certain line bundle (c.f. Definition 7.5.1), such that for any sheaf $F \in Sh(T^n, \overline{\Lambda}_\Sigma)$, its stalk at the point θ can be computed by

$$F_\theta \cong hom(\overline{P}_{[\theta]}[-n], F). \tag{7.1.1}$$

This immediately implies that if F satisfies $hom(\overline{\kappa}(L), F) = 0$ for all the line bundles L on X_Σ, then $F = 0$. In other words, the stalk functors in $Sh(T^n, \overline{\Lambda}_\Sigma)$ are co-represented by $\overline{\kappa}$-images of line bundles on X_Σ. We thank David Treumann for the suggestion of co-representing the stalk functors.

The quasi-isomorphism (7.1.1) is due to a non-characteristic deformation argument for constructible sheaf. We define a 1-parameter family of sheaves $\{P_t\}_{t\in[0,1]}$, such that

(1) $P_0 = j_{B!}\mathbb{C}_B$, where B is a small enough convex open set around θ, such that $F_\theta \cong \Gamma(B, F) \cong hom(P_0, F)$.

(2) $P_1 = \overline{P}_{[\theta]}[-n]$.

(3) For $t \in (0, 1)$, take the linear interpolation between P_0 and P_1, and show that $SS^\infty(P_t) \cap \overline{\Lambda}_\Sigma^\infty = \emptyset$.

By the non-characterstic deformation lemma[1], $hom(P_t, F)$ is invariant during the deformation, hence we get (7.1.1).

Example 7.1.4 (Expanding family of twisted polytope sheaves)**.** The example of Hirzebruch surface $\mathbb{F}_2$, see Figure 7.1. Here we describe the sheaf $P_{[x]}$ upstairs in $\mathbb{R}^2$, where $\overline{P}_{[\pi(x)]} := \pi_* P_{[x]}$ and $\pi : \mathbb{R}^2 \to T^2$ is the quotient map. The point we want to probe is at $x = (-0.5, 0)$, marked in black. The green, blue, red and black curves are the boundaries of the twisted polytopes in the interpolating family P_t. The green and blue ones are still open convex polytopes, the red and black ones are twisted. We marked the direction of the singular support for the sheaf P_{red} corresponding red curve, and note that $SS^\infty(P_{\mathrm{red}}) \cap \Lambda^\infty = \emptyset$. △

[1]One needs to be careful about the endpoint $t = 1$, since $SS^\infty(P_1) \cap \overline{\Lambda}_\Sigma^\infty \neq \emptyset$. The non-characteristic deformation lemma for sections over open sets, Proposition 3.5.1, avoids this problem.

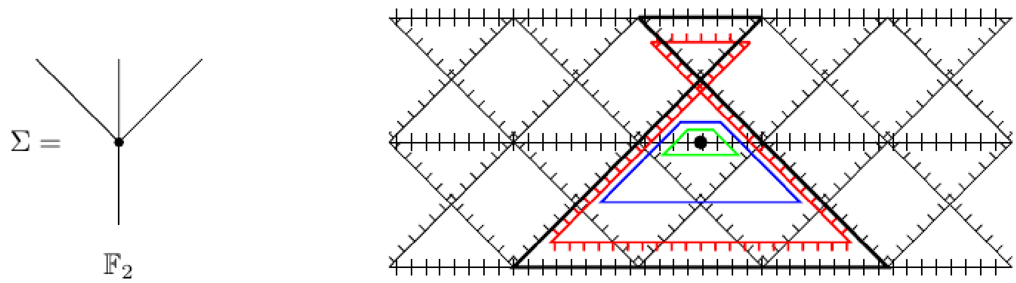


Figure 7.1. Expanding family of twisted polytope sheaves on $\mathbb{R}^2$.

Remark 7.1.5. The collection of line bundles as the $\overline{\kappa}$-preimages of $\{\overline{P}_{[\theta]}\}$ is a finite collection, since sheaves in $Sh(T^n, \overline{\Lambda}_\Sigma)$ admits a finite stratification depending only on $\overline{\Lambda}_\Sigma$. In the case of $\mathbb{P}^n$, they turn out to be $\mathcal{O}(1), \cdots, \mathcal{O}(n+1)$, and form an exceptional collection. However, for general toric variety, even smooth Fano ones, the collection of line bundles cannot always be an exceptional collection [**?**, **?**].

However, for any collection of line bundles $L_1, \cdots, L_N$, Craw-Smith [CS] considered the endomorphism algebra $A = \mathrm{End}(\oplus_{i=1}^N L_i)$ and defined a bound quiver of sections (Q, R) associated with A. It would be interesting to study our collection of line bundles using this quiver approach.

7.2. Review of Toric Geometry

An n-dimensional smooth projective complex manifold X is toric if there is a holomorphic $(\mathbb{C}^*)^n$-action with an open dense orbit X^o on which $(\mathbb{C}^*)^n$ acts freely. The complement of the open orbit $D = X \backslash X^o$ is a simple normal crossing divisor with irreducible torus-invariant components.

We first review the standard setup and notation for the combinatorial data used for defining a toric variety. Then we explain the relationship between equivariant line bundles, toric divisors, and twisted polytopes (as a collection of labeled vertices). Finally, we review 'twisted polytope sheaves', the corresponding constructible sheaves for equivariant line bundles under the equivariant CCC.

The data of a toric manifold can be expressed combinatorically using a fan. Let $N \cong \mathbb{Z}^n$ be a rank n lattice, with $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. Let $M = \mathrm{Hom}(N, \mathbb{Z})$ be the dual lattice and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ be the dual vector space. Let $\langle -, - \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$ be the dual pairing. Let $T_M = M_{\mathbb{R}}/M$ be a real n-dimensional torus, and $\pi : M_{\mathbb{R}} \to T_M$ be the quotient map. We recall the following definitions.

(1) A convex polyhedral cone $\sigma \subset N_{\mathbb{R}}$ is a set of the form $\sigma = \mathrm{cone}(S) = \{\sum_{u \in S} \lambda_u u \mid \lambda_u \geq 0\}$, where the cone generator $S \subset N_{\mathbb{R}}$ is a finite subset. A cone σ is *rational* if there is a generator S for σ such that $S \subset N$. A cone is *strongly convex* if it does not contain any non-trivial linear subspace of $N_{\mathbb{R}}$.

(2) Let $\sigma \in \Sigma$ be a cone, we define the dual (closed) cone $\sigma^\vee$ as

$$\sigma^\vee := \{x \in M_{\mathbb{R}} \mid \langle x, y \rangle \geq 0, \forall y \in \sigma\}.$$

We also define $\sigma^\perp = \{x \in M \mid \langle x, y \rangle = 0, \forall y \in \sigma\} \subset M_{\mathbb{R}}$, and σ^o (resp. $(\sigma^\vee)^o$) as the relative interior of σ (resp. $\sigma^\vee$).

(3) A face of a cone σ is the subset $H_m \cap \sigma$ for some $m \in \sigma^\vee$ and $H_m = m^\perp$. We use $\sigma(r)$ to denote the collection of r-dimensional faces of σ.

(4) A fan Σ in $N_{\mathbb{R}}$ is a finite collection of strongly convex rational polyhedral cones $\sigma \subset N_{\mathbb{R}}$, such that (a) if $\sigma \in \Sigma$ then any face of σ is in Σ, and (b) if σ_1, σ_2 are cones in Σ then $\sigma_1 \cap \sigma_2$ is a face in both σ_1 and σ_2. We use $\Sigma(r)$ to denote the collection of r-dimensional cones in Σ.

(5) A fan Σ in $N_{\mathbb{R}}$ is *complete*, if its support $|\Sigma| := \cup_{\sigma\in\Sigma}\sigma$ is the entire $N_{\mathbb{R}}$. A complete fan Σ is *smooth*, if each maximal cone $\sigma \in \Sigma(n)$ is generated by a lattice basis of N.

(6) A smooth complete fan Σ is *projective*, if there exists a convex piecewise linear function $\varphi : N_{\mathbb{R}} \to \mathbb{R}$, such that the maximal linearity domains of φ are the maximal cones of Σ. (cf. Proposition 7.3.6)

See Example 7.4.10 for a fan of $\mathbb{P}^2$.

Assumption: We will always assume Σ to be a smooth projective fan.

The affine toric variety X_σ is then defined by

$$X_\sigma = \operatorname{Spec}(\mathbb{C}[\sigma^\vee \cap M])$$

where $\mathbb{C}[\sigma^\vee \cap M])$ is the group ring of the abelian semi-group $\sigma^\vee \cap M$. If $\tau \subset \sigma$ is a face of σ, then $\sigma^\vee \subset \tau^\vee$, hence $\mathbb{C}[\sigma^\vee \cap M]) \hookrightarrow \mathbb{C}[\tau^\vee \cap M])$, and $X_\tau \hookrightarrow X_\sigma$ is an open inclusion. We may equip Σ with a partial ordering, for $\tau, \sigma \in \Sigma$, $\tau \le \sigma \iff \tau \subset \sigma$. Then X_Σ can be glued together from affine open pieces X_σ, as a colimit of schemes

$$X_\Sigma = \operatorname{colim}_{\sigma\in\Sigma} X_\sigma.$$

7.3. Toric Divisors, Support Functions and Twisted Polytopes

For each ray $\rho \in \Sigma(1)$, let $v_\rho \in \rho \cap N$ be a minimal ray generator, $\lambda^{v_\rho} : \mathbb{C}^* \to N \otimes_{\mathbb{Z}} \mathbb{C}^*$ as the one-parameter subgroup, and $D_\rho = \overline{\{\lim_{t\to 0} \lambda^{v_\rho}(t) \cdot x \mid x \in X^o\}}$ the torus-invariant divisor, or, a *toric divisor*. We write $\sum_\rho$ for a summation over the rays $\rho \in \Sigma(1)$ when there is no danger of confusion.

