Fermionic Isocurvature Perturbations

by

Peng Zhou

A dissertation submitted in partial fulfillment of
the requirements for the degree of

Doctor of Philosophy
(Physics)

at the

University of Wisconsin – Madison
2012

Defended on July 20th, 2012
Dissertation approved by the following members of the Final Oral Committee:
Daniel Chung · Associate Professor of Physics
Peter Timbie · Professor of Physics
Aki Hashimoto · Associate Professor of Physics
Lisa Everett · Associate Professor of Physics
Jean-Luc Thiffeault · Associate Professor of Mathematics

Abstract

Isocurvature perturbations generated within the context of inflationary cosmology typically involve fluctuations of bosonic condensates. Here we consider isocurvature perturbations from fermion fields in inflation. With suitable long range non-gravitational interaction, scale invariant perturbations can be generated in the number density of the gravitationally produced fermion. In this thesis, the fermion relic density, isocurvature correlator, and cross correlator with curvature perturbations are determined. The results provide new probes for theories with stable massive fermions.

Acknowledgements

First, I would like to thank my advisor Daniel Chung. He encouraged me to explore what inspired me and also guided me through the many obstacles in research and graduate life. I benefited much from his creative and rigorous approach in research, and learned a lot also from his deep insights in physics.

I would also like to thank my fellow students, Andrew Long, Hojin Yoo, Hiren Patel, Paul Mcguirk. Without numerous discussions with them, I can hardly imagine solving all the technical difficulties on my own.

Finally, I would like to thank my family, for inspiring me as a young kid into the field of science, and for their support in my pursuit to understand the secret of nature.

Contents

Chapter 1

Introduction

1.1 Standard Cosmology

Cosmology, which is a study of the origin and the evolution of the universe, has witnessed fast development during the last 30 years. Since Wilson and Penzias discovered the existence of Cosmic Microwave Background (CMB) in the mid-1960s, the range and precision of the measurements have improved rapidly. Launched in 1989, the Cosmic Background Explorer Satellite (COBE) established the thermal nature of the CMB using its spectrophotometer and determined its temperature to a high precision of 2.725 ± 0.002K(95% confidence) [1]. Later, the angular anisotropy of CMB was measured for the first time with the radiometer on COBE. After the effort of many ground based and balloon-borne experiments, the measurement of the temperature fluctuation by the Wilkinson Microwave Anisotropy Probe (WMAP) established the scale-invariant power spectrum of the fluctuation.

In the meantime, astronomical measurements provide important independent constraints on many cosmological parameters. The relic abundance of the light elements, D^3, He^4, He^7, Li measured from interstellar absorptions agrees with the calculation from Big Bang Nucleosynthesis (BBN) and provide a precise measurement of the baryon density. The discrepancy between the baryon density Ω_B and the total matter density Ω_M, indicates the existence of non-baryonic dark matter. In the late 1990s, the Type Ia supernovae as a standard candle led to the discovery of accelerated expansion [2, 3], which introduced the even more mysterious dark energy into the picture.

Currently, the available observational data are well explained by standard inflationary cosmology. The universe begins with an accelerated expansion phase, where the energy density was dominated by the potential energy of a scalar field (the inflaton). After inflation ends, the energy in the inflaton was converted into standard model particles during the reheating phase. The universes then cools down, during which a number of spontaneous symmetry breaking phase transitions take place, such as the electroweak phase transition. When the temperature drops to around $10 \sim 0.1$MeV, the primordial nucleosynthesis occurs. Finally, during the recombination era the ions and electrons combine to form atoms, and photons can propagate without much scattering and bring us the picture of the universe at that time. Afterwards, inhomogeneities in the energy density grow nonlinearly due to the attractive nature of gravity. These early inhomogeneities would then form large scale structures, such as voids, superclusters, cluster of galaxies, galaxies, etc.

One active research area in theoretical cosmology involves the modeling of inflation. Inflation presents an elegant solution to some theoretical puzzles, such as spatial flatness, large-scale smoothness and unwanted relics. It also explains the origin of the primordial density inhomogeneities that form large scale structures. In the minimal setting of inflation, the inflaton is a single scalar field. The quantum fluctuations of the inflaton generate scale-invariant, Gaussian and adiabatic perturbations[1], which are confirmed by the CMB anisotropy measurements. However, it is possible that during inflation there is more than just one inflaton field. The presence of these additional fields can also generate density perturbations. Due to the extra degrees of freedom, in addition to the total energy density fluctuation, the composition of the constituents in the energy density could vary. Such kinds of perturbations in the mixture are called isocurvature perturbations. In this thesis,we study isocurvature perturbation generated by fermion fields.

1.2 Review of Isocurvature Models

First, we give a more precise definition for isocurvature perturbations in the fluid description of the universe. If there are N species of fluid elements, the total energy density perturbations causes curvature perturbations. The remaining $N-1$ degrees of freedom can be tuned arbitrarily while keeping the curvature perturbation constant, hence the isocurvature perturbations. More precisely, we may define the isocurvature perturbations between the i-th and j-th fluid element as

$$S_{ij} = \frac{\delta\rho_i}{\bar{\rho}_i + \bar{p}_i} - \frac{\delta\rho_j}{\bar{\rho}_j + \bar{p}_j} \tag{1.1}$$

[1]Scale invariance means the perturbations do not have a characteristic scale; Gaussian means the two-point correlation function of the fluctuation determines higher order correlation functions;adiabaticity means the primordial fractional energy density perturbations $\delta\rho/\bar{\rho}$ are the same for all the constituents in the universe.

where ρ_i and p_i are the energy density and pressure respectively of the i-th element [2]. While curvature perturbations depend on the total energy density fluctuation of all the species, isocurvature perturbations depend on the difference of the energy density fluctuation. Here we consider the isocurvature perturbations between the gravitationally produced fermionic cold dark matter (CDM) and radiation. During the radiation dominated (RD) era, the total energy density fluctuations receive little contribution from the matter components, hence the isocurvature perturbations do not source gravitational potential perturbations. However after matter domination (MD) occurs, isocurvature perturbations will source gravitational perturbation and leave an imprint on the CMB spectrum.

Isocurvature perturbations were proposed in the late 80's as an alternative mechanism to generate the primordial density perturbations [5, 6]. From the measurement of the CMB acoustic peak and the TE correlation, we know the initial condition for the cosmological perturbation is consistent with the adiabatic initial condition [7]. However, the current bounds from CMB, large scale structure and supernovae still allow for order 10% fraction of isocurvature perturbation in the initial condition [7, 8, 9, 10, 11, 12, 13].

Isocurvature perturbations have been studied in various scenarios, such as double inflation [6, 14], axion [15, 16], curvaton [17, 18] and superheavy dark matter [19, 20]. The isocurvature power spectrums are computed in these models and are used to constrain the parameter space. Furthermore, as the precision of the CMB measurement improves, we can hope to measure or constrain non-Gaussianities in the CMB spectrum. The simplest single field inflation does not have large non-Gaussianities [21], whereas isocurvature perturbations in a variety of models can have large non-Gaussianities[22, 23, 24, 25]. Hence, isocurvature perturbation and its phenomenology has become ever more important.

Here we minimally extend the single field inflation by adding in a stable massive fermion field. Previously, the stable massive scalar field case (superheavy dark matter) has been considered in [20, 25], in which the isocurvature 2-point and 3-point functions were computed. In that case, large non-Gaussianities can be produced with existing observational bounds satisfied. Since the superheavy dark matter model only requires gravitational coupling, it provides probes to any theory with a stable massive scalar. Here, we use the same idea and investigate the cosmological effects of a generic stable massive fermion.

[2] Our definition for isocurvature does not apply to the neutrino velocity isocurvature. See e.g. [4].

1.3 Organization of the thesis

The thesis is organized as follows. In the first three chapters, we review the formal tools. In Chapter 2, we setup the model and introduce the in-in formalism. In Chapter 3, we review the renormalization of composite operators in curved spacetime. In Chapter 4, we review the diffeomorphism invariance and present the gravitational Ward identity. Then in the next three chapters, we study the properties of the gravitational fermion productions. In Chapter 5, we study the homogeneous average energy density. In Chapter 6, we look at the density inhomogeneities by computing the isocurvature two point function . In Chapter 7, we investigate the cross correlation between the isocurvature and curvature perturbations. We consider the phenomenology constraint and give a preliminary discussion on non-Guassianities in Chapter 8. Finally, we summarize our findings and conclude in Chapter 9.

Chapter 2

Setup

2.1 The Model

Here we consider an inflationary model with an inflaton sector and a hidden sector. The inflaton sector contains of a single scalar field ϕ, and the hidden sector contains a light scalar field σ and a Dirac fermion field ψ. The mass of σ is less than the Hubble expansion rate. The action can be written as

$$\begin{aligned} S &= S_{EH} + S_\phi + S_\sigma + S_\psi \\ &= \int d^4x\sqrt{g}\left\{\frac{1}{2}M_p^2 R + [-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)] \right. \\ & \left. +[-\frac{1}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - \frac{1}{2}m_\sigma^2\sigma^2] + \bar{\psi}(i\gamma^a\nabla_{e_a} - m_\psi)\psi - \lambda\sigma\bar{\psi}\psi\right\} \end{aligned} \tag{2.1}$$

where $M_p^2 = \frac{1}{8\pi G} = 1$ is the reduced Planck constant.

The metric can be parametrized in ADM formalism [26], [1]

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + h_{ij}N^iN^j & h_{ij}N^j \\ h_{ij}N^j & h_{ij} \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} -N^{-2} & N^iN^{-2} \\ N^iN^{-2} & h^{ij} - N^iN^jN^{-2} \end{pmatrix} \tag{2.2}$$

where h_{ij} is the metric tensor on the constant time hypersurface, with the inverse metric as h^{ij}. We use Latin indices $i, j\cdots$ for objects on the 3-dimensional constant time hypersurface, and we use h_{ij} and h^{ij} to raise and lower the indices. The Ricci curvature and the metric determinant can be

[1] We use $(-+++)$ sign convention for the metric, and physical time t .