Let $D = \sum_\rho a_\rho D_\rho$ be a toric $\mathbb{R}$-divisor on X_Σ, $a_\rho \in \mathbb{R}$. If $a_\rho \in \mathbb{Z}$ for all ρ, then D is an integral toric divisor, or toric $\mathbb{Z}$-divisor. There are two equivalent ways to describe a toric divisor, either using a support function φ_D on $N_\mathbb{R}$, or a twisted polytope χ_D on $M_\mathbb{R}$.

Definition 7.3.1 (Support function)**.** A *support function* for Σ is a continuous piecewise linear function $\varphi : N_\mathbb{R} \to \mathbb{R}$, such that for each maximal cone $\sigma \in \Sigma(n)$, the restriction $\varphi|_\sigma$ is linear.

- A support function φ is integral if it sends N to $\mathbb{Z}$.
- A support function φ is *convex*, if for any $x, y \in N_\mathbb{R}$

$$\varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y).$$

- Furthermore, we say φ is *strictly convex*, if the strict inequality holds whenever x, y is not contained in the same cone.

Definition 7.3.2 (Twisted polytope)**.** A *twisted polytope* for Σ is an assignment of element in $M_\mathbb{R}$ to top-dimensional cones in Σ,

$$\chi : \Sigma(n) \to M_\mathbb{R}, \quad \sigma \mapsto \chi_\sigma,$$

such that if $\sigma, \tau \in \Sigma(n)$ then $\langle \chi_\sigma, \cdot \rangle = \langle \chi_\tau, \cdot \rangle$ on $\sigma \cap \tau$.

- A twisted polytope χ is integral if the function $\chi_\sigma \in M$ for all $\sigma \in \Sigma(n)$.
- If χ is a twisted polytope, then for any cone $\sigma \in \Sigma$, we define $\chi_\sigma \in M_\mathbb{R}/\sigma^\perp$ by

$$\chi_\sigma = \text{ Affine Hull}(\{\chi_\tau \mid \ \tau \text{ is a maximal cone containing } \sigma\}) \subset M_\mathbb{R}.$$

- For any $x \in M_{\mathbb{R}}$, let $\chi + x$ denote the translated twisted polytope that sends $\sigma \mapsto \chi_\sigma + x$ for any $\sigma \in \Sigma(n)$.

Remark 7.3.3. The data for a twisted polytope is a collection of the vertices, labelled by $\Sigma(n)$.

Proposition 7.3.4. *Let Σ be an n-dimensional smooth projective fan. Then we have a canonical equivalences among the following three types of objects. (1) A toric $\mathbb{R}$-divisor $D = \sum_\rho a_\rho D_\rho$, $a_\rho \in \mathbb{R}$.*

(2) A twisted polytope, $\chi : \Sigma(n) \to M_{\mathbb{R}}$.

(3) A support function, $\varphi : N_{\mathbb{R}} \to \mathbb{R}$.

In particular, integral toric divisors corresponds to integral twisted polytopes and integral support functions.

Proof. (2) $\Leftrightarrow$ (3). Given χ, we may define φ by $\varphi(x) = \langle \chi_\sigma, x \rangle$ if $x \in \sigma$ for some maximal cone $\sigma \in \Sigma(n)$. This is well-defined since if $x \in \sigma \cap \tau$, then $\langle \chi_\sigma, x \rangle = \langle \chi_\tau, x \rangle$. Conversely, given φ, then for each maximal cone σ, the linear function $\varphi|_\sigma$ determines an element in $M_{\mathbb{R}}$, denoted by χ_σ. The continuity of φ ensures $\langle \chi_\sigma, \cdot \rangle = \langle \chi_\tau, \cdot \rangle$ on $\sigma \cap \tau$.

(1) $\Leftrightarrow$ (3). Given a toric $\mathbb{R}$-divisor $D = \sum_\rho a_\rho D_\rho$, for each $\rho \in \Sigma(1)$, we define $\varphi|_\rho : \rho \to \mathbb{R}$ by $v_\rho \mapsto a_\rho$. Since the cones of Σ are simplicial, there is a unique piecewise linear extension of φ to $N_{\mathbb{R}}$ that is linear in each cone of Σ. Conversely, given φ, let $a_\rho = \varphi(v_\rho)$ for each $\rho \in \Sigma(1)$.

The claim on integrality is straightforward to verify. This finishes the proof of the Proposition. $\square$

Remark 7.3.5. (1) If D is a toric $\mathbb{R}$-divisor, we let χ_D and φ_D be the corresponding twisted polytope and support function. (2) If χ is a twisted polytope for Σ, and φ is the corresponding support function, then $\chi_\sigma \in M_\mathbb{R}/\sigma^\perp$ corresponds to the linear function $\varphi|_\sigma : \sigma \to \mathbb{R}$.

Proposition 7.3.6. *Let X_Σ be a smooth complete toric variety. Let $D = \sum_\rho a_\rho D_\rho$ be an integral toric divisor. Then*

(1) D is base-point free if and only if φ_D is convex.

(2) D is ample if and only if φ_D is strictly convex.

Proof. [CLS], Chapter 4 and 6. □

Definition 7.3.7. If $D = \sum_\rho a_\rho D_\rho$ is ample, we define the open convex polytope Δ_D as the interior of the convex hull of $\{\chi_\sigma \mid \sigma \in \Sigma(n)\}$. Equivalently, we have

$$\Delta_D = \{x \in M_\mathbb{R} \mid \langle x, v_\rho \rangle < a_\rho, \text{ for all } \rho \in \Sigma(1)\}.$$

7.4. Constructible Sheaves and Twisted Polytope Sheaves

Let M, N be dual rank-n lattices, and Σ a smooth complete fan in $N_\mathbb{R}$. We define the conical Lagrangians Λ_Σ in $T^*M_\mathbb{R}$ as [2]

$$\Lambda_\Sigma = \bigcup_{\sigma \in \Sigma} (\sigma^\perp + M) \times \sigma \subset M_\mathbb{R} \times N_\mathbb{R} = T^*M_\mathbb{R}. \tag{7.4.1}$$

[2]Our definition differs in sign convension from that in [FLTZ1]. If we change Σ to $-\Sigma$ in this paper, then the definition agrees.

We denote the push-forward of Λ_Σ to T^*T_M by $\overline{\Lambda}_\Sigma$, or directly we have

$$\overline{\Lambda}_\Sigma = \bigcup_{\sigma\in\Sigma} (\sigma^\perp/\sigma^\perp \cap M) \times \sigma \subset T_M \times N_\mathbb{R} = T^*T_M. \tag{7.4.2}$$

Definition 7.4.1 (Standard Shard Sheaves)**.** For any cone $\sigma \in \Sigma$, $c \in M_\mathbb{R}/\sigma^\perp$, we define the closed subset $Q(\sigma, c) \subset M_\mathbb{R}$ and the standard sheaf $P(\sigma, c)$ as

$$Q(\sigma, c) := c + \sigma^\vee \subset M_\mathbb{R}, \quad P(\sigma, c) := j_{Q(\sigma,c)*}\mathbb{C}_{Q(\sigma,c)}.$$

Definition 7.4.2 (Twisted Polytope Sheaves on $M_\mathbb{R}$)**.** Let χ be a twisted polytope for Σ, let D be the corresponding toric $\mathbb{R}$-divisor. The *twisted polytope sheaf* $P(\chi)$ *on* $M_\mathbb{R}$ is defined by the following chain complex of sheaves, with $\mathbb{C}_{M_\mathbb{R}}$ at degree $-n$,

$$P(\chi) := \left(\mathbb{C}_{M_\mathbb{R}} \xrightarrow{d_1} \bigoplus_{\sigma_1\in\Sigma(1)} P(\sigma_1, \chi_{\sigma_1}) \xrightarrow{d_2} \bigoplus_{\sigma_2\in\Sigma(2)} P(\sigma_2, \chi_{\sigma_2}) \xrightarrow{d_3} \cdots \xrightarrow{d_n} \bigoplus_{\sigma_n\in\Sigma(n)} P(\sigma_n, \chi_{\sigma_n})\right)$$

where d_k for $k = 1, \cdots, n$ is given in the following way:

$$d_k = \sum_{\sigma_{k-1}\subset\sigma_k} sgn(\sigma_{k-1}, \sigma_k)\rho_{\sigma_k\to\sigma_{k+1}}$$

where the sum is over $\sigma_{k-1} \in \Sigma(k-1)$, $\sigma_k \in \Sigma(k)$, and

$$\rho_{\sigma_{k-1}\to\sigma_k} : P(\sigma_{k-1}, \chi_{\sigma_{k-1}}) \to P(\sigma_k, \chi_{\sigma_k})$$

is the canonical restriction, and the sign $sgn(\sigma_{k-1}, \sigma_k) = \pm 1$ is chosen such that $d^2 = 0$ (see the following remark).

If D is any toric $\mathbb{R}$-divisor, $\chi = \chi_D$ the corresponding twisted polytope, we sometimes write $P(D)$ for $P(\chi_D)$.

Remark 7.4.3. We can be more concrete about the sign choices $sgn(\sigma_{k-1}, \sigma_k)$. One way is to fix a linear ordering of the rays $\Sigma(1)$, then a k-dimensional simplicial cone σ_k can be identified with the ordered set $\sigma_k(1) = \{\rho_1 < \rho_2 < \cdots < \rho_k\}$. If $\sigma_{k-1} = \sigma_k - \{\rho_j\}$, then we set $sgn(\sigma_{k-1}, \sigma_k) = (-1)^{j-1}$. Another way is to fix the orientations of all cones in Σ once and for all, and $sgn(\sigma_{k-1}, \sigma_k) = \pm 1$ depending on if σ_{k-1} agrees with the induced boundary orientation of σ_k.

Example 7.4.4. Consider the example of $\mathbb{P}^1$, where $\Sigma(1) = \{\mathbb{R}v_1, \mathbb{R}v_2\}$, where $v_1 = 1$ and $v_2 = -1$. We still need to fix the 'offset parameters' χ_i for each v_i. We consider the following three cases

(1) $\chi_1 = -1, \chi_2 = 1$, then

$$P(\chi) \cong (\mathbb{C}_{\mathbb{R}} \to \mathbb{C}_{[-1,\infty)} \oplus \mathbb{C}_{(-\infty,1]}) \cong \mathbb{C}_{[-1,1]}$$

(2) $\chi_1 = 0, \chi_2 = 0$, then

$$P(\chi) \cong (\mathbb{C}_{\mathbb{R}} \to \mathbb{C}_{[0,\infty)} \oplus \mathbb{C}_{(-\infty,0]}) \cong \mathbb{C}_{\{0\}}$$

(3) $\chi_1 = 1, \chi_2 = -1$, then

$$P(\chi) \cong (\mathbb{C}_{\mathbb{R}} \to \mathbb{C}_{[1,\infty)} \oplus \mathbb{C}_{(-\infty,-1]}) \cong \mathbb{C}[1]_{(-1,1)}$$

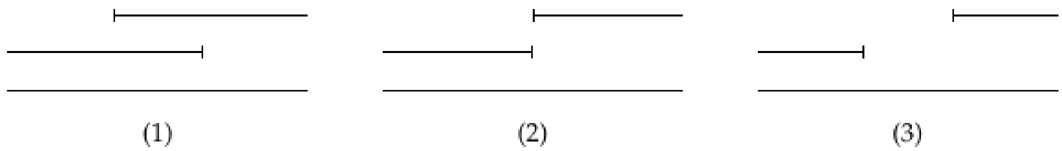


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Figure 7.2. Twisted Polytope Sheaves for $\mathbb{P}^1$

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where we briefly abuse notation and denote by $\mathbb{C}_A$ the constant sheaf supported on the subset A. The supports of the standard sheaves in the chain complexes also shown in Figure 7.2. △

Since $M_{\mathbb{R}}$ is a vector space, we have the addition operation $v : M_{\mathbb{R}} \times M_{\mathbb{R}} \to M_{\mathbb{R}}$. The addition operation induces the convolution product $\star$ for sheaves $Sh(M_{\mathbb{R}})$

$$F_1 \star F_2 := v_!(F_1 \boxtimes F_2).$$

We have the following properties of twisted polytope sheaves.

Proposition 7.4.5. *Let Σ be a smooth projective fan, $D = \sum_\rho a_\rho D_\rho$ a toric $\mathbb{R}$-divisor, and $P(D)$ the twisted polytope sheaves on $M_{\mathbb{R}}$. Then*

(1) *If D is integral, then there is a unique up to isomorphism equivariant line bundle $\mathcal{O}_X(D)$ on X_Σ, and*

$$\kappa(\mathcal{O}_X(D)) = P(D).$$

In particular $P(0) = j_{\{0\}*}\mathbb{C}_{\{0\}}$ is the skyscraper sheaf at point 0.

(2) *If D is an ample divisor, then $P(D)$ is a costandard sheaf supported on a simplicial convex polytope, with each facet corresponding to a ray $\rho \in \Sigma(1)$, and each vertex corresponding to a maximal cone $\sigma \in \Sigma(n)$.*

(3) *If $-D$ is an ample divisor, then $P(D)$ is a standard sheaf supported on a simplicial convex polytope. $P(D) = a_* \mathbb{D}(P(-D))$, where $a : M_{\mathbb{R}} \to M_{\mathbb{R}}$ sends $x \mapsto -x$.*

(4) *$P(D)$ has compact support in $M_{\mathbb{R}}$. For any $x \in M_{\mathbb{R}}$, the stalk $P(D)_x$ has cohomology in degrees between $-n$ and 0.*

(5) *If $D = D_1 + D_2$, then $P(D) = P(D_1) \star P(D_2)$, where $\star$ is the convolution product on $M_{\mathbb{R}}$.*

(6) *Let χ be any twisted polytope, then $(-) \star P(\chi) : Sh(M_{\mathbb{R}}, \Lambda_\Sigma) \to Sh(M_{\mathbb{R}}, \Lambda_\Sigma)$ is an equivalence of cateogry. The functor $(-) \star P(\chi)$ has an inverse $(-) \star P(-\chi)$*

Proof. The results are given in [**?, Tr**], with straightforward adaptations from integer to real coefficients. □

Lemma 7.4.6. *Let $D = \sum_\rho a_\rho D_\rho$. If a_ρ is not an integer for any $\rho \in \Sigma(1)$, then $SS^\infty(P(D)) \cap \Lambda_\Sigma^\infty = \emptyset$.*

Proof. From the chain complex definition for $P(\chi)$, we have

$$SS(P(\chi)) \subset \bigcup_{\sigma \in \Sigma} SS(P(\sigma, \chi_\sigma)) = \bigcup_{\sigma \in \Sigma} (\chi_\sigma + \sigma^\perp) \times \sigma.$$

If $(x, p) \in SS(P(\chi)) \cap \Lambda_\Sigma$ and $p \neq 0$, then there are non-zero cones $\sigma, \tau \in \Sigma$, such that

$$(x, p) \in \left((\chi_\sigma + \sigma^\perp) \times \sigma\right) \bigcap \left((M + \tau^\perp) \times \tau\right).$$

Hence $p \in \sigma \cap \tau$.Thus $\sigma \cap \tau$ contains at least a ray $\rho \in \Sigma(1)$, otherwise $p = 0$. Consider $\langle x, v_\rho \rangle$. Since $x \in \chi_\sigma + \sigma^\perp$, we have $\langle x, v_\rho \rangle = a_\rho$. On the other hand, $x \in M + \tau^\perp$, hence $\langle x, v_\rho \rangle \in \mathbb{Z}$. This contradicts with $a_\rho \notin \mathbb{Z}$ for any $\rho \in \Sigma(1)$. Thus the lemma is proven. □

Definition 7.4.7 (Twisted Polytope Sheaves on T_M)**.** Let χ be a twisted polytope for Σ, $P(\chi)$ the twisted polytope sheaf for χ on $M_{\mathbb{R}}$, then the twisted polytope sheaf for χ on T_M is

$$\overline{P}(\chi) := \pi_* P(\chi) = \pi_! P(\chi).$$

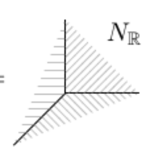
where $\pi_* = \pi_!$ since π is proper on $\operatorname{Supp} P(\chi)$.

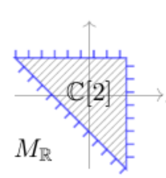
Remark 7.4.8. For any lattice point $x \in M$, the shifted polytope $\chi + x$ defines the same twisted polytope sheaf, $\overline{P}(\chi) = \overline{P}(\chi + x)$, since $\pi = \pi \circ (\cdot + x) : M_{\mathbb{R}} \to T_M$.

The following is stated in [Tr].

Proposition 7.4.9. *If D is an integral twisted polytope, then the non-equivariant CCC functor $\overline{\kappa}$ sends $\mathcal{O}_X(D)$ to $\overline{P}(D)$.*

Example 7.4.10. Consider the following two dimensional fan Σ, with ray generators $v_1 = (1,0), v_2 = (0,1), v_3 = (-1,-1)$. Let $D = D_1 + D_2 + D_3$ where D_i is the toric divisor for the ray v_i, then φ_D is a strictly positive function on $N_{\mathbb{R}}$, such that $\varphi_D(v_i) = 1$. The vertices for the twisted polytope χ_D are $(1,1), (-2,1), (1,-2)$ in $M_{\mathbb{R}}$. The twisted polytope sheaf $P(\chi)$ is the costandard sheaf supposed on the interior of the shaded region. The blue hairs indicate the singular support $SS^\infty(P(\chi))$ at infinity.

$\Sigma =$ 

, $P(\chi) =$ 

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△

7.5. Proof of the Main Theorem

Let $\mathcal{P}$ be the full dg subcategory of $Sh(M_{\mathbb{R}}, \Lambda_\Sigma)$ spanned by the *integral* twisted polytope sheaves on $M_{\mathbb{R}}$, and let $\overline{\mathcal{P}}$ be the full dg subcategory of $Sh(T_M, \overline{\Lambda}_\Sigma)$ spanned by the *integral* twisted polytope sheaves on T_M.

The follow proposition is the heart of this paper. We first define the probe sheaves for the stalk over $x \in M_{\mathbb{R}}$ and $\theta \in T_M$.

Definition 7.5.1. For any $x \in M_{\mathbb{R}}$, let the integral toric divisor $D_{[x]}$ be defined by

$$D_{[x]} := \sum_{\rho} (\lfloor \langle x, v_\rho \rangle \rfloor + 1) D_\rho, \quad P_{[x]} := P(D_{[x]}).$$

For any $\theta \in T_M$, we may fix any lift x of θ in $M_{\mathbb{R}}$, then define

$$\overline{P}_{[\theta]} := \pi_* P_{[x]}.$$

Since different lifts of x differ by an element in M, hence the push-forward is independent of the choice of the lift. (cf. Remark 7.4.8.)