expressed as

$$R = R^{(3)} + \frac{1}{N^2}(E_{ij}E^{ij} - E^2) \tag{2.3}$$

$$\det(g_{\mu\nu}) = -N^2 \det(h_{ij}) \tag{2.4}$$

where

$$E_{ij} = \frac{1}{2}(\dot{h}_{ij} - \nabla_i^{(3)} N_j - \nabla_j^{(3)} N_i). \tag{2.5}$$

$$E = E_{ij}h^{ij}. \tag{2.6}$$

In the fermion sector, the Dirac matrices are taken as

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{2.7}$$

to simplify the derivation of the second order differential equation of the spinor mode functions. $\{e_a\}_{a=0,\cdots,3}$ are the vielbein fields, where Latin indices $a, b \cdots$ are the vierbein labels. In a coordinate system x^μ, they are written as $e_a = e_a^\mu(x)\partial_\mu$. The covariant derivative is defined as

$$\nabla_{e_a}\psi = [e_a^\mu \partial_\mu + \frac{1}{2}\omega_{a;bc}\Sigma^{bc}]\psi \tag{2.8}$$

where $\omega_{a;bc}$ is the spin connection, defined as

$$\omega_{a;bc} = \langle e_b, \nabla_{e_a} e_c \rangle \tag{2.9}$$

and Σ^{bc} is the generator of the Lorentz group on spinor field, defined as

$$\Sigma^{bc} = -\frac{1}{4}[\gamma^b, \gamma^c]. \tag{2.10}$$

Next, we consider the perturbative expansion of the action. Consider a spatially homogeneous background configuration

$$\phi^{(0)} = \bar{\phi}(t) \tag{2.11}$$

$$\sigma^{(0)} = 0 \tag{2.12}$$

$$g_{\mu\nu}^{(0)} = \begin{pmatrix} -1 & 0 \\ 0 & a^2(t)\delta_{ij} \end{pmatrix} \tag{2.13}$$

with the background equation of motion

$$3H^2 = \frac{1}{2}\dot{\bar{\phi}}^2 + V(\bar{\phi}) \tag{2.14}$$

$$\dot{H} = -\frac{1}{2}\dot{\bar{\phi}}^2 \tag{2.15}$$

$$\ddot{\bar{\phi}} + 3H\dot{\bar{\phi}} + V'(\bar{\phi}) = 0. \tag{2.16}$$

Thus, we can get action about the perturbations,

$$\delta S(\delta g, \delta\phi, \sigma, \psi) = S(\bar{g} + \delta g, \bar{\phi} + \delta\phi, \sigma, \psi) - S(\bar{g}, \bar{\phi}, 0, 0) \tag{2.17}$$

In the remaining part, we shall write δS as S when there is no ambiguity.

2.2 Gauge Fixing

To quantize this action with diffeomorphism gauge symmetry, one can add gauge fixing terms and Faddeev-Popov ghost terms[2]. For example, in comoving gauge, we have [3]

$$\delta\phi = 0, \quad \gamma_{ii} = 0, \quad \partial_i \gamma_{ij} = 0 \tag{2.18}$$

where

$$h_{ij} = a^2(t)[e^{\Gamma}]_{ij}, \quad \Gamma_{ij} = 2\zeta\delta_{ij} + \gamma_{ij}. \tag{2.19}$$

We may use the following gauge fixing function

$$G_1 = \delta\phi, \quad G_{2,j} = \partial_i(\ln h)_{ij} - \frac{1}{3}\partial_j(\ln h)_{ii} \tag{2.20}$$

and add the the gauge-fixing and ghost field Lagrangian

$$\mathcal{L}_{gf} = \frac{1}{2\alpha}(G_1^2 + \sum_j G_{2,j}^2) \tag{2.21}$$

$$\mathcal{L}_{gh} = \bar{c}_1 \frac{\delta G_1}{\delta X^\mu}\theta^\mu + \bar{c}_{2,j}\frac{\delta G_{2,j}}{\delta X^\mu}\theta^\mu \tag{2.22}$$

where the vector field X^μ is the infinitesimal gauge transformation parameter. The gauge transformation rule for $\delta\phi$ and δg are

$$\Delta_X \delta\phi = \mathcal{L}_X(\bar{\phi} + \delta\phi), \quad \Delta_X(\delta g)_{\mu\nu} = [\mathcal{L}_X(\bar{g} + \delta g)]_{\mu\nu} \tag{2.23}$$

which can be expanded as

$$\Delta_X \delta\phi = X^0 \dot{\bar{\phi}} + \cdots, \qquad \Delta_X h_{ij} = a^2(2X^0 H\delta_{ij} + \partial_i X^j + \partial_j X^i) + \cdots \tag{2.24}$$

where $\cdots$ represents interactions terms between $\bar{c}, \theta$ and $\delta g, \delta\phi$. Plug these into $\delta G/\delta X^\mu$ in $\mathcal{L}_{gh}$, we have

$$\mathcal{L}_{gh} = \bar{c}_1 \theta^0 \dot{\bar{\phi}} + \bar{c}_{2,j}(\theta^j_{,ii} + \frac{1}{3}\theta^i_{,ij}) + \cdots. \tag{2.25}$$

[2] Covariant gauge fixing has been widely employed, see e.g. [27, 28]

[3] In this section, latin indices i, j are raised and lowered by δ_{ij}, and repeated indices are contracted.

It is clear that the ghost fields are non-dynamical. In the limit $\alpha \to 0$, we shall enforce $G_1 = 0, G_{2,j} = 0$, and recover the comoving gauge.

Alternatively, we may use the uniform curvature gauge, which is defined by $\zeta = 0$ instead of $\delta\phi = 0$. We only need to replace gauge fixing function G_1 by $G_1 = [\ln(a^{-2}h)]_{ii}$. It is clear that the ghost fields in this gauge are non-dynamical as well.

Actually, in this work, the graviton loop and inflaton loop do not appear at order of the loopwise expansion we are considering, we may ignore the ghost fields in general. In other word, we use tree level approximation for the inflaton-graviton sector.

2.3 Interaction Lagrangian

Under the tree-level approximation for the inflaton-graviton sector, we may solve the constraint equations for N, N^i in terms of $h_{ij}, \phi, \cdots$ and plug the solution back into the action to get the gravitational interactions. In any gauge, the constraint equations are

$$0 = \frac{1}{N}[R^{(3)} - \frac{1}{N^2}(E_{ij}E^{ij} - E^2)] - 2NT^{00} \tag{2.26}$$

$$0 = \frac{2}{N}\nabla_i^{(3)}[\frac{1}{N}(E^{ij} - Eh^{ij})] + 2N^jT^{00} + 2T^{0j} \tag{2.27}$$

where $T^{\mu\nu}$ is the total matter stress tensor. The above constraint equations can be solved iteratively. We use $O^{(n)}$ to denote the n-th level contribution to O, i.e. it contains products of n simple field variable. To get the matter-gravity interaction action at cubic order, we only need to solve the equations for $N^{(1)}, N^{i(1)}$, since $N^{(2)}, N^{i(2)}$ are coupled to the constraint equation at linear order. Under the above consideration, the matter-gravity interaction is given by

$$S^{(3)} = \frac{1}{2}\int d^4x\sqrt{\bar{g}}T^{\mu\nu(2)}g^{(1)}_{\mu\nu}. \tag{2.28}$$

For simplicity, we only consider the scalar metric perturbation in $g^{(1)}_{\mu\nu}$.

In comoving gauge, denoted by superscript (C), the ADM constraints gives

$$N^{(1,C)} = \frac{\dot{\zeta}}{H}, \quad N_i^{(1,C)} = \partial_i[-\frac{\zeta}{H} + \epsilon\frac{a^2}{\nabla^2}\dot{\zeta}] \tag{2.29}$$

the metric perturbations are

$$\delta g^{(C)}_{\mu\nu} = \begin{pmatrix} -2\frac{\dot{\zeta}}{H} & (-\frac{\zeta}{H} + \epsilon\frac{a^2}{\nabla^2}\dot{\zeta})_{,i} \\ (-\frac{\zeta}{H} + \epsilon\frac{a^2}{\nabla^2}\dot{\zeta})_{,i} & a^2\delta_{ij}2\zeta \end{pmatrix} \tag{2.30}$$

The free action and the matter-ζ cubic interaction action are

$$S^{(2,C)} = \int dtd^3xa_x^3\epsilon(\dot{\zeta}^2-(\frac{\nabla}{a}\zeta)^2) \tag{2.31}$$

$$S^{(3,C)} = i\int d^4xa_x^3[T^{ij}a^2\delta_{ij}\zeta+T^{0i}(-\frac{\zeta}{H}+\epsilon\frac{a^2}{\nabla^2}\dot{\zeta})_{,i}-T^{00}\frac{\dot{\zeta}}{H}]. \tag{2.32}$$

In uniform curvature gauge, denoted by superscript (U), the inflaton degree of freedom is in $\delta\phi$. However, this degree of freedom can be represented using the gauge-invariant variable

$$\zeta = -\frac{H}{\dot{\bar{\phi}}}\phi^{(1,U)}. \tag{2.33}$$

In this gauge, the ADM constraint renders

$$N^{(1,U)} = -\epsilon\zeta, \quad N_i^{(1,U)} = \partial_i[\epsilon\frac{a^2}{\nabla^2}\dot{\zeta}]. \tag{2.34}$$

We get the linear metric perturbation as

$$\delta g_{\mu\nu}^{(U)} = \begin{pmatrix} 2\epsilon\zeta & \epsilon\frac{a^2}{\nabla^2}\dot{\zeta}_{,i} \\ \epsilon\frac{a^2}{\nabla^2}\dot{\zeta}_{,i} & 0 \end{pmatrix}. \tag{2.35}$$

The free action is the same as in Eq.(2.31), and matter-ζ cubic interaction action is

$$S^{(3,U)} = \int d^4xa_x^3[T^{00}\epsilon\zeta+T^{0i}\epsilon\frac{a^2}{\nabla^2}\dot{\zeta}_{,i}]. \tag{2.36}$$

The stress tensor for scalar field is

$$T_\sigma^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}\partial_\alpha\sigma\partial_\beta\sigma+g^{\mu\nu}\mathcal{L}_\sigma \tag{2.37}$$

and the stress tensor for fermion field is (in vierbein indices)

$$T_\psi^{ab} = -\frac{i}{2}[\bar{\psi}\gamma^{(a}\nabla^{b)}\psi-\nabla^{(b}(\bar{\psi})\gamma^{a)}\psi]+\eta^{ab}\mathrm{Re}(\mathcal{L}_\psi) \tag{2.38}$$

as derived in Appendix A.4.

2.4 In-In Formalism

In-In formalism is a framework to compute the expectation value of a given observable with a given initial state. It is often used in cosmology, where one need to compute the late time equal time correlator for a given initial vacuum state. The in-in formalism can be formulated using path integral and using canonical quantization. Here we only give the prescription following the reference [29, 30]

In the path integral formulation, we consider the following partition function

$$Z[J^+,J^-] = \int D\Psi^{\pm} e^{iS^+ - iS^- + i\int J^+\Psi^+ - i\int J^-\Psi^-} \tag{2.39}$$

where the $\pm$ indices indicate the forward and backward time branches. Consider a 4-dimensional spacetime manifold M with boundary Σ_f at very late time. Consider two copies of field configurations Ψ^+ and Ψ^- on M that agrees on Σ_f, and define the associated action as

$$S^{\pm} = \int_M d^4x\sqrt{g^{\pm}}\mathcal{L}(\Psi^{\pm}). \tag{2.40}$$

We may introduce two copies of M, noted as M^+ (the forward branch) and M^- (the backward branch), on which Ψ^+ and Ψ^- live respectively. And we can glue together M^+ and M^- together along Σ_f, to have single manifold $\mathcal{M}$. There is a partial ordering of points on $\mathcal{M}$: if $x, y \in M^+$, then x precedes y if x is in the past lightcone of y; if $x, y \in M^-$, then x precedes y if x is in the future lightcone of y; if $x \in M^+$ and $y \in M^-$, then x always precedes y.