Proposition 7.5.2. *For any point $\theta \in T_M$, there is a unique twisted polytope sheaf $\overline{P}_{[\theta]}$ on T_M, such that for any sheaf $\overline{F} \in Sh(T_M, \overline{\Lambda}_\Sigma)$, the stalk at θ can be computed by*

$$\overline{F}_\theta \cong hom(\overline{P}_{[\theta]}[-n], \overline{F}).$$

Proof. Fix any $x \in \pi^{-1}(\theta)$. For any sheaf $\overline{F} \in Sh(T_M, \overline{\Lambda}_\Sigma)$, let $F = \pi^{-1}\overline{F} = \pi^! \overline{F}$. Then we have canonical isomorphisms

$$F_x \cong \overline{F}_\theta,$$

and

$$hom(\overline{P}_{[\theta]}[-n], \overline{F}) \cong hom(\pi_! P_{[x]}[-n], \overline{F}) \cong hom(P_{[x]}[-n], \pi^! \overline{F}) = hom(P_{[x]}[-n], F).$$

Hence it suffices to prove that for any fixed $x \in M_{\mathbb{R}}$, we have

(*) $$F_x \cong hom(P_{[x]}[-n], F).$$

Since Σ is smooth projective, there exists an integral ample toric divisor

$$A = \sum_{\rho} a_\rho D_\rho, \quad a_\rho \in \mathbb{Z}_{>0}.$$

Then the twisted polytope sheaf $P(A)$ is supported on Δ_A, with stalk $\mathbb{C}[n]$. Since $a_\rho > 0$, we have $0 \in \Delta_A$.

Fix $\epsilon_0 > 0$ small enough, depending only on x and Λ_Σ, such that for any $0 < \epsilon \leq \epsilon_0$,

$$F_x \cong \Gamma(\Delta_{D_{(x)}+\epsilon A}, F)$$

where $\Delta_{D_{(x)}+\epsilon A}, = x + \epsilon\Delta_A$ is a shifted open convex polytope around x. This is possible since F is a polyhedral constructible sheaf, and Δ_A is a convex set. In particular, we may shrink ϵ_0 and further assume that

$$\epsilon_0 a_\rho + \langle x, v_\rho \rangle < \lfloor \langle x, v_\rho \rangle \rfloor + 1, \quad \text{for all } \rho \in \Sigma(1).$$

Fix $R > 0$ a large enough integer, such that $D_{[x]} + RA$ is an ample integral toric divisor. Let $\Delta_{D_{[x]}+RA}$ be the corresponding open convex polytope.

For any $s \in [0,1]$, we define a 1-parameter family of ample divisors D_s, interpolating between $x + (R+\epsilon_0)A$ and $D_{[x]} + RA$.

$$D_s = \sum_\rho a_{\rho,s} D_\rho, \quad a_{\rho,s} = (1-s)(\langle x, v_\rho \rangle + (R+\epsilon_0)a_\rho) + s(\lfloor \langle x, v_\rho \rangle \rfloor + 1 + Ra_\rho),$$

and let

$$\Delta_s := \Delta_{D_s}, \quad \text{and} \quad P_s := P(D_s).$$

Since $\lfloor \langle x, v_\rho \rangle \rfloor + 1 > \langle x, v_\rho \rangle + \epsilon_0 a_\rho > \langle x, v_\rho \rangle$, and there is no integer in the open interval $(\langle x, v_\rho \rangle, \lfloor \langle x, v_\rho \rangle \rfloor + 1)$, hence for any $s \in (0,1)$,

$$\langle x, v_\rho \rangle + \epsilon_0 a_\rho < a_{\rho,s} - Ra_\rho < \lfloor \langle x, v_\rho \rangle \rfloor + 1, \text{ and } a_{\rho,s} \notin \mathbb{Z}.$$

Thus from Lemma 7.4.5 ,

$$SS^\infty(P_s) \cap \Lambda_\Sigma^\infty = \emptyset \quad \text{for all } \ s \in (0,1).$$

Apply the non-characteristic deformation result in Proposition 3.5.1, let

$$U_t = \begin{cases} \Delta_0, & t \leq 0 \\ \Delta_{t/(1+t)}, & t > 0 \end{cases}$$

we have for any sheaf $G \in Sh(M_\mathbb{R}, \Lambda_\Sigma)$,

$$\Gamma(\Delta_1, G) = \Gamma(\cup_{s \in (0,1)} \Delta_s, G) \cong \Gamma(\Delta_t, G) \quad \text{for all } \ t \in [0,1).$$

Since D_s are ample divisor for all $s \in [0,1]$ (since ample cone is convex), and $P(D_s) = j_{\Delta_s !}\omega_{\Delta_s} \cong j_{\Delta_s !}\mathbb{C}_{\Delta_s}[n]$, we have

$$\Gamma(\Delta_s, G) = hom(j_{\Delta_s !}\mathbb{C}_{\Delta_s}, G) \cong hom(P(D_s)[-n], G), \quad \text{for all } \ s \in [0,1].$$

Finally, we use convolution $\star$ is an equivalence of category on $Sh(M_{\mathbb{R}})$ to get

$$\begin{aligned} F_x \cong \Gamma(x+\epsilon\Delta_A, F) &\cong hom(j_{x+\epsilon\Delta_A !}\mathbb{C}_{x+\epsilon\Delta_A}, F) \\ &\cong hom(j_{x+\epsilon\Delta_A !}\mathbb{C}_{x+\epsilon\Delta_A} \star P(RA), F \star P(RA)) \\ &\cong hom(P(D_0)[-n], F \star P(RA)) \\ &\cong hom(P(D_1)[-n], F \star P(RA)) \\ &\cong hom(P(D_1 - RA)[-n], F) \\ &\cong hom(P(D_{[x]})[-n], F). \end{aligned}$$

This finishes the proof of the Proposition. □

Now we prove the main theorem stated in the introduction.

Proof of the Main Theorem. First we claim that there exists a semi-orthogonal expansion

$$Sh(T_M, \overline{\Lambda}_\Sigma) \cong \langle \langle \overline{\mathcal{P}} \rangle^{\perp}, \langle \overline{\mathcal{P}} \rangle \rangle.$$

Since $\overline{\kappa}$ is an quasi-embedding, hence

$$Coh(X_\Sigma) \xrightarrow{\sim} \overline{\kappa}(Coh(X_\Sigma)).$$

Since line bundles generates $Coh(X_\Sigma)$, hence

$$\overline{\kappa}(Coh(X_\Sigma)) \cong \overline{\kappa}(\langle\{\mathcal{L} : \text{ line bundles}\}\rangle) \cong \langle\overline{\mathcal{P}}\rangle.$$

Kawamata proved that $Coh(X_\Sigma)$ admits an exceptional collection [?], hence $\langle\overline{\mathcal{P}}\rangle$ also admits an exceptional collection. By Proposition 2.6 and Corollary 2.10 in [BK], $\langle\overline{\mathcal{P}}\rangle$ is saturated and is left and right admissible. In other words, the semi-orthogonal decomposition in the claim exists.

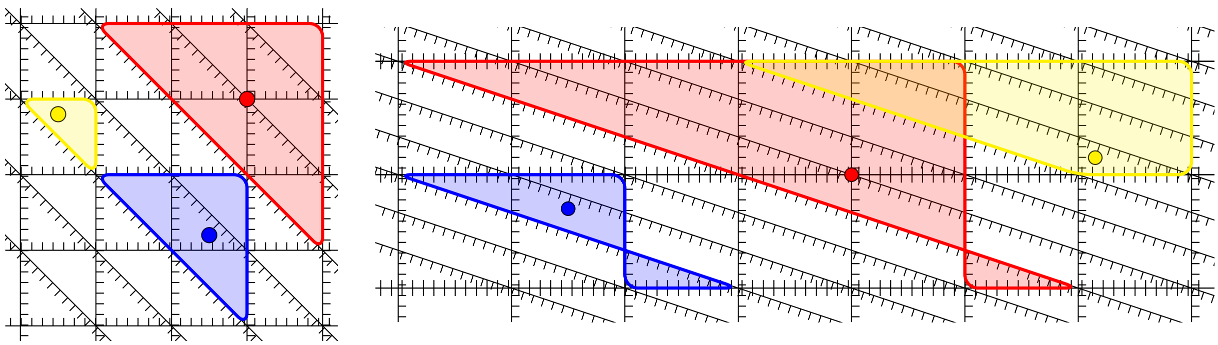
From Proposition 7.5.2, we have $\langle\overline{\mathcal{P}}\rangle^\perp = 0$. Hence $Sh(T_M, \overline{\Lambda}_\Sigma) \cong \langle\overline{\mathcal{P}}\rangle \cong Coh(X_\Sigma)$. $\square$

Example 7.5.3. We consider two toric surfaces, with some twisted polytopes $P_{(x)}$ shown in Figure 7.3.

(1) Let $X_\Sigma = \mathbb{P}^2$. The red, blue and yellow twisted polytopes are $\mathcal{O}(3), \mathcal{O}(2), \mathcal{O}(1)$ respectively.

(2) Let $X_\Sigma = \mathbb{F}_3$, with ray generators $(1,0), (0,1), (-1,-3), (-1,0)$. This is a smooth non-Fano projective toric surface. The red polytope corresponds to the anti-canonical bundle, with all $a_\rho = 1$. Indeed, it is non-Fano since the anti-canonical bundle is twisted. The yellow polytope is ample.