If we use the free action $S_0^{\pm}$ in the partition function Eq. (2.39), we can have the following four kinds of free correlation functions (here we use a scalar field $\phi(x)$ as an example):

$$\langle in|\phi^+(x)\phi^+(y)|in\rangle_0 = \langle in|T\{\phi(x)\phi(y)\}|in\rangle_0 \tag{2.41}$$

$$\langle in|\phi^-(x)\phi^-(y)|in\rangle_0 = \langle in|\bar{T}\{\phi(x)\phi(y)\}|in\rangle_0 \tag{2.42}$$

$$\langle in|\phi^+(x)\phi^-(y)|in\rangle_0 = \langle in|\phi(y)\phi(x)|in\rangle_0 \tag{2.43}$$

$$\langle in|\phi^-(x)\phi^+(y)|in\rangle_0 = \langle in|\phi(x)\phi(y)|in\rangle_0 \tag{2.44}$$

In other word, the field operators ϕ are ordered according to the partial ordering of its location of $\mathcal{M}$, if x precedes y, then $\phi^a(x)$ is placed to the left of $\phi^b(y)$. We call this ordering *path ordering*.

The expectation value of some observable (say, equal-time n-point function) can be expanded perturbatively in interaction picture

$$\langle O[\phi]\rangle = \sum_{N=0}^{\infty}\int D\phi^{\pm} e^{i(S_0^+ - S_0^-)} O[\phi]\frac{(iS_{int}^+ - iS_{int}^-)^N}{N!}. \tag{2.45}$$

Next, we turn to the canonical formulation of In-In formalism, following [30]. Consider the observable $O[t;\phi^H] = \phi^H(x_1,t)\phi^H(x_2,t)\cdots$, a products of Heisenberg picture operators, then

$$\begin{aligned} &\langle in|O[t;\phi^H]|in\rangle \\ = \;&\langle in|O[t;\phi^I]|in\rangle - i\int_{-\infty}^{t} dt_1\langle in|[O^I(t), H_{int}^I(t_1)]|in\rangle \\ &+(-i)^2\int_{-\infty}^{t} dt_1\int_{-\infty}^{t_1} dt_2\langle in|[[O^I(t), H_{int}^I(t_1)], H_{int}^I(t_2)]|in\rangle + \cdots \end{aligned} \tag{2.46}$$

where $H^I_{int}(t_1)$ is the interaction Hamiltonian in the interaction picture.

We shall find that for formal manipulations, the path integral formulation is more convenient. For actual computation, the canonical formulation is more efficient, since the causal structure of the Green's function is more manifest.

Chapter 3

Regularization and Renormalization of Composite Operator

In the loopwise expansion of QFT calculation, there are ultraviolet (UV) divergences in the loop integral. To make such integral is well-defined, we can introduce a UV regulator with parameter Λ to soften the UV behavior of the propagator. Now the loop integrals are finite, but the result is regulator dependent. This regulator dependence can then be removed by introducing counter-terms in the Lagrangian with coefficients that depend on Λ. After considering the diagrams with counter-terms, we can take the limit that $\Lambda \to \infty$ and have a finite renormalized result. The residual freedoms in the finite part of the renormalization constant can be fixed by the renormalization conditions. However, there are additional subtleties in the regularization and renormalization procedure in curved spacetime and with composite operator insertion.

In curved spacetime, the classical action enjoys the diffeomorphism invariance. To preserve it at quantum level, one need to use a covariant regulator (e.g. Schwinger proper time regulator, zeta function regulator, Pauli-Villars regulator. See [31]). Here we shall adopt the Pauli-Villars (PV) regulator, following [32].

Observables associated with energy density fluctuation are usually bilinear in the field operator, e.g. σ^2. To define such a composite operator and have finite renormalized correlation function, one need to have extra counter-terms other than those for renormalizing the Lagrangian.

This chapter is organized as follows. We first review PV regularization in Section 3.1. Then we give prescription for renormalizing a composite operator in curved spacetime in Section 3.2. Finally, we show how to renormalize the operator σ^2 in Section 3.3.

3.1 Pauli-Villars Regularization

We first review the Pauli-Villars regulator for flat spacetime following [32]. Consider a free massive scalar field ϕ with the following time ordered propagator

$$\langle \mathcal{T}\phi(x)\phi(y)\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{-i}{p^2+m^2}e^{ip\cdot x} \tag{3.1}$$

If we consider $\langle \phi^2(x)\rangle$, then it has quadratic UV divergence

$$\langle \phi_x^2\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2+m^2} \tag{3.2}$$

To tame such a divergence, we can modify the propagator in such a way that it decreases faster at large momentum. For example, we can replace

$$(p^2+m^2)^{-1} \to (p^2+m^2+\alpha_2\frac{p^4}{\Lambda^2}+\alpha_3\frac{p^6}{\Lambda^4}+\cdots)^{-1} \tag{3.3}$$

and choose the degree n to make all diagrams convergent. The parameter Λ is the cut-off. In the large cut-off limit the original propagator is recovered.

In practice, to regulate the UV divergences caused by scalar field loop, we introduce a set of scalar regulator fields χ_n for $n = 1, \cdots, s$ with the following free Lagrangian

$$\mathcal{L}_{PV} = \sum_{n=1}^{s} C_n(-\frac{1}{2}g^{\mu\nu}\partial_\nu\chi_n\partial_\nu\chi_n - \frac{1}{2}M_n^2\chi_n^2). \tag{3.4}$$

and introduce appropriate coupling between χ_n and the other fields, such that χ loop contribution would cancel out the ϕ loop divergences. The number of regulator fields s depends on how many independent divergences one need to remove. In order to eliminate UV divergences up to some even order $2D$,we must take the C_n and regulator masses M_n to satisfy

$$\sum_n C_n^{-1} = -1, \sum_n C_n^{-1}M_n^2 = -m^2, \cdots \sum_n C_n^{-1}M_n^{2D} = -m^{2D}. \tag{3.5}$$

For instance, if there were only logarithmic divergences then $D = 0$, and we would only need one regulator field with $C_1 = -1$. Under such a change, the free field propagator in momentum space becomes

$$\frac{i}{p^2+m^2} \quad\to\quad \frac{i}{p^2+m^2} + \sum_{n=1}^{s} C_n^{-1}\frac{i}{p^2+M_n^2} \tag{3.6}$$

$$\to\quad i\sum_{N=0}^{s} C_N^{-1}\frac{1}{p^2}\left[1-\frac{M_N^2}{p^2}+\left(\frac{M_N^2}{p^2}\right)^2-\cdots\right] \tag{3.7}$$

$$\to\quad i\sum_{N=0}^{s} C_N^{-1}\frac{1}{p^2}\left[(-1)^D\left(\frac{M_N^2}{p^2}\right)^D-\cdots\right] \tag{3.8}$$

where we used the notation $M_0^2 = m^2$ and $C_0 = 1$, and we shall let $N = 0$ denote the original ϕ field. In the last step, we used the condition in Eq. (3.5) to cancel the first D terms.

In fixed curved spacetime, the same regulator Lagrangian in Eq. (3.4) can be used. For example, on a homogeneous FRW background metric, the physical and regulator scalar field can be quantized as [32]

$$[\chi_N, \dot{\chi}_M] \;=\; ia^{-3}(t)\delta^3(\vec{x}-\vec{y})\delta_{NM}C_N^{-1} \tag{3.9}$$

with the following mode decomposition

$$\chi_N(\vec{x},t) \;=\; \int d^3p(a_{N,\vec{p}}u_{N,\vec{p}}(t)+c.c) \tag{3.10}$$

$$[a_{N,\vec{p}}, a^\dagger_{M,\vec{k}}] \;=\; C_N^{-1}\delta_{NM}\delta^3(\vec{k}-\vec{p}) \tag{3.11}$$

where $u_{N,\vec{p}}(t)$ satisfies the usual equation of motion and normalization conditions as Eq. (A.8).[1] In dynamical curved spactime, with metric perturbations, we can still use the Lagrangian in Eq. (3.4) . In that case, the regulator fields are minimally coupled to gravity.

3.2 Renormalization of Composite Operator

In cosmology, we are often interested in the correlator of composite operators, e.g. the stress tensor $T^{\mu\nu}(x)$ or the mass term $\frac{1}{2}m^2\phi^2$. The insertion of such composite operators would cause new divergences. For example, the Fourier transform of $\langle\phi^2(x)\phi^2(y)\rangle$ is logarithmic divergent even in the free theory. Thus to have finite correlators, we need to introduce new counter-terms and introduce new renormalization conditions.

The renormalization of composite operators in fixed curved spacetime is a straightforward generalization of renormalization on the flat spacetime. Here we follow the treatment of [33] and give the prescription. Consider the action of a matter field, for concreteness, we take it to be a massive scalar field with quartic coupling λ

$$S = \int (dx)(-\frac{1}{2}(\partial\sigma)^2 - \frac{1}{2}m^2\sigma^2 - \frac{1}{4!}\lambda\sigma^4) \tag{3.12}$$

where $(dx) = d^4x\sqrt{|g|}$. Suppose we have some UV regulator with scale Λ, and we denote regu-

[1] Our treatment here differs from [32] in that the physical scalar field ϕ here has no background solution, and the regulator field χ_n does not mix with ϕ by mass term.

lated action as S_Λ. Using the Pauli-Villars regulraization we have

$$\begin{aligned} S_\Lambda &= \int (dx)\{-\frac{1}{2}(\partial\sigma)^2 - \frac{1}{2}m^2\sigma^2 - \frac{1}{4!}\lambda(\sigma + \sum_n \chi_n)^4 \\ &\quad + \sum_{n=1}^{s} C_n(-\frac{1}{2}g^{\mu\nu}\partial_\nu\chi_n\partial_\nu\chi_n - \frac{1}{2}M_n^2\chi_n^2)\} \end{aligned} \tag{3.13}$$

where Λ denote the set of M_n. We can define the bare partition function $Z_\Lambda[J]$ as:

$$Z_\Lambda[J] = \int D\sigma e^{\frac{i}{\hbar}S_\Lambda + \frac{i}{\hbar}\int(dx)J(x)\sigma(x)} \tag{3.14}$$

and the generating functional of connected diagrams $W_\Lambda[J]$ as:

$$W_\Lambda[J] = -i\hbar \ln Z_\Lambda[J]. \tag{3.15}$$

We keep the Λ subscript explicit to show its regulator dependence.