$\triangle$



(A) $\mathbb{P}^2$ (B) Hirzebruch surface $\mathbb{F}_3$

Figure 7.3. Various probe sheaves $P_{[x]}$ (shown as colored twisted polytopes) on $M_{\mathbb{R}}$ for different x (shown as solid dots).

CHAPTER 8

Lagrangian Thimbles and Vanishing Cycles

In this chapter, we will build a fully-faithful functor from the Fukaya-Seidel category for a Laurent polynomial (with certain condition) to a category of constructible sheaves on a real torus with certain singular support condition. Then we will show that this is an equivalence of category.

Results from previous chapters will be used here, the result about Liouville skeleton for a regular fiber of the Picard-Lefschetz fibration will be used to push the vanishing cycles into a neighborhood of the skeleton; the autoequivalence of the constructible sheaf category induce by changing singular support conditions will be used to study the effect of changing the phase angles of the coefficients of the Laurent polynomial.

More precisely, we will proceed in the following three steps. We use the same notation of lattices M, N and Newton polytope Q as in the introduction. For clarify, we chose an identification $M \cong \mathbb{Z}^n$ and hence $T_M \cong T^n$.

(1) Prove that any Lagrangian thimble ending in $\mathcal{H}$ can be extended canonically to a asymptotically conical Lagrangian, which after an identification between $(\mathbb{C}^*)^n \cong T^*T^n$, gives an object in $\mathrm{Fuk}(T^*T^n, \Lambda_{\mathcal{T}})$. Consequently, we prove there is a canonical embedding of

$$\Phi_{\mathcal{T}} : FS((\mathbb{C}^*)^n, f) \hookrightarrow \mathrm{Fuk}(T^*T^n, \Lambda_{\mathcal{T}})$$

(2) We show that there is a monodromy action of $\mathbb{Z}^A = \pi_1(T^A)$ on both $FS((\mathbb{C}^*)^n, f)$ and $\mathrm{Fuk}(T^*T^n, \Lambda_{\mathcal{T}}) \cong Sh(T^n, \Lambda_{\mathcal{T}})$, compatible with the above embedding $\Phi_{\mathcal{T}}$.

(3) We show that the image of the distinguished real thimble, under the monodromy action of $\mathbb{Z}^A$, generates the target category $Sh(T^n, \Lambda_{\mathcal{T}})$, hence $\Phi_{\mathcal{T}}$ is an equivalence of categories.

8.1. From Thimble to Asymptotically Conical Lagrangian

Recall that our tropical polynomial is defined as

$$f_{R,h,\theta}(z) = \sum_{\alpha \in \partial A} R^{-h(\alpha)} e^{-i\theta(\alpha)} z^\alpha.$$

We choose the regular value at $R^{-h(0)}e^{-i\theta(0)} = e^{-i\theta(0)}$. Without loss of generality, we may choose $\theta(0) = 0$. Let

$$\mathcal{H}_{R,h,\theta} := f^{-1}_{R,h,\theta}(1)$$

denote this regular fiber. Let $f_{R,s,h,\theta}$ be Abouzaid's tropical localization for value 1, and $\mathcal{H}_{R,s,h,\theta} = f^{-1}_{R,s,h,\theta}(1)$. For simplicity of notation, we will drop h, θ from the subscript.

First, we introduce the parameter space $(\mathbb{C}^*)^A$ of hypersurfaces defined by equations with monomials from the set A, and define the descrimant locus, secondary polytope, distribution of the critical values following the book [GKZ] and the paper [DKK]. We will explain the reason for introducing the triangulation when Q is not a facet-simplicial polytope.

Then, we begin the construction of an asymptotically conical Lagrangian from a thimble. Here by a thimble we mean the sweep-out by the symplectic parallel transport from a singular fiber of f_R to the regular fiber $\mathcal{H}_R = f_R^{-1}(1)$ along a vanishing path in $\mathbb{C}$. We now begin a sequence of deformation of the vanishing cycle $\mathcal{S}$ and then the thimble $\mathcal{D}$:

(1) First, we deform the fiber $\mathcal{H}_R = \mathcal{H}_{R,0}$ to $\mathcal{H}_{R,1}$, and parallel transport $\mathcal{S}$ into $\mathcal{H}_{R,1}$.

(2) Then, we apply the contracting Liouville flow in $\mathcal{H}_{R,1}$ to contract the vanishing cycle to a small enough tubular neighborhood neighborhood of the skeleton $\mathrm{Skel}(\mathcal{H}_{R,1})$. Here we use the special Kähler potential, hence the skeleton is homeomorphic to the RSTZ-skeleton.

(3) Finally, we prove that the thimbles can be deformed along the way through admissible Lagrangians, following [**Ab1**] section 2.

Finally, we will extend the deformed thimble to infinity through a one-parameter family of hypersurfaces $\mathcal{H}_{R,1}$ with R running from the fixed value to ∞. We note that this is different from let the regular value of f runs to infinity. We illustrate the difference by the following example.

Example 8.1.1. Let Q be the convex polygon in $\mathbb{R}^3$ defined by

$$Q = \mathrm{conv}(\partial A), \quad \partial A := \{(1,0,1), (-1,0,1), (0,1,1), (0,-1,1), (0,0,-1)\},$$

i.e an upside-down pyramid. For any Laurent polynomial $f(x,y,z)$ with Newton polytope Q, if the value R of f runs to infinity along the real line, then rescaled amoeba of $f^{-1}(R)$, i.e.

$$\mathcal{A}_R = \frac{1}{\log R}(\{\log|z_1|, \log|z_2|, \log|z_3|) \mid f(z) = R\} \subset \mathbb{R}^3$$

tends to the tropical amoeba corresponding to the PL function $\widehat{h}: Q \to \mathbb{R}$ such that $\widehat{h}(0) = 0, \widehat{h}|_{\partial Q} = 1$. This $\widehat{h}$ induces a star subdivision of Q based at 0, but not a triangulation.

On the other hand, if we choose a coherent star subdivision $\mathcal{T}$ of Q induced by h, and $f_{R,h} = \sum_{\alpha \in \partial A} R^{-h(\alpha)} z^{\alpha}$, then as $R \to \infty$, the rescaled amoeba

$$\mathcal{A}'_R = \frac{1}{\log R}(\{\log|z_1|, \log|z_2|, \log|z_3|) \mid f_{R,h}(z) = 1\} \subset \mathbb{R}^3$$

will tend to the tropical amoeba corresponding to h. There is a nice skeleton supported on $f^{-1}_{R,h}(1)$ for all large R. $\triangle$

Remark 8.1.2. If the polytope Q is facet simplicial (we don't require the verties of the facet to be a $\mathbb{Z}$-basis), and we take $\mathcal{T}$ to be the star triangulation generated by faces of Q. Then the above construction can be much simplified. We do not have to take the regular fiber of f at R and deform the thimble ending on the fiber, instead one can let the thimble runs to ∞ and prove that the vanishing cycle automatically concentrate to the skeleton of the fiber of ∞. This method will break down, if the cone over faces Q is not simplicial, since as the value of f goes to infinity, one is essentially considering the equation

$$1 = \sum_{\alpha} \frac{R^{-h(\alpha)} e^{-i\theta(\alpha)}}{f(z)} z^{\alpha}$$

we may view $f(z)$ itself as a tropicalizing parameter, and the weight of the vertices are all 1. The resulting polytope subdivision of Q is not a triangulation, and the tropical skeleton is not defined.

The (new) method adopted here has two additional advantages. First, we can choose the triangulation $\mathcal{T}$, and the resulting sheaf category should be all be equivalent (through a non-canonical equivalence). This change of skeleton (by changing h, or equivalently, the triangulation $\mathcal{T}$) is less well understood on the constructible sheaf side. Secondly, by extending a Lagrangian thimble with boundary to a conical Lagrangian, we may consider the sequence of nested Fukaya-Seidel categories $FS_1 \hookrightarrow FS_2 \hookrightarrow \cdots$ (the inclusion is not canonical due to auto-equivalences), by considering more and more critical values, as done in [**DKK**]. One can build corresponding conical Lagrangians $\Lambda_1, \Lambda_2, \cdots$ and get equivalence on the filtered level. This would not be possible if we were to allow the value of f to run to ∞.

8.1.1. Deformation of Lagrangian Thimbles with boundary

Let (M, ω) be a symplectic manifold. First, we state a result of Weinstein neighborhood theorem for Lagrangian with boundary.

Proposition 8.1.3 ([**Ab1**], Lemma 2.1)**.** *Let $(L, \partial L)$ be an exact Lagrangian with boundary ∂L, and E be an oriented rank* 1 *sub-bundle fo the symplectic orthogonal complement of $T\partial L$ such that the pairing*

$$TL|_{\partial L} \otimes E \to \mathbb{R}$$

induced by the symplectic form is non-degenerated and yields the appropriate co-orientation on ∂L. In side a sufficiently small neighborhood of L in N, there exists a full dimensional submanifold with boundary $(V_L, \partial V_L)$, such that the inclusion $(L, \partial L) \subset (V_L, \partial V_L)$ satisfies the following properties:

(a) The restriction of $T\partial V_L$ to ∂L contains the sub-bundle E.

(b) There exists a symplectomorphism $(V_L, \partial V_L) \to (T^*L, T^*L|_{\partial L})$ identifying L with the zero section of its cotangent bundle, such that the following diagram commutes

$$\begin{tikzcd}
(V_L, \partial V_L) \arrow[rr] \arrow[dr] & & (T^*L, T^*L|_{\partial L}) \arrow[dl] \\
& (L, \partial L) &
\end{tikzcd}$$

Proposition 8.1.4 ([Ab1], Lemma 2.2)**.** *Let L' and L be two Lagrangian manifolds of M which have the same boundary. Let V_L be a submanifold of L in Proposition 8.1.3. If L' is transverse to ∂V_L and there is a neighborhood of $\partial L'$ in L' which is contained in V_L, then there exists a Lagrangian submanifold L'' which satisfies the following conditions:*
(a) L'' is Hamiltonian isotopic to L.
(b) L'' agrees with L' in a sufficiently small neighborhood of $\partial L'$.
(c) L'' agrees with L away from a larger neigbhorhood of the boundary.
Moreover, L'' is independent, up to Hamiltonian isotopy, of the choices made in its construction.