We can compute the time ordered connected n point functions by

$$\langle T\{\sigma(x_1)\cdots\sigma(x_n)\}\rangle_{conn,\Lambda} = \left[\prod_i \frac{1}{\frac{i}{\hbar}\sqrt{g(x_i)}}\frac{\delta}{\delta J(x_i)}\right]_{J(x)=0} \frac{i}{\hbar}W_\Lambda[J]. \tag{3.16}$$

However, these bare correlators are divergent as $\Lambda \to \infty$ while keeping m, λ fixed. Thus, we introduce the renormalized field and parameters as such

$$\sigma = Z_\sigma(\Lambda, \lambda_r, m_r)\sigma_r \tag{3.17}$$

$$\lambda = Z_\lambda(\Lambda, \lambda_r, m_r)\lambda_r \tag{3.18}$$

$$m = Z_m(\Lambda, \lambda_r, m_r)m_r \tag{3.19}$$

where the renormalization constants Z_i are expressed as formal power series of $\hbar$:

$$Z_i = 1 + Z_i^{(1)}(\Lambda, \lambda_r, m_r)\hbar + Z_i^{(2)}(\Lambda, \lambda_r, m_r)\hbar^2 + \cdots. \tag{3.20}$$

The goal is to choose Z_i order by order in $\hbar$, such that correlators of σ_r are finite at each order of $\hbar$ when $\Lambda \to \infty$ while keeping m_r, λ_r fixed.

The renormalized action is given by

$$\begin{aligned} & S_r(\sigma_r; m_r, \lambda_r) \\ = & \int (dx)\{-\frac{1}{2}(\partial\sigma_r)^2 - \frac{1}{2}m_r^2\sigma_r^2 - \frac{1}{4!}\lambda_r(\sigma_r + \sum_n \chi_n)^4 \\ & -\frac{1}{2}(Z_\sigma^2 - 1)(\partial\sigma_r)^2 - \frac{1}{2}(Z_m^2 Z_\sigma^2 - 1)m_r^2\sigma_r^2 - \frac{1}{4!}(Z_\lambda Z_\sigma^4 - 1)\lambda_r(\sigma_r + \sum_n \chi_n)^4 \\ & + \sum_{n=1}^{s} C_n(-\frac{1}{2}g^{\mu\nu}\partial_\nu\chi_n\partial_\nu\chi_n - \frac{1}{2}M_n^2\chi_n^2)\} \end{aligned} \tag{3.21}$$

where the regulator fields is added *after* we split the bare action into the renormalized one and the counter term. The renormalized $W_r[J]$ is given by

$$W_r[J] = -i\hbar \ln \int D\sigma e^{\frac{i}{\hbar}S_r(\sigma_r;m_r,\lambda_r)+\frac{i}{\hbar}\int(dx)J(x)\tilde{\sigma}_r(x)}. \tag{3.22}$$

where we define

$$\tilde{\sigma} = \sigma + \sum_n \chi_n, \quad \tilde{\sigma}_r = \sigma_r + \sum_n \chi_n \tag{3.23}$$

Next, we consider correlators that contains composite operators. We first do it with the bare, regulated action. For example, consider a composite operator $O(x) = \sigma^2(x)$, we introduce a source $K(x)$ that couples to it, then we get

$$Z_\Lambda[J,K] = e^{\frac{i}{\hbar}W_\Lambda[J,K]} = \int D\sigma e^{\frac{i}{\hbar}S_\Lambda+\frac{i}{\hbar}\int(dx)J(x)\tilde{\sigma}(x)+\frac{i}{\hbar}\int K\tilde{\sigma}^2(x)} \tag{3.24}$$

To renormalize σ^2 operator, we must include in the action all local terms that is constructed from $K(x)$ and other field with dimension 4 or less. Here the dimension of K is

$$[K(x)] = 4 - [\sigma^2] = 2 \tag{3.25}$$

thus the source term is changed to

$$\begin{aligned}\int(dx)K(x)\tilde{\sigma}^2(x) \mapsto S_K \quad &= \quad \int(dx)[K(x)\tilde{\sigma}_r^2(x)(1+\delta Z_1) \\ &\quad +K(x)\delta Z_2 + K(x)^2\delta Z_3 + K(x)R(x)\delta Z_4]\end{aligned} \tag{3.26}$$

where the δ sign reminds ourselves that at $\hbar^0$ order they are zero.

We can construct the renormalized W_r by

$$W_r[J,K] \quad = \quad -i\hbar \ln \int D\sigma \exp\frac{i}{\hbar}\{S_r + \int(dx)J\sigma_r + S_K\} \tag{3.27}$$

From the Taylor expansion of $W_r[J,K]$ w.r.t. the source, we get

$$\begin{aligned}W_r[J,K] \quad &= \quad \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}(\frac{i}{\hbar})^{n+l-1}\int[(dx_1)\cdots(dx_n)][(dy_1)\cdots(dy_l)] \\ &\quad \frac{1}{n!l!}[J(x_1)\cdots J(x_n)][K(y_1)\cdots K(y_l)]W_r^{(n,l)}(\{x_i\};\{y_j\})\end{aligned} \tag{3.28}$$

where $W_r^{(n,l)}(\{x_i\};\{y_j\})$ is the connected diagram with n external σ vertices and l external σ^2 vertices.

We list the first few coefficients in the above expansion:

$$W_r^{(0,1)}(;y_1) \quad = \quad \langle\tilde{\sigma}_r^2(y_1)\rangle_c(1+\delta Z_1) + \delta Z_2 + R(y_1)\delta Z_4 \tag{3.29}$$

$$W_r^{(0,2)}(;y_1,y_2) \quad = \quad \langle\tilde{\sigma}_r^2(y_1)\tilde{\sigma}_r^2(y_2)\rangle_c(1+\delta Z_1)^2 + \delta Z_3\frac{\hbar}{i}\frac{2\delta_{y_1,y_2}}{\sqrt{g(y_2)}} \tag{3.30}$$

$$W_r^{(2,1)}(x_1,x_2;y_1) \quad = \quad \langle\tilde{\sigma}_r(x_1)\tilde{\sigma}_r(x_2)\tilde{\sigma}_r^2(y_1)\rangle_c(1+\delta Z_1) \tag{3.31}$$

We can determine the divergent parts of δZ_i at $O(\hbar)$ order, by requiring the M_n dependent parts in $W_r^{(0,1)}(;y_1), W_r^{(0,2)}(;y_1,y_2), W_r^{(2,1)}(x_1,x_2;y_1)$ cancel out:

$$0 \sim \langle \tilde{\sigma}_r^2(y_1)\rangle + \delta Z_2^{div} + R(y_1)\delta Z_4^{div} \tag{3.32}$$

$$0 \sim \int_{y_2\sim y_1}(dy_2)\langle T\tilde{\sigma}_r^2(y_1)\tilde{\sigma}_r^2(y_2)\rangle_c + \delta Z_3^{div}\frac{\hbar}{i}2 \tag{3.33}$$

$$\begin{aligned} 0 \sim{}& \langle \tilde{\sigma}_r(x_1)\tilde{\sigma}_r(x_2)\tilde{\sigma}_r^2(y_1)\rangle_{conn,tree}\delta Z_1^{div} \\ & -\frac{\lambda_r}{4!}\frac{i}{\hbar}\int_{z\sim y_1}(dz)\langle T\tilde{\sigma}_r(z)^4\tilde{\sigma}_r(x_1)\tilde{\sigma}_r(x_2)\tilde{\sigma}_r^2(y_1)\rangle_c \end{aligned} \tag{3.34}$$

We can determine the finite part of δZ_i by imposing renormalization conditions. For δZ_i which are not coupled to $R, R_{\mu\nu}, R^{\alpha}_{\beta\mu\nu}$ and their derivatives, (e.g. $\delta Z_1, \delta Z_2, \delta Z_3$), we can go to the Minkowski space and impose the renormalization conditions there; for δZ_i which are coupled to $R, R_{\mu\nu}, R^{\alpha}_{\beta\mu\nu}$ or their derivatives, (e.g. δZ_4), there is no prefered choices of renormalization conditions. We expect the ambiguity associated with curvature related renormalization condition should have negligible effect, if we evaluate the correlators in an asymptotically flat region.

3.3 Example: σ^2 operator

In this section, we determine the renormalization constants δZ_i to $O(\hbar)$, using Eq. (3.32, 3.33, 3.34).

First look at Eq. (3.32). To compute $\langle \tilde{\sigma}_r^2(y_1)\rangle_c^{div}$, we can use the short distance expansion of the propagator. From Eq. (182) (199) (207) (208) of work ([34]), we have

$$\begin{aligned} i\langle out|T\phi(x)\phi(y)|in\rangle ={}& -\frac{\Delta(x,y)^{1/2}}{8\pi}\exp\left[\sum_{n=1}^{\infty} a_n(x,y)\left(-\frac{\partial}{\partial m^2}\right)^n\right] \\ & \times\frac{m^2 H_1^{(2)}(\sqrt{-2m^2\sigma})}{\sqrt{-2m^2\sigma}} \end{aligned} \tag{3.35}$$

where σ is half the square of the geometric distance between x, y, and

$$\Delta(x,y) = g^{-1/2}(x)g^{-1/2}(y)D(x,y) \tag{3.36}$$

$$D(x,y) = -\det(-\partial^2\sigma/\partial x^{\mu}\partial y^{\nu}) \tag{3.37}$$

$$g = -\det g_{\mu\nu} \tag{3.38}$$

In the limit $y \to x$, we get

$$a_1(x,x) = (\frac{1}{6}-\xi)R \tag{3.39}$$

$$a_2(x,x) = -\frac{1}{180}R_{\mu\nu}R^{\mu\nu} + \frac{1}{180}R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - \frac{1}{6}(\frac{1}{5}-\xi)R_{;\mu}^{\;\mu} \tag{3.40}$$

If we let $y \to x$ from a spacelike separation, then we get

$$
\begin{aligned}
& i\langle out|T\phi(x)\phi(y)|in\rangle \\
\sim\ & -\frac{\Delta(x,y)^{1/2}}{8\pi}[1-(\frac{1}{6}-\xi)R\frac{\partial}{\partial m^2}] \\
& \frac{1}{\pi i}\left\{\frac{1}{\sigma+i0}-m^2\left(\gamma+\frac{1}{2}\ln(m^2/2)+\frac{1}{2}\ln(\sigma+i0)\right)+\cdots\right\} \\
\sim\ & -\frac{\Delta(x,y)^{1/2}}{8\pi^2 i}\left\{\frac{1}{\sigma+i0}-m^2\left(\gamma+\frac{1}{2}\ln(m^2/2)+\frac{1}{2}\ln(\sigma+i0)\right)\right. \\
& \left.+(\frac{1}{6}-\xi)R\left(\gamma+\frac{1}{2}\ln(m^2/2)+\frac{1}{2}\ln(\sigma+i0)+\frac{1}{2}\right)+\cdots\right\} \qquad (3.41)
\end{aligned}
$$

where $\cdots$ represents terms that are non-divergent when $\sigma \to 0$.

After these preparation, we can now compute $\langle\tilde{\sigma}_r^2(y_1)\rangle_c^{div}$:

$$
\begin{aligned}
& \langle out|\tilde{\sigma}_r^2(y_1)|in\rangle_c^{div} \\
\sim\ & \lim_{y\to x}\langle T\sigma(x)\sigma(y)\rangle+\sum_{n=1}^{s}\langle T\chi_n(x)\chi_n(y)\rangle \\
=\ & \lim_{y\to x}\frac{\Delta(x,y)^{1/2}}{8\pi^2}\sum_{N=0}^{s}C_N^{-1}\left\{\frac{1}{\sigma+i0}-M_N^2\left(\gamma+\frac{1}{2}\ln(M_N^2/2)+\frac{1}{2}\ln(\sigma+i0)\right)\right. \\
& \left.+(\frac{1}{6}-\xi)R\left(\gamma+\frac{1}{2}\ln(M_N^2/2)+\frac{1}{2}\ln(\sigma+i0)+\frac{1}{2}\right)+\cdots\right\} \\
=\ & \frac{1}{16\pi^2}\sum_{N=0}^{s}C_N^{-1}\left\{-M_N^2\ln M_N^2+(\frac{1}{6}-\xi)R\ln M_N^2\right\} \qquad (3.42)
\end{aligned}
$$

Thus, we can eliminate M_n dependence by choosing δZ_2 and δZ_4 by

$$
\delta Z_2 = \frac{1}{16\pi^2}\left(\sum_{N=0}^{s}C_N^{-1}M_N^2\ln M_N^2+\mu_2\right) \qquad (3.43)
$$

$$
\delta Z_4 = \frac{1}{16\pi^2}\left(\sum_{N=0}^{s}C_N^{-1}(\xi-\frac{1}{6})\ln M_N^2+\mu_4\right) \qquad (3.44)
$$

where μ_2, μ_4 are finite constants.

Next we use Eq. (3.33) to compute Z_3. Since δZ_3 is dimensionless, it contain log divergence with curvature independent coefficients. Hence, we can perform the computation in flat spacetime

$$
\begin{aligned}
& \int d^4x[\langle T(\sigma(0)+\sum_n\chi_n(0))^2(\sigma(x)+\sum_n\chi_n(x))^2\rangle_c \qquad (3.45) \\
=\ & (-i)^2\int\frac{d^4p}{(2\pi)^4}\left(\sum_N C_N^{-1}\frac{1}{p^2+M_N^2}\right)^2 \qquad (3.46) \\
=\ & \frac{i}{32\pi^2}\sum_{N,M}C_N^{-1}C_M^{-1}\frac{M_N^2+M_M^2}{M_N^2-M_M^2}\ln\frac{M_N^2}{M_M^2} \qquad (3.47)
\end{aligned}
$$

To cancel out such M_N dependence, we can choose

$$\delta Z_3 = \frac{1}{64\pi^2}(\sum_{N,M} C_N^{-1} C_M^{-1} \frac{M_N^2 + M_M^2}{M_N^2 - M_M^2} \ln \frac{M_N^2}{M_M^2} + \mu_3) \tag{3.48}$$

where μ_3 is a finite constant.

To compute δZ_1 at one-loop level, we can use Eq. (3.34). As the case for Z_1, we can perform the computation in flat spacetime. We can get

$$\delta Z_1 = \frac{\lambda_r}{64\pi^2}(\sum_{N,M} C_N^{-1} C_M^{-1} \frac{M_N^2 + M_M^2}{M_N^2 - M_M^2} \ln \frac{M_N^2}{M_M^2} + \mu_1) \tag{3.49}$$

To summarize, we have fixed the divergent part of the renormalization constants. The remaining finite part μ_i should be determined by renormalization conditions, which reflects the physical degrees of freedom in defining this operator. We note that due to the diffeomorphism invariance, these counter-terms would also work to cancel out the 1-loop divergence in the correlator $\langle \sigma_x^2 \sigma_y^2 \rangle$ or $\langle \sigma_x^2 \zeta_y \rangle$.

Chapter 4

Diffeomorphism Invariance and Ward Identity

Diffeomorphism invariance reflects the simple fact that the form of physical laws does not depend on the coordinate choice. Just as Ward Identity in QED ensures the cancellation of the radiative correction to the photon mass, Ward Identity in general relativity ensures the shift-symmetry of ζ variable. In this chapter, we first review the diffeomorphism invariance for classical field theory at full order and at perturbative order, then we review the implication for correlation function in quantum field theory on a fixed curved background.

4.1 Classical Diffeomorphism Invariance - Gauge transformation

Classical differmorphism invariance requires that the action

$$S[g,\sigma] = S_{EH}[g] + S_M[g,\sigma] \tag{4.1}$$

to be invariant under the transformation

$$g \mapsto \varphi^* g \tag{4.2}$$

$$\sigma \mapsto \varphi^* \sigma \tag{4.3}$$

where g is the metric tensor $g = g_{\mu\nu} dx^\mu dx^\nu$, σ is a matter field and $\varphi : \mathcal{M} \to \mathcal{M}$ is a diffeomorphism of the spacetime manifold $\mathcal{M}$. We use Ψ to denote the metric and field configurations, i.e. $\Psi = (g,\sigma)$. Two configurations are equivalent, noted as $\Psi \sim \Psi'$, if we can find diffeomorphism φ such that $\Psi = \varphi^* \Psi'$.

A special case of diffeomorphism is the exponential map $\exp(\epsilon X)$, where ϵ is a small real number and X is a vector field. If T is a tensor, we can Taylor expand $\exp(\epsilon X)^* T$ with respect to ϵ:

$$\exp(\epsilon X)^* T = T + \epsilon \mathcal{L}_X T + \frac{1}{2}\epsilon^2 \mathcal{L}_X \mathcal{L}_X T + \cdots . \tag{4.4}$$

In perturbation theory, we choose a background configuration $\Psi^{(0)}$ and consider perturbations $\delta\Psi$ on top of it. The action for the perturbation can be obtained as

$$S[\delta\Psi; \Psi^{(0)}] = S[\Psi^{(0)} + \delta\Psi] - S[\Psi^{(0)}]. \tag{4.5}$$

Under the diffeomorphism φ, the perturbation $\delta\Psi$ transforms as

$$\delta\Psi \mapsto \varphi^*(\Psi^{(0)} + \delta\Psi) - \Psi^{(0)}. \tag{4.6}$$

To be able to do power counting in perturbation theory, we introduce a formal parameter λ. The λ dependent configuration Ψ_λ can be written as

$$\Psi_\lambda = \Psi^{(0)} + \lambda \Psi^{(1)} + \lambda^2 \Psi^{(2)} + \cdots \tag{4.7}$$

and the λ dependent diffeomorphism φ_λ can be written as [35],

$$\varphi_\lambda = \cdots \exp(\frac{\lambda^3}{3!} X^{(3)}) \circ \exp(\frac{\lambda^2}{2!} X^{(2)}) \circ \exp(\lambda X^{(1)}) \tag{4.8}$$

where $X^{(i)}$ is a vector field. The diffeomorphism can be worked out order by order in λ,

$$\begin{aligned} \varphi^*_\lambda \Psi_\lambda &= \Psi^{(0)} + \lambda[\Psi^{(1)} + \mathcal{L}_{X^{(1)}} \Psi^{(0)}] \\ &+ \lambda^2[\Psi^{(2)} + \mathcal{L}_{X^{(1)}} \Psi^{(1)} + \frac{1}{2}\mathcal{L}_{X^{(1)}} \mathcal{L}_{X^{(1)}} \Psi^{(0)} + \frac{1}{2}\mathcal{L}_{X^{(2)}} \Psi^{(0)}] + \cdots \end{aligned} \tag{4.9}$$

And the action $S[\Psi_\lambda]$ is invariant under the above diffeomorphism at each order of λ.

A local object $f[x; \delta\Psi]$ is said to be n-th order gauge-invariant, if for any diffeomorphism φ_λ

$$f[x; \varphi^*_\lambda \Psi_\lambda - \Psi^{(0)}] - f[x; \Psi_\lambda - \Psi^{(0)}] = O(\lambda^{n+1}). \tag{4.10}$$

4.2 Quantum Diffeomorphism Invariance - Ward Identity

A symmetry in a classical field theory is preserved at the quantum level, if the regulator preserves this symmetry and if the functional measure is invariant under this transformation. The quantum symmetry is reflected in the transformation of the correlation functions.

For example, consider a scalar field on a given manifold $(\mathcal{M}, g)$. The two point function is

$$\langle\phi(x)\phi(y)\rangle_g = \int D\phi e^{iS(\phi;g)}\phi(x)\phi(y) \tag{4.11}$$

The two point function only depends on the metric field g and points x, y. Intuitively, the symmetry says for any diffeomorphism $\varphi : \mathcal{M} \mapsto \mathcal{M}$, the metric field and the points changes as

$$g \mapsto \tilde{g} = (\varphi^{-1})^* g,\ x \mapsto \tilde{x} = \varphi(x),\ y \mapsto \tilde{y} = \varphi(y) \tag{4.12}$$

then the two-point function should remain invariant, i.e.

$$\langle\phi(x)\phi(y)\rangle_g = \langle\phi(\tilde{x})\phi(\tilde{y})\rangle_{\tilde{g}}. \tag{4.13}$$

Ward identity is the infinitesimal version of this relation.

Let $\varphi = \exp(\epsilon X)$, then

$$\begin{aligned}\tilde{g} &= \exp(-\epsilon X)^* g = g - \epsilon\mathcal{L}_X g + \cdots & (4.14)\\ S(\tilde{g}, \phi) &= S(g, \phi) - \epsilon\int d^4x\sqrt{g}\frac{1}{2}T^{\mu\nu}\mathcal{L}_X(g)_{\mu\nu} + \cdots & (4.15)\\ \phi(\tilde{x}) &= \phi(x) + \epsilon\mathcal{L}_X\phi(x) + \cdots & (4.16)\end{aligned}$$

Plug this into Eq.(4.13) and Taylor expand with respect to ϵ, one get

$$-i\int d^4z\sqrt{g}\frac{1}{2}\mathcal{L}_X(g)_{\mu\nu}(z)\langle T_z^{\mu\nu}\phi_x\phi_y\rangle_g + \langle\mathcal{L}_X(\phi)_x\phi_y\rangle_g + \langle\phi_x\mathcal{L}_X(\phi)_y\rangle_g = 0. \tag{4.17}$$

Or equivalently, using

$$\mathcal{L}_X(g)_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu \tag{4.18}$$

and perform integration by part, we get

$$i\nabla_\mu\langle T_z^{\mu\nu}\phi_x\phi_y\rangle_g = \frac{1}{\sqrt{g_x}}\delta^4(x-z)g^{\alpha\nu}\frac{\partial}{\partial x^\alpha}\langle\phi_x\phi_y\rangle_g + \frac{1}{\sqrt{g_y}}\delta^4(y-z)g^{\alpha\nu}\frac{\partial}{\partial y^\alpha}\langle\phi_x\phi_y\rangle_g \tag{4.19}$$

Eq.(4.19) is derived for path ordered vacuum expecation value, we can specialize to the in-in case (we keep the external operator inserted on the forward branch)

$$\begin{aligned}i\nabla_\mu\langle in|T_z^{\mu\nu+}\phi_x^+\phi_y^+|in\rangle_g &= \frac{1}{\sqrt{g_x}}\delta^4(x-z)g_x^{\alpha\nu}\frac{\partial}{\partial x^\alpha}\langle in|\phi_x^+\phi_y^+|in\rangle_g\\ &\quad + \frac{1}{\sqrt{g_y}}\delta^4(y-z)g_y^{\alpha\nu}\frac{\partial}{\partial y^\alpha}\langle in|\phi_x^+\phi_y^+|in\rangle_g & (4.20)\\ i\nabla_\mu\langle in|T_z^{\mu\nu-}\phi_x^+\phi_y^+|in\rangle_g &= 0 & (4.21)\end{aligned}$$

The fact that Eq.(4.21) has no contact term is easy to understand, since $T_z^{\mu\nu-}$ is inserted on the backward time branch of the manifold, it can never *contact* points x and y.