Next, we start moving the fiber $\mathcal{H}_{R,s} = f_{R,s}^{-1}(1)$ from $s = 0$ to $s = 1$. By Abouzaid's result, the 1-parameter family of hypersurfaces $\{\mathcal{H}_{R,s}\}_s$ are all symplectic hypersurfaces, and in fact for each s, $f_{R,s}$ is a symplectic fibratio near value 1. In particular, we obtain a symplectic connection. To be more precise, let U be a small enough neighborhood around 1, and we consider the fibration

$$f_{R,-} : X = (\mathbb{C}^*)^n \times [0,1] \to \mathbb{C} \times [0,1],$$

restricted to $U \times [0,1]$. The ambient two-form ω is the pull-back from the one $(\mathbb{C}^*)^n$ defined by $2i\partial\bar{\partial}\varphi$. The path in the base, along which we define the parallel transport is $1 \times [0,1]$. Let $Y_{R,s}$ be the horizontal lift of $-\frac{\partial f_{R,s}}{\partial s}$. We have the following bound on $Y_{R,s}$.

Proposition 8.1.5 ([Ab1], Lemma 4.11, 4.12, 4.13)**.** *$Y_{R,s}$ is a bounded vector field, hence integrates to a locall diffeomorphism near the fiber $\mathcal{H}_{R,s}$. The flow generated by $Y_{R,s}$ restricts to a symplectomorphism between $\mathcal{H}_{R,0}$ and $\mathcal{H}_{R,1}$.*

More over, there exists a Hamiltonian time-dependent vector field $Y'_{R,s}$ on $(\mathbb{C}^*)^n$, which is supported in a neighborhood of $\mathcal{H}_{R,s}$, and which integrate to a symplectic flow ψ_s that maps $\mathcal{H}_{R,0}$ to $\mathcal{H}_{R,s}$.

Definition 8.1.6. Let $f : X \to \mathbb{C}$ be a symplectic fibration near value 0. We say a Lagrangian $(L, \partial L)$ is an admissible Lagrangian with respect to f ending in $f^{-1}(0)$ with ending direction $\nu \in (-\pi, \pi)$, if the following holds:

(1) $\partial L \subset f^{-1}(0)$.

(2) $f(L)$ agrees with the half-line $\mathbb{R}_{<0} \cdot e^{i\nu}$ near 0.

Let L, L' be two admissible Lagrangians with respect to f, ending in $f^{-1}(0)$ with the same ending direction $\nu \in (-\pi, \pi)$. We say L, L' are Hamiltonian isotopic through admissible Lagrangians, if for any small $\delta > 0$, there exists a Hamiltonian isotopy $\{L(t)\}_{t\in[0,1]}$ with $L(0) = L$, $L(1) = L'$, such that

(1) $\partial L(t) \subset f^{-1}(0)$.

(2) $f(L(t))$ is contained in the cone $\{\mathbb{R}_{<0} \cdot e^{i\theta} \mid \theta \in (\nu - \delta, \nu + \delta)\}$ near 0.

Proposition 8.1.7. *Given a Lagrangian thimble $\mathcal{D}_{R,0}$ with respect to the Lefschetz fibration $f_{R,0}$, ending on vanishing spheres $\mathcal{S}_{R,0}$ in the fiber $\mathcal{H}_{R,0}$, with vanishing path*

$\gamma : [0,1] \to \mathbb{C}$ from a critical value to 1, such that $\dot{\gamma}(1) = \nu \in T_1\mathbb{C}$ pointing to the right half-plane, there exists an admissible Lagrangians disks $\mathcal{D}_{R,1}$ with respect to the Lefschetz fibration $f_{R,1}$, ending on an exact Lagrangian sphere $\mathcal{S}_{R,1}$ in the fiber $\mathcal{H}_{R,1}$, with the same admissable direction ν, and $\mathcal{D}_{R,1}$ is Hamiltonian isotopic to $\psi_1(\mathcal{D}_{R,0})$.

Proof. Recall $\psi_s : ((\mathbb{C}^*)^n, \mathcal{H}_{R,0}) \to ((\mathbb{C}^*)^n, \mathcal{H}_{R,s})$ is a symplectomorphism defined in Proposition 8.1.5. Let $F_{R,s} = f_{R,s} \circ \psi_s$, then $F_{R,s}^{-1}(1) = \mathcal{H}_{R,0}$for all s. The Lagrangian disk $\mathcal{D}_{R,0}$ is admissible for $F_{R,0}$, but cutting $[0,1]$ into sufficiently fine intervals with end points $0 = s_0 < s_1 < \cdots < s_N = 1$, and apply Proposition 8.1.4, we can define a family of Lagrangians $\widetilde{\mathcal{D}}_{R,s_i}$ with the same boundary $\mathcal{S}_{R,0}$ that is admissible for all F_{R,s_i}.

More precisely, for each $s \in [0,1]$, we symplectic parallel transport the vanishing sphere $\mathcal{S}_{R,0}$ along a small straight line segment from 1 to $1 - \epsilon\nu$ with respect to the fibration $F_{R,s}$, define that as $\mathcal{L}_s$. The rank 1 sub-bundles E_s along $\mathcal{S}_{R,0}$, used in the construction of the neighorhood of $\mathcal{L}_s$ in Proposition 8.1.3, can be chosen to be the horizontal lift of tangent vector $J\nu$ with respect to $F_{R,s}$. By successively apply Proposition 8.1.4 for a sufficiently fine partition of $[0,1]$, we can get $\widetilde{\mathcal{D}}_{R,s_i}$ to agree with $\mathcal{L}_{s_i}$.

Finally, we define $\mathcal{D}_{R,1} = \psi_1(\widetilde{\mathcal{D}}_{R,1})$. This finshes the proof of the proposition. □

Proposition 8.1.8. *Any Hamiltonian isotopy of $\mathcal{S}_{R,1}$ inside $\mathcal{H}_{R,1}$ induces an Hamiltonian isotopy of the Lagrangian $\mathcal{D}_{R,1}$ through admissible Lagrangians.*

Proof. Let $t \in [0,1]$ parameterize the Hamiltonian isotopy. Let $H_t : \mathcal{S}_{R,1} \to \mathbb{R}$ be the time-dependent Hamiltonian inducing the isotopy. Using the Weinstein neighborhood Proposition 8.1.3, we can extend the Hamiltonian isotopy of the boundary to a compactly supported Hamiltonian isotopy of the zero-section. □

Remark 8.1.9. Since $\mathcal{S}_{R,1}$ is diffeomorphic to S^{n-1}, for $n > 2$ any Lagrangian isotopy of $\mathcal{S}_{R,1}$ is automatically a Hamiltonian isotopy. The Lagrangian isotopy induced by Liouville flow on any exact Lagrangian L is also automatically Hamiltonian, since $\iota_{X_\lambda}\omega|_L = \lambda|_L = dH$ by the definition of exact Lagrangian.

Combining the results in this section, we have the following deformation result of thimbles.

Proposition 8.1.10. *Let $\mathcal{D}_{R,1}$ be an admissible Lagrangian with respect to $f_{R,1}$ ending on the fiber $\mathcal{H}_{R,1}$, with ending direction $\nu \in T_1\mathbb{C}$. Then for any $\epsilon > 0$, $\mathcal{D}_{R,1}$ is Hamiltonian isotopic to an admissible Lagrangians $\mathcal{D}'_{R,1}$ with ending direction ν where the boundary is contained in an ϵ-neighborhood of the skeleton* $\mathrm{Skel}(\mathcal{H}_{R,1})$*.*

Proof. We apply the contractible Liouville flow $\Phi^t_{-X_\lambda}$ on the compact Lagrangian $\mathcal{S}_{R,1}$ for time $t \in [0, T]$, such that $\Phi^T_{-X_\lambda}(\mathcal{S}_{R,1})$ is contained in an ϵ-neighborhood of the skeleton $\mathrm{Skel}(\mathcal{H}_{R,1})$. Since $\mathcal{S}_{R,1}$ is an exact Lagrangian, the Lagrangian isotopy generated by the Liouville flow is an Hamiltonian isotopy. Using Proposition 8.1.8, we can extend the Hamiltonian isotopy of the boundary to the Lagrangian $\mathcal{D}_{R,1}$. By a similar argument as in Proposition 8.1.7, we can partition the time segment $[0, T]$ to sufficiently small segments, such that the Lagrangian at the endpoints time are admissible. $\square$

8.1.2. Extension of the Deformed Lagrangian Thimble to Infinity

In the previous subsection, we have deformed a Lagrangian thimble $\mathcal{D}_{R,0}$ with respect to the holomorphic Picard-Lefschetz fibration $f_{R,0}$ to an admissible Lagrangian $\mathcal{D}_{R,1}$ with respect to $f_{R,1}$, such that the boundary $\mathcal{S}_{R,1} = \partial\mathcal{D}_{R,1}$ is close to the skeleton of $f_{R,1}$.

In this section, we will consider a family of hypersurface $\mathcal{H}_{r,1}$ parametrized by $r \in [R, \infty)$, and parallel transport $\mathcal{S}_{R,1}$ along the way. We will prove that $\cup_{r\in[R,\infty)}\mathcal{S}_{r,1}$ sweeps out an asymptotically conical Lagrangians.

We recall the result from Abouzaid, adapted to our notation here.

Proposition 8.1.11 ([Ab1] Proposition 4.9)**.** *After rescaling the symplectic form by* $\log(r)$*, the pairs* $((\mathbb{C}^*)^n, \mathcal{H}_{r,1})$ *are symplectomorphic for all* $r \in [R, \infty)$*.*

We will refine the result on the symplectic parallel transport in the neighborhood of the Liouville skeleton $\Lambda_{r,1} \subset \mathcal{H}_{r,1}$. Let $Z_{r,1}$ be the horizontal lift of $-\frac{\partial f_{r,1}}{\partial \log r}$, that is, we want

$$(f_{r,1})_*(Z_{r,1}) = -\frac{\partial f_{r,1}}{\partial \log r} = \sum_{\alpha\in\partial A}[-\frac{\partial \chi_{r,1,\alpha}(z)}{\partial \log r} + \chi_{r,1,\alpha}(z)h(\alpha)]e^{-i\theta(\alpha)}r^{-h(\alpha)}z^{\alpha} \tag{8.1.1}$$

and $Z_{r,1}(p) \in (T_p\mathcal{H}_{r,1})^{\perp}$, where $\perp$ means the ω-orthogonal. First we find the parallel transport vector $Z_{r,1}$ on skeleton $\Lambda_{r,1}$.

For any $p \in \mathcal{H}_{r,1}$ and $v \in T_p(\mathbb{C}^*)^n$, we define the following decomposition

$$v = v^{\parallel} + v^{\perp}, \quad v^{\parallel} \in T_p\mathcal{H}_{r,1}, \quad v^{\perp} \in (T_p\mathcal{H}_{r,1})^{\perp}.$$

Recall the Liouville structure on $(\mathbb{C}^*)^n$ is given by $\lambda = -d^c\varphi_P = \sum_i \partial_{\rho_i}\varphi_P(\rho)d\theta_i$, which corresponds by Legendrian transformation to $\lambda = \sum_i p_i d\theta_i$ on T^*T^n. The Liouville flow on T^*T^n is then radial $\sum_i p_i\partial_{p_i}$, and since the Legendre transformation takes rays to rays, the Liouville flow on $(\mathbb{C}^*)^n$ also has integral curves as rays, i.e $X_\lambda = b(\rho)\sum_i \rho_i\partial_{\rho_i}$ for some positive function $b(\rho)$.

Proposition 8.1.12. *On skeleton* $\Lambda_{r,1}$*, we have* $Z_{r,1} \propto X_\lambda^{\perp}$*.*

Proof. First, we claim that if $p \in \Lambda_{r,1}$, then $-\frac{\partial f_{r,1}}{\partial \log r}|_p \in \mathbb{R}_{>0}$ and $X_\lambda(f_{r,1})|_p > 0$. Recall that on skeleton, all the 'relevant' summand in $f_{r,1}$ are positive and real. And the cut-off function $\chi_{r,1,\alpha}(z) = \chi_\alpha(\rho/\log r)$. Hence

$$\begin{aligned}
-\frac{\partial f_{r,1}}{\partial \log r} &= \sum_{\alpha \in \partial A} \left[\frac{\rho_i}{(\log r)^2} \partial_{\rho_i} \chi_\alpha|_{\frac{\rho}{\log r}} + \chi_\alpha(\frac{\rho}{\log r}) h(\alpha) \right] e^{\langle \rho, \alpha \rangle - \log r h(\alpha)} \\
&\geq \inf_{\alpha \in \partial A} h(\alpha) - \frac{1}{\log r} \sup_\alpha \|D\chi_\alpha\|_{C^0} \\
&\geq \inf_{\alpha \in \partial A} h(\alpha) - \frac{C}{\log r}
\end{aligned}$$

hence is positive for $r \geq R$ for large enough R. Since X_λ is proportional to the radial vector field $\sum \rho_i \partial_{\rho_i}$

$$\begin{aligned}
&(\frac{1}{\log r} \sum_i \rho_i \partial_{\rho_i})(f_{r,1})|_p \\
= &\sum_{\alpha \in \partial A} \left[\frac{\rho_i}{(\log r)^2} \partial_{\rho_i} \chi_\alpha|_{\frac{\rho}{\log r}} + \chi_\alpha(\frac{\rho}{\log r}) \langle \frac{\rho}{\log r}, \alpha \rangle \right] e^{\langle \rho, \alpha \rangle - \log r h(\alpha)} \\
\geq &\inf_{\alpha \in \partial A} h(\alpha) + \sum_{\alpha \in \partial A} \left[\chi_\alpha(\frac{\rho}{\log r})(\langle \frac{\rho}{\log r}, \alpha \rangle - h(\alpha)) \right] e^{\langle \rho, \alpha \rangle - \log r h(\alpha)} - \frac{C}{\log r} \\
\geq &\inf_{\alpha \in \partial A} h(\alpha) - \frac{C}{\log r} - \epsilon_\chi
\end{aligned}$$

where $0 < \epsilon_\chi \ll 1$ depending on the transition-width of the cut-off functions χ_α and can be made arbitrarily small.

This shows $Z_{r,1}$ is proportional to $X_\lambda^\perp$ at the point p. $\square$

Corollary 8.1.13. *The flow-out of skeleton $\Lambda_{r,1}$ is a conical Lagrangian $\mathbb{R}_{>1} \cdot \Lambda_{r,1}$, in particular the flow $Z_{r,1}$ takes skeleton to skeleton.*

Proof. The flow out of skeleton by $Z_{r,1} = X_\lambda^\perp = X_\lambda - X_\lambda^\parallel$, however $X_\lambda^\parallel$ is the Liouville flow of the hypersurface, which is tangent to the skeleton, hence the flow-out by $Z_{r,1}$ is the same as the flow-out by X_λ, again the same as the flow-out by the radial vector field, denoted as $\rho\partial_\rho$. Hence the flow-out of the skeleton $\Lambda_{r,1}$ by $Z_{r,1}$ is a conical Lagrangian $\mathbb{R}_{>1} \cdot \Lambda_{r,1}$.

Under the Legendre transformation defined by the quadratic potential φ_P, the conical Lagrangian $\mathbb{R}_{>0} \cdot \Lambda_{r,1}$ is exactly $\Lambda_{\mathcal{T},\theta}$.

Since skeleton in $\mathcal{H}_{r',1}$ is the intersection $\mathcal{H}_{r',1} \cap \mathbb{R}_{>0} \cdot \Lambda_{r,1}$, we see the flow takes skeleton to skeleton. $\square$

Next, we show that a point in $\mathcal{H}_{r,1}$ in the neighborhood of the skeleton $\Lambda_{r,1}$ will tend towards the skeleton under the flow, after proper rescaling. Let $\Lambda^\infty_{\mathcal{T},\theta} \subset M^\infty_\mathbb{R} \times T_M$ be the skeleton at infinity defined by $(\mathbb{R}_{>0} \cdot \Lambda_{r,1})/\mathbb{R}_{>0}$ for any r. Choose any Riemannian metric on $M^\infty_\mathbb{R} \times T_M$, with distance function denoted by $\mathrm{dist}^\infty(-,-)$. And let $\pi^\infty : M_\mathbb{R} \backslash \{0\} \times T_M \to M^\infty_\mathbb{R} \times T_M$ denote the projection map.

Proposition 8.1.14. *For any point (ρ_R, ϕ_R) of $\mathcal{H}_{R,1}$ in a tubular neighborhood of the Liouville skeleton of $\Lambda_{R,1}$, let (ρ_r, ϕ_r) denote its image in $\mathcal{H}_{r,1}$ under the symplectic parallel transport for the family of hypersurfaces $\{\mathcal{H}_{r,1}\}_{r>R}$. We have*

$$\lim_{r\to\infty} \mathrm{dist}^\infty(\pi^\infty(\rho_r, \phi_r), \Lambda^\infty_{\mathcal{T},\theta}) = 0.$$

Proof. We first change the notation a bit. Let $\hbar = 1/\log r$ and $\widehat{\rho} = \hbar\rho$. Let $S_\hbar : M_\mathbb{R} \times M_T \to M_\mathbb{R} \times M_T$ be the rescale map, $(\rho, \phi) \mapsto (\hbar\rho, \phi)$. Then we define

$$\mathcal{H}_\hbar := S_\hbar \mathcal{H}_{r,1} = \{(\widehat{\rho}, \phi) \in M_\mathbb{R} \times M_T \mid 1 = \sum_{\alpha \in \partial A} \chi_\alpha(\widehat{\rho}) e^{i\langle \phi, \alpha\rangle - i\theta(\alpha)} e^{\hbar^{-1}[\langle \widehat{\rho}, \alpha\rangle - h(\alpha)]}\}.$$

Suppose $\hbar$ decreases with t as $\frac{d\hbar}{dt} = -\hbar$, we will first produce a curve $(\rho(t), \phi(t)) \in \mathcal{H}_{\hbar(t)}$, then we will project the tangent vector at $t = 0$ to its symplectic-orthogonal component.