The above Ward identity was derived for bare correlation function. To get renormalized version, we note that

$$\begin{aligned}\phi_B(x) &= Z_\phi \phi_r(x) &(4.22)\\ T_B^{\mu\nu}(x) &= \frac{2}{\sqrt{g(x)}}\frac{\delta S_\Lambda}{\delta g_{\mu\nu}(x)} = \frac{2}{\sqrt{g(x)}}\frac{\delta(S_r + S_{P.V.} + S_{c.t.})}{\delta g_{\mu\nu}(x)} \\ &= T^{\mu\nu}[\phi_r] + T^{\mu\nu}[\chi] + T^{\mu\nu}_{c.t.} \equiv (T^{\mu\nu})_r(x) &(4.23)\end{aligned}$$

where the renormalized $(T^{\mu\nu})_r$ includes the regulator contribution and the counter-term contribution. Plug them into Eq.(4.19), and cancel out factors of Z_ϕ, we get

$$i\nabla_\mu \langle (T_z^{\mu\nu})_r \phi_{x,r}\phi_{y,r}\rangle_g = \frac{\delta_{xz}}{\sqrt{g_x}} g_z^{\alpha\nu} \frac{\partial}{\partial x^\alpha}\langle \phi_{x,r}\phi_{y,r}\rangle_g + \frac{\delta_{yz}}{\sqrt{g_y}} g_z^{\alpha\nu}\frac{\partial}{\partial y^\alpha}\langle\phi_{x,r}\phi_{y,r}\rangle_g \qquad (4.24)$$

Chapter 5

Dark Matter Relic Abundance

Gravitational particle production (as reviewed e.g in [36, 34]) and string production (see e.g. [37, 38, 39, 40, 41, 42, 43, 44]) are generic phenomena for quantum fields in a curved spacetime background and are analogs of particle creation in strong electric fields (see e.g. [45, 46]). In the case of Friedmann-Robertson-Walker (FRW) cosmology without inflation, it was found [47, 48, 49, 50, 51] that the production of fermion and conformally coupled scalar fields near the radiation dominated (RD) universe singularity occurs when the particle masses m are comparable to the Hubble expansion rate H, with a number density $n \sim m^3$ that dilutes as a^{-3} due to expansion. The fractional relic density of these particles at the time of radiation-matter equality is $\Omega_X \sim (m_X/10^9\text{GeV})^{5/2}$ [52]. Hence, the requirement of $\Omega_X < 1$ puts an upper bound of 10^9 GeV on the stable particle mass.[1]

In contrast, in inflationary cosmology the previously unbounded rapid growth of H as one moves backward in time towards the RD singularity is replaced by a nearly constant H_e during the quasi-de Sitter (dS) era. In such cases, the possibility of superheavy dark matter in a wide range of masses including $m > H_e$ was emphasized in [19, 55]. In fact, natural superheavy dark matter candidates existed in the context of string phenomenology before the gravitational production mechanism was appreciated [56, 57]. Furthermore, many extensions of the Standard Model also possess superheavy dark matter candidates (see e.g. [58, 59, 60, 61, 62, 63, 64, 65, 66, 67]), which can have interesting astrophysical implications (see e.g. [62, 68, 69, 70, 71, 72]). In such contexts, analytic relic density formulae have been computed in the heavy and the light mass regimes for conformally coupled scalars [73, 74]).

In this chapter, we turn our attention to the gravitational particle production of long-lived Dirac

[1] Physics quite similar to this is reported in [53, 54].

fermions in inflationary cosmology. Gravitational particle production of Dirac fermions has been studied numerically within the context of specific chaotic inflationary models [55]. Our purpose is to clarify the analytic computation and to derive a universal result for the light mass scenario that is nearly independent of the details of the inflationary model. Our result is identical up to an overall $O(1)$ multiplicative factor to that obtained for conformally coupled light scalar fields in [74], despite the fact that the Dirac structure naively imposes a different spectral (momentum scaling) property on the equations governing the particle production.[2] In comparison to the conformally coupled scalar case, no special non-renormalizable coupling to gravity nor possibility of tadpole instabilities concern the fermionic scenario in the light mass limit because the fermion kinetic operator is conformally invariant and fermions cannot obtain a nonvanishing vacuum expectation value.

We also derive the particle production spectrum for the heavy mass scenario and find it to be identical to the result of [73] (again up to an $O(1)$ multiplicative constant) despite a different momentum dependence of the starting point of the equations. As expected, the heavy mass number density falls off exponentially. In contrast with the light mass limit, this case is sensitive to the details of the transition out of the inflationary era. To emphasize the simplicity and the novel analytic arguments of the light mass scenario, we relegate the heavy mass results to an appendix.

It should be noted that the production of fermions in inflationary cosmology has been extensively considered during the recent past, but most analyses have focused on the non-gravitational interactions. For example, [75, 76, 77, 78, 79, 80, 81, 82] focused on both numerical and analytic analyses of fermion production during preheating. [83] considered the production effects when the fermion mass passes through a zero during the quasi-dS phase. The effects of radiative corrections that modify the fermion dispersion relationship and its connection to particle production were considered in [84]. Gravitino production has also been considered by many authors (see e.g. [85, 86, 87, 88, 89, 90]). The main thrust of this work differs in that it focuses on the minimal gravitational coupling and derives a simple bound analogous to Eq. (44) of [74]. Indeed, our results will aid in future investigations similar to [65] which would benefit from a more accurate simple analytic estimate of the dark matter abundance.

In Sec. 5.1, we discuss the intuition behind the general formalism for the gravitational produc-

[2] Although the aim of [74] is to consider a hybrid inflationary scenario, it also contains a universal result, equation (44), applicable to generic inflationary scenarios. There is also a misprint in [74] in stating that the situation is for minimal coupling rather than for conformal coupling.

tion of massive Dirac fermions in curved spacetime. In Sec. 5.2, we discuss the generic features of the spectrum and derive the main result, which is that for a given mode with comoving wave number k, the Bogoliubov coefficient magnitude $|\beta_k|^2 \sim O(1/2)$ if $H(\eta) > m$ when $k/a(\eta) \sim m$. We test this analytic result within a toy inflationary model in Sec. 5.3, and discuss the dependence on reheating and the implications for the relic density in Sec. 5.4. Appendix A contains a collection of useful results for fermionic Bogoliubov transformation computations. Appendix B.1 contains a complementary argument (which relies more on the spinorial picture of the fermions) for the universality of the Bogoliubov coefficient in the light mass region. Appendix B.2 contains the particle density spectrum for the heavy mass limit.

5.1 Fermion Particle Production: Background and Intuition

To compute the particle production of Dirac fermions in curved spacetime, we follow the standard procedure as outlined for example in [36, 34] to calculate the Bogoliubov coefficient β_k between the in-vacuum corresponding to the inflationary adiabatic vacuum and the out-vacuum corresponding to the adiabatic vacuum defined at post-inflationary times. The details of this formalism and our conventions are presented in Appendix A, with the expression for β_k given in Eq. (A.45).

However, to obtain a better intuitive picture of the particle production mechanism, here we present general physical arguments regarding the expected features of the spectrum. We begin by considering a Dirac fermion field Ψ described by

$$\mathcal{L} = i\bar{\Psi}\gamma^\mu \nabla_\mu \Psi - m\bar{\Psi}\Psi \tag{5.1}$$

minimally coupled to gravity. As the action $S = \int d^4x\sqrt{g}\mathcal{L}$ is conformally invariant in the $\{m \to 0, \hbar \to 0\}$ limit (with $\delta g_{\mu\nu}(x) = -2\sigma(x)g_{\mu\nu}(x)$), physical quantities are necessarily independent of the FRW scale factor a to leading order in $\hbar$. Hence, the leading $\hbar$ order Bogoliubov coefficient β_k is zero in the $ma/k \to 0$ limit, since it is the metric that drives the particle production (*i.e.*, it plays the role of the electric field in the analogy of particle creation by strong electric fields). This implies that particle production can only occur in significant quantities for non-relativistic modes.[3]

We next point out that the Dirac equation with a time-dependent mass term results in mixing between positive and negative frequency modes, similar to the case of the conformally coupled

[3] We neglect possible conformal symmetry breaking effects associated with preheating [77]. In that sense, there is a mild implicit model dependence here.

Klein-Gordon system with a time-dependent mass. To see this explicitly, consider the Dirac equation for the spinor mode functions $u_{A,B}$ that follows from Eq. (5.1):

$$i\partial_\eta \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} am & k \\ k & -am \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix}, \tag{5.2}$$

which is our Eq. (A.23) from Appendix A. Here $u_{A,B}$ span the complete solution space (they contain both approximate positive and negative frequency solutions in the adiabatic regime). Here we are working in conformal time, which is related to the comoving observer's proper time via $dt \equiv a(\eta)d\eta$. From Eq. (5.2), we see that the rotation matrix that diagonalizes the right hand side is a function of the time-dependent quantity am. Hence, the Dirac equation *as a function of time* mixes approximate positive and negative frequency solutions leading to non-vanishing particle production.

To estimate the Bogoliubov coefficient, we can compute the effects of the time-dependent mixing matrix $\mathcal{U} \in O(2)$ as follows. We begin by inserting $1 = \mathcal{U}^T\mathcal{U}$ into Eq. (5.2) to obtain

$$i\mathcal{U}\partial_\eta \left[\mathcal{U}^T\mathcal{U} \begin{pmatrix} u_A \\ u_B \end{pmatrix}\right] = \mathcal{U} \begin{pmatrix} am & k \\ k & -am \end{pmatrix} \mathcal{U}^T\mathcal{U} \begin{pmatrix} u_A \\ u_B \end{pmatrix} \tag{5.3}$$

$$\implies i\mathcal{U}\partial_\eta\mathcal{U}^T \begin{pmatrix} u'_A \\ u'_B \end{pmatrix} + i\partial_\eta \begin{pmatrix} u'_A \\ u'_B \end{pmatrix} = \begin{pmatrix} \omega_c & 0 \\ 0 & -\omega_c \end{pmatrix} \begin{pmatrix} u'_A \\ u'_B \end{pmatrix}, \tag{5.4}$$

in which $\omega_c = \sqrt{k^2 + m^2a^2}$ and the primed basis is defined to be

$$\begin{pmatrix} u'_A \\ u'_B \end{pmatrix} \equiv \mathcal{U} \begin{pmatrix} u_A \\ u_B \end{pmatrix}. \tag{5.5}$$

The Dirac equation is diagonal in the primed basis except for the appearance of the mixing term

$$\mathcal{U}\partial_\eta\mathcal{U}^T = \frac{a}{2} \begin{pmatrix} 0 & \frac{mHk_p}{k_p^2+m^2} \\ -\frac{mHk_p}{k_p^2+m^2} & 0 \end{pmatrix}, \tag{5.6}$$

with $k_p \equiv k/a$. From this result, we see that during inflation the mixing term approximately vanishes for a fixed comoving wave number k as $a \to 0$, while after inflation it is the largest when H is the largest. Using this result, it is straightforward to show that the Bogoliubov coefficients due to mixing take the following form:

$$\beta_k^{\text{mix}} \sim \int dt \frac{mk_p}{k_p^2 + m^2} H e^{-2i\int dt\, \omega_k}, \tag{5.7}$$

in which $\omega_k = \sqrt{k_p^2 + m^2}$. One may still ask whether there are any other sources of positive and negative frequency mixing since the diagonal terms of Eq. (5.4) are time dependent, just as conformally coupled scalar fields contain $\omega^2 = k^2 + m^2 a^2$ in their mode equations. The answer is no *if* the fermionic particles are *defined* as modes that exactly satisfy the condition

$$i\partial_\eta \begin{pmatrix} u'_A \\ u'_B \end{pmatrix} = \begin{pmatrix} \sqrt{k^2 + m^2 a^2} & 0 \\ 0 & -\sqrt{k^2 + m^2 a^2} \end{pmatrix} \begin{pmatrix} u'_A \\ u'_B \end{pmatrix}. \tag{5.8}$$

For example, the adiabatic vacuum positive frequency modes are defined to be

$$\begin{pmatrix} u'_A \\ u'_B \end{pmatrix} \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\int dt \sqrt{\frac{k^2}{a^2} + m^2}}. \tag{5.9}$$

Eq. (5.9) corresponds to a zeroth order adiabatic vacuum in which the adiabaticity parameter ϵ_A is defined as

$$\epsilon_A \equiv \frac{mHk_p}{\left(k_p^2 + m^2\right)^{3/2}}, \tag{5.10}$$

in accordance with the usual conventions [36, 51, 19, 91]. This parameter vanishes in the asymptotically far past (near when the in-vacuum is defined) and in the far future (near when the out-vacuum is defined). Eq. (5.9) coincides with

$$\begin{pmatrix} u_A \\ u_B \end{pmatrix} \rightarrow \begin{pmatrix} \sqrt{\frac{\omega + am}{2\omega}} \\ \sqrt{\frac{\omega - am}{2\omega}} \end{pmatrix} e^{-i\int^\eta d\eta' \omega} \tag{5.11}$$

in the basis of Eq. (5.2).

To summarize, the zeroth adiabatic order vacuum Bogoliubov coefficient is approximately given by Eq. (A.57). Compared to the conformally coupled bosonic case (see e.g. [73]), the long wavelength fermionic particle production is suppressed due to the appearance of k_p in the numerator.

5.2 Light Mass Case and Generic Features of the Spectrum

In this section, we present a universal result for the spectrum in the light mass scenario that is nearly independent of the details of the inflationary model. We shall show that under a specific set of conditions, the Bogoliubov spectral amplitude (evaluated with observable particle state basis defined at time t) takes the approximate form

$$|\beta_k(t)|^2 \sim O(1/2). \tag{5.12}$$

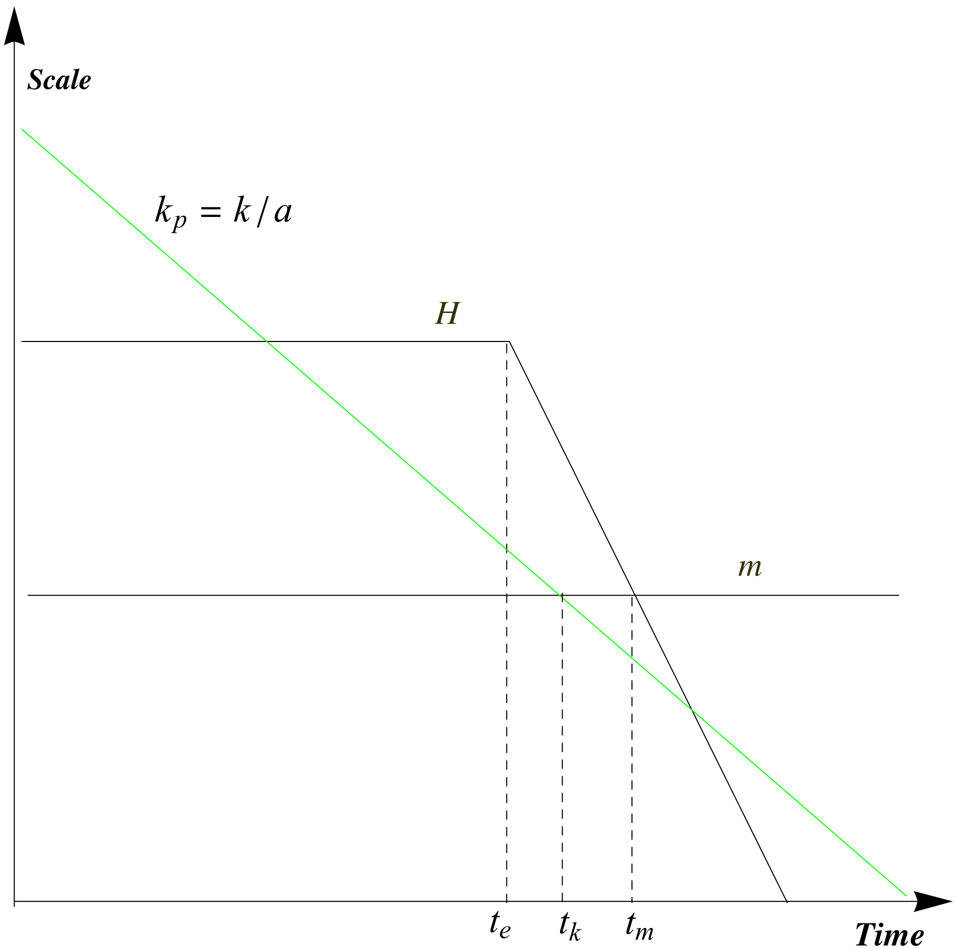

Figure 5.1: The evolution of the physical scales $H(t)$, $k_p(t)$ and the corresponding time points. Modes with comoving wavenumber k make the transition from relativistic to non-relativistic at time t_k. The Hubble rate drops below m at t_m, and the end of inflation is at t_e.

An alternate argument emphasizing more of the spinorial nature of the fermions is presented in Appendix B.1.

For Eq. (5.12) to hold generically, the following conditions must simultaneously be satisfied. The fermions that are produced must be light (to be made precise below). After the end of inflation, the modes that are produced must become non-relativistic during the time when the expansion rate is the dominant mass scale. Finally, t must be a time when particles with $k_p = k/a$ are non-relativistic.

The evolution of the relevant physical scales is shown for clarity in Fig. (5.1). Here t_e denotes the time of the end of inflation, t_m is defined by $H(t_m) = m$, and t_k stands for the time when $k_p(t) = m$. The two conditions under which Eq. (5.12) holds are $t_m > t_k > t_i$ and $t > t_k$, in which t_i marks the beginning of inflation (not shown in the figure).

To show this more explicitly, we begin by noting that the modes that can be significantly produced by the FRW expansion satisfy $k_p \lesssim m$, since relativistic modes are approximately conformally invariant. Furthermore, during the time that $k_p \lesssim m$, Eq. (A.57) takes the form

$$\beta_k(t) \quad \sim \quad \int^t dt' \frac{k_p(t')}{m} H(t') e^{-2i\int^{t'} dt'' \omega_k(t'')}. \tag{5.13}$$

Let us consider Eq. (5.13) for the time period with $H(t') > m$, such that $H^2 > \omega_k^2$. Here we take k to be consistent with $k_p \lesssim m$; more precisely, $ma(t) > k > ma(t_i)$, where t_i is the time when the initial vacuum is defined, which is typically at the beginning of inflation.[4] In this regime, the largest contribution to β_k arises from the time t_k when $k_p = k/a$ is at its largest while remaining non-relativistic: i.e., $k/a(t_k) = m$. When these conditions are satisfied, Eq. (5.13) results in

$$\beta_k(t) \sim O\left(\frac{k/a(t_k)}{m}\right), \tag{5.14}$$

which we see is indeed of $O(1)$.

Our result indicates that the fermion creation saturates the Pauli exclusion principle, since $|\beta_k|^2$ represents the phase space density of the fermions created. The conditions leading to this result can be intuitively explained as follows. To have such a maximal production, we cannot excite $k_p \gg m$ modes because of conformal symmetry. Furthermore, we cannot excite $k_p \ll m$ modes because the violation of energy conservation is of order $F\Delta x \sim (Hk_p)k_p/(mH) \sim (k_p/m)k_p$, where F is the force due to the expansion of the universe and Δx is the distance over which the particle travels under this force. In addition, the force can act on the virtual particle only on a time scale shorter than the lifetime of the virtual state, which is of order $1/m$. This is equivalent to the condition that $H > m$ for this picture of particle production.

As Eq. (5.14) is independent of H, the result is insensitive to the details of the inflationary model. This insensitivity holds as long as the dominant contribution to $\beta_k(t)$ arises from the time period with $H(t')/m > 1$. However, $H(t')/m > 1$ clearly fails if $t' > t_m$. Thus, there is a mild inflationary model dependence, although it is largely insensitive. This is clear because the fermion mass can be made arbitrarily small compared to the expansion rate for any inflationary model. As we shall see in Sec. 5.4, a stronger inflationary model dependence arises from the dilution factor $a(t_m)/a(t)$, which typically is a function of the reheating temperature.

[4] The condition $k > ma(t_i)$ comes from the requirement of setting the adiabatic vacuum condition, which only applies for modes with subhorizon wavelengths.

Given that there is a general restriction that $|\beta_k|^2 < 1$ from quantization conditions, here $O(1)$ must mean a number less than unity.[5] To remind ourselves of this fact, we will refer to this $O(1) < 1$ number as $O(1/\sqrt{2})$. Putting all the conditions together with Eq. (5.14), we find

$$|\beta_k(t)|^2 \sim O(1/2) \quad \text{for } t_m > t_k > t_i \text{ and } t > t_k. \tag{5.15}$$

A more explicit restriction on k that is consistent with the requirements of Eq. (5.15) can be written as follows:

$$ma(t_m) \gtrsim k > ma(t_i) \text{ and } ma(t) \gtrsim k. \tag{5.16}$$

Eqs. (5.15) and (5.16) are the main results of this section.

For modes with $k > ma(t_m)$, $|\beta_k|^2$ is smaller since Eq. (5.13) is suppressed by an additional factor of H/m. The exact high k behavior of β_k is sensitive to the adiabatic order of the vacuum boundary condition as well as the details of the scale factor during the transition out of the quasi-dS era. However, what is generic is that the spectral contribution to the particle density no longer grows appreciably when $k > ma(t_m)$. Hence, we can define the critical momentum $k_* \equiv ma(t_m)$, which satisfies

$$k_*/a_e = (H_e/m)^{2/n_a} m, \tag{5.17}$$

where we have parameterized the energy density after the end of inflation as $\rho \propto a^{-n_a}$. Integrating over $d^3k/(2\pi a)^3$ to obtain the energy density of the fermions, for an order of magnitude estimate we can introduce a step function $\Theta(k_* - k)$ as follows:

$$\rho_\Psi(t) \sim 4 \times \frac{m}{4\pi^2} \frac{1}{a^3} \int dk k^2 \Theta(k_* - k), \qquad t_{ma(t_i)} \ll t_m < t, \tag{5.18}$$

in which t_{ma_i} is the time at which $k = ma(t_i)$. Assuming that the lower limit of Eq. (5.18) contributes negligibly to the integral, we obtain

$$\rho_\Psi(t) \sim 4 \times \frac{m^4}{12\pi^2} \left(\frac{a(t_m)}{a(t)} \right)^3, \tag{5.19}$$

which contains the mild inflationary scenario dependence discussed previously.

5.3 Example of Fermion Production in a Toy Inflationary Model

To test the analytic estimation of Sec. 5.2, we now numerically compute the particle production in a toy inflationary model with instantaneous reheating occurs (*i.e.*, in which the quasi-dS phase

[5] The Bogoliubov coefficients satisfy $|\alpha_{\vec{k},s}|^2 + |\beta_{-\vec{k},s}|^2 = 1$, while Eq. (5.14) effectively neglects this constraint.

connects instantaneously to the RD phase). As is well known, such non-analytic models have unphysical large momentum behavior [36], which for our purposes can be dealt with simply by cutting off the integration of the spectrum. We find there is an upper bound on the fermion mass if $m < H_e$ during inflation, similar to the case of fermion production in pure RD cosmology [52]. We shall turn to the more realistic case in which the inflationary era exits to a transient pressureless era during reheating in Sec. 5.4.

Let us consider a background spacetime which is initially dS with a Hubble constant H_e that is followed by RD spacetime. Although the junction between the dS and RD eras is instantaneous, the scale factor $a(t)$ and the Hubble rate $H(t)$ are continuous across the junction. In particular, if we set the junction time at the conformal time $\eta = 0$ and we set the scale factor at the junction time to be a_e, the scale factor and Hubble rates can be written as

$$a(\eta) = \begin{cases} \left(\left(\frac{1}{a_e H_e} - \eta\right) H_e\right)^{-1} & \eta \leq 0 \text{ (dS)} \\ a_e^2 H_e \left(\eta + \frac{1}{a_e H_e}\right) & \eta > 0 \text{ (RD)}, \end{cases} \tag{5.20}$$

in which

$$H(\eta) = \begin{cases} H_e & \eta \leq 0 \text{ (dS)} \\ H_e \left(\frac{a_e}{a(\eta)}\right)^2 & \eta > 0 \text{ (RD)}, \end{cases} \tag{5.21}$$

indicating that the leading discontinuity in a occurs at second order in the conformal time derivative.

To compute β_k using Eq. (A.45), it is necessary to fix the boundary conditions for the in-modes and the out-modes. For the in-modes, we require that in the infinite past, when a certain given mode's wavelength is within the horizon radius, its mode function must agree with the flat space positive frequency mode function. In other words, as $a(\eta) \to 0$,

$$\begin{pmatrix} u_A \\ u_B \end{pmatrix}_{k,\eta}^{in} \to \begin{pmatrix} \sqrt{\frac{\omega + a(\eta)m}{2\omega}} \\ \sqrt{\frac{\omega - a(\eta)m}{2\omega}} \end{pmatrix} e^{-i\int^{\eta} \omega(\eta')d\eta'}. \tag{5.22}$$

The in-modes' analytic expressions during the dS era thus take the form

$$\begin{pmatrix} u_A \\ u_B \end{pmatrix}_{k,\eta}^{in} = \begin{pmatrix} \sqrt{\frac{\pi}{4}(\frac{k}{aH_e})} e^{i\frac{\pi}{2}(1-i\frac{m}{H_e})} H^{(1)}_{\frac{1}{2}-i\frac{m}{H_e}}(\frac{k}{aH_e}) \\ \sqrt{\frac{\pi}{4}(\frac{k}{aH_e})} e^{i\frac{\pi}{2}(1+i\frac{m}{H})} H^{(1)}_{\frac{1}{2}+i\frac{m}{H_e}}(\frac{k}{aH_e}) \end{pmatrix} \tag{5.23}$$

(5.24)

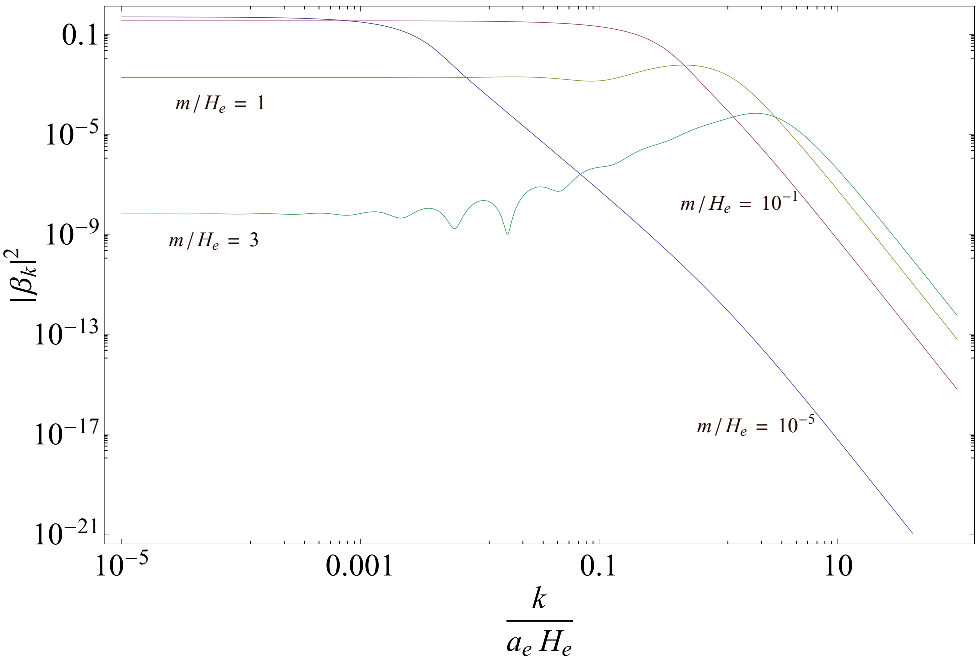

Figure 5.2: The Bogoliubov coefficient amplitude $|\beta_k|^2$ as a function of $k/(a_e H_e)$ for various ratios of the fermion mass to the Hubble expansion rate during the dS era.

where $H_\nu^{(1)}$ are Hankel functions of the first kind. Similarly, for the out-modes, as $k/a > H(\eta)$ in the RD era, we require the mode functions to agree with the flat space positive frequency mode functions, *i.e.*, as $a(\eta) \to +\infty$,

$$\begin{pmatrix} u_A \\ u_B \end{pmatrix}_{k,\eta}^{out} \to \begin{pmatrix} \sqrt{\frac{\omega + a(\eta)m}{2\omega}} \\ \sqrt{\frac{\omega - a(\eta)m}{2\omega}} \end{pmatrix} e^{-i\int^\eta \omega(\eta')d\eta'}. \tag{5.25}$$

The out-mode analytic expressions during the RD era are given by

$$\begin{pmatrix} u_A \\ u_B \end{pmatrix}_{k,\eta}^{out} = \begin{pmatrix} e^{-\frac{\pi}{4}C} D_{-iC}(e^{i\pi/4}\sqrt{\frac{2m}{H(\eta)}}) \\ \sqrt{C} e^{-\frac{\pi}{4}C + i\frac{\pi}{4}} D_{-iC-1}(e^{i\pi/4}\sqrt{\frac{2m}{H(\eta)}}) \end{pmatrix}, \tag{5.26}$$

in which $C \equiv (k^2/a_e^2)/(2mH_e)$ characterizes the ratio of the momentum to the dynamical mass scale and the $D_v(x)$ are parabolic cylinder functions.

The numerical results for $|\beta_k|^2$ are shown as a function of $k/(a_e H_e)$ for various choices of the fermion masses in Fig. 5.2. From these results, we first note that it can be determined that for heavy masses $m > H_e$, e.g. $m/H_e = 1$ or 3, the infrared end of the spectrum behaves as $|\beta_k|^2 \sim (1 + \exp(2\pi m/H_e))^{-1}$. Further details of the heavy mass case are given in Appendix B.2. As the heavy mass situation is likely to be more sensitive to the abrupt transition approximation made in this section, we restrict our attention here to the light mass case in which $m < H_e$.

For the light mass case (e.g. $m/H_e = 10^{-5}$ in Fig. (5.2)), we can see there are three ranges of k that each have qualitatively different behavior. For $k/a_e > H_e$, the modes are still inside the horizon at the end of inflation, and the spectrum falls off as $|\beta_k|^2 \propto k^{-6}$. In contrast, for $\sqrt{mH_e} < k/a_e < H_e$, the modes are outside of the horizon at the end of inflation and remain relativistic at the time when $m = H(\eta)$ during RD. In this case, the spectrum falls off as $|\beta_k|^2 \propto k^{-4}$. Finally, for $k/a_e < \sqrt{mH_e}$, the modes are outside the horizon at the end of inflation and have become non-relativistic before $m = H(\eta)$ during RD. This results in a constant spectrum of $|\beta_k|^2 \approx \frac{1}{2}$, in agreement with the results of Sec. 5.2. Generically, if the scale factor $a(\eta)$ is sufficiently continuous [50, 91], the spectrum will fall off in the ultraviolet region faster than k^{-3}, such that the total number density $n \sim \int d^3k|\beta_k|^2$ is finite. The majority of the contribution arises from the region in which $k/a_e < \sqrt{mH_e}$ where $|\beta_k|^2 \approx \frac{1}{2}$, as anticipated in Sec. 5.2. The number density for particle masses in the range of $m < 0.1H_e$ is numerically determined to be (recall that η_m is defined by $H(\eta_m) = m$)

$$n(\eta) = 4 \times 0.005\, m^3 \left(\frac{a(\eta_m)}{a(\eta)}\right)^3 , \tag{5.27}$$

which again agrees