Since on the skeleton, we have

$$1 = \sum_{\alpha \in \partial A} \chi_\alpha(\widehat{\rho}) e^{\hbar^{-1}[\langle \widehat{\rho}, \alpha\rangle - h(\alpha)]}$$

hence if $e^{\hbar^{-1}[\langle \widehat{\rho}, \alpha\rangle - h(\alpha)]} > 1/10$, then $\chi_\alpha(\widehat{\rho}) = 1$, and we have $|\langle \widehat{\rho}, \alpha\rangle - h(\alpha)| = O(\hbar)$. We will find a curve $(\widehat{\rho}(t), \phi(t))$, such that for α-term dominating[1], we have

$$\begin{aligned} 0 &= \frac{d}{dt}[\hbar(t)^{-1}(\langle \widehat{\rho}(t), \alpha\rangle - h(\alpha))] \\ &= \hbar(t)^{-1}(\langle \widehat{\rho}(t), \alpha\rangle - h(\alpha)) + \hbar(t)^{-1}\frac{d\langle\widehat{\rho}(t), \alpha\rangle}{dt} \Rightarrow \frac{d\langle\widehat{\rho}(t), \alpha\rangle}{dt} = -(\langle \widehat{\rho}(t), \alpha\rangle - h(\alpha)) = O(\hbar) \end{aligned}$$

Hence, we may choose a solution for $\widehat{\rho}(t)$, such that $(d/dt)\widehat{\rho}(t) = O(\hbar)$. Thus, we have

$$\frac{d}{dt}\rho(t) = \frac{d}{dt}[\hbar^{-1}(t)\widehat{\rho}(t)] = \hbar^{-1}(\widehat{\rho}(t) + O(\hbar)).$$

Next, we can take the symplectic orthogonal complement.

$$[\frac{d}{dt}\rho(t)]^\perp = \hbar^{-1}(\widehat{\rho}(t)^\perp + O(\hbar)) = \hbar^{-1}(cX_\lambda^\perp|_{(\widehat{\rho},\phi)} + O(\hbar))$$

[1]Here we are being loose in definition. One way of defining α-term dominating is $\chi_\alpha > 0$, but this is too large. A more useful definition is to have $\chi_\alpha > 0$ and $0 > h(\alpha) - \langle \rho, \alpha\rangle < \hbar^\epsilon$ for some $\epsilon < 1$. We also ignore $d\chi_\alpha$ terms, since when χ_α drops from 1 to 0, the corresponding term $e^{\hbar^{-1}[\langle \widehat{\rho}, \alpha\rangle - h(\alpha)]} = O(\hbar^\infty)$.

Here $X_\lambda^\perp$ is symplectic orthogonal projection for the rescaled hypersurface $\mathcal{H}_\hbar$, and

$$X_\lambda^\perp = X_\lambda - X_\lambda^\parallel.$$

Hence the symplectic transport vector $Z_{r,1}$ at (ρ, ϕ), when push-forward by $(S_\hbar)_*$, can be written as $X_\lambda - X_\lambda^\parallel + O(\hbar)$, note that X_λ is radial hence does change the projection image to $M_\mathbb{R}^\infty$, hence only the contracting Liouville vector field $X_\lambda^\perp$ on the fixed size amoeba $\mathcal{A}_\hbar = \{\widehat{\rho} \in M_\mathbb{R} \mid 1 = \sum_\alpha \chi_\alpha(\widehat{\rho}) e^{\hbar^{-1}[\langle \widehat{\rho}, \alpha\rangle - h(\alpha)]}\}$ will affect the direction.

Thus, we have shown outside of $O(\hbar)$ small neighborhood of critical manifold of $X_\lambda^\parallel$ on $\mathcal{H}_\hbar$, we have $X_\lambda^\parallel$ dominating the $O(\hbar)$ term. Hence for any fixed arbitrarily small $\delta > 0$, we may first take $\hbar$ small enough, such that for a point $p \in \mathcal{H}_\hbar$ that is δ-distance away from the rescaled skeleton $\Lambda_\hbar$, the vector $X_\lambda^\parallel$ dominate the $O(\hbar)$ term, then we flow p for large enough time till it is contained in the δ-neighborhood of the skeleton $\Lambda_\hbar$. $\square$

8.2. Monodromy Action

In this section, we allow the coefficients in the polynomial f to have arbitrary phase angle, i.e.

$$f_{R,h,\theta}(z) = \sum_{\alpha \in \partial A} e^{-i\theta(\alpha)} R^{-h(\alpha)} z^\alpha.$$

Recall A is the vertices of the star triangulation $\mathcal{T}$, and $\partial A = A \backslash \{0\}$. In particular, we will consider loops of $\theta \in T^A \cong Map(A, T)$ based on $0 \in T^A$ and the generated monodromy.

It is conceptually clear to think of the parameter space of the coefficients $A \to \mathbb{C}^*$. Consider the fibration

$$\pi_A : (\mathbb{C}^*)^n \times (\mathbb{C}^*)^A \to (\mathbb{C}^*)^A$$

where we have a sub-bundle, such that over the fiber $c = (c_\alpha) \in (\mathbb{C}^*)^A$, we have the hypersurface $\{f_c(z) = 0\} \subset (\mathbb{C}^*)^n$, denote the fiber by H_c.

Let $\mathfrak{D} \subset (\mathbb{C}^*)^A$ be the descriminant loci where the fiber H_c is singular. Given a Laurent polynomial, is to give an affine line embedding $\iota_f : \mathbb{C}^* \hookrightarrow (\mathbb{C}^*)^A$, the critical value are the intersection of $\iota_f(\mathbb{C}^*) \cap \mathfrak{D}$.

We identify $(\mathbb{C}^*)^A \cong \mathbb{R}^A \times T^A$. By choosing $c_\alpha = R^{-h(\alpha)}e^{-i\theta(\alpha)}$ with fixed h and varying $\theta(\alpha)$, we are considering a point in $\mathbb{R}^A$ near infinity, and a torus T^A. By theorem of GKZ, the projection of the discriminant loci $\mathfrak{D}$ to $\mathbb{R}^n$ cut the boundary $\mathbb{R}^n$ into conical chambers, each chamber corresponds to a triangulation of Q with vertices in A, and chamber-crossing occurs when the triangulation changes. Hence for large enough R, we are deep in one chamber and the torus $\{h\} \times T^A$ by varying θ will be disjoint from $\mathfrak{D}$. The embedded affine line $\iota_f(\mathbb{C}^*)$ is given by an affine linear embedding

$$\iota_f : \mathbb{R} \times T \hookrightarrow \mathbb{R}^A \times T^A.$$

For large R, it intersects the walls of $\mathfrak{D}$ at well separated places. In other words, the critical values of f shows up in almost concentric circles [**DKK**].

As we vary $\theta \in T^A$ though a loop, the embedding ι_f has invariant $\mathbb{R}$-factor, hence there is no collision of critical values, and R remains a regular value. Thus, any vanishing path from a critical value of f to the regular value R can be uniquely deformed as vanishing path with fixed regular value when θ varies.

Thus, we have defined the monodromy action of $\pi_1(T^A)$ on the set of Lagrangian thimbles.

8.3. Essential Surjectivity of $\Phi_{\mathcal{T}}$.

In this section, we work with $Sh(T^n, \Lambda_{\mathcal{T},\theta})$ instead of $\mathrm{Fuk}(T^*T^n, \Lambda_{\mathcal{T},\theta})$, since the sheaf language is more convenient.

Consider $\theta = 0$ first. There is a distinguished real thimble in $FS((\mathbb{C}^*)^n, f_{R,h})$, which goes to the skyscraper sheaf $\mathbb{C}_0$. We apply the monodromy action $\mathbb{Z}^A$ to $\mathbb{C}_0$ to get a class of sheaves, called 'twisted-polytope-sheaves'.

First, we will show that twisted polytope sheaves co-represent the stalks in $Sh(T^n, \Lambda_{\mathcal{T}})$, that is for any point $x \in T^n$, there is canonically a twisted polytope sheaf $P_{[x]} \in Sh(T^n, \Lambda_{\mathcal{T}})$, such that for any sheaf $F \in Sh(T^n, \Lambda_{\mathcal{T}})$, we have

$$F_x \cong hom(P_{[x]}, F).$$

The proof proceeds by taking a costandard sheaf of a small open ball around x, then expand it non-characteristically with respect to the conical Lagrangian $\Lambda_{\mathcal{T}}$ until it cannot be expanded further, then the probe sheaf also changes from the costandard sheaf to a twisted polytope sheaf. It is entirely analagous to Propositionp:stalk.

On the other side, we define the set of thimbles obtained by applying the monodromy operation of $\mathbb{Z}^A$ on the real thimble as 'monodromy-generated-thimbles', denoted as $\{\mathbb{Z}^A \cdot \mathrm{Thimble}_{\mathbb{R}}\}$. At this moment, we do not know whether $\langle\{\mathbb{Z}^A \cdot \mathrm{Thimble}_{\mathbb{R}}\}\rangle \cong FS$. However, since the FS category is generated by an exceptional collection formed by a distinguished set of vanishing cycles, then by Proposition 2.6 and Corollary 2.10 in [BK], hence $\langle\{\mathbb{Z}^A \cdot \mathrm{Thimble}_{\mathbb{R}}\}\rangle$ is saturated. Hence the image under the embedding, $\langle\{\mathbb{Z}^A \cdot \mathbb{C}_0\}\rangle \subset Sh(T^n, \Lambda_{\mathcal{T}})$ is saturated and is left and right admissible. Since $\langle\{\mathbb{Z}^A \cdot \mathbb{C}_0\}\rangle^{\perp} = 0$ by the co-representing stalk property, we have

$$Sh(T^n, \Lambda_{\mathcal{T}}) \cong \langle \{\mathbb{Z}^A \cdot \mathbb{C}_0\} \rangle \subset \Phi_{\mathcal{T}}(FS) \subset Sh(T^n, \Lambda_{\mathcal{T}}).$$

Hence the above inclusions are actually equivalences.

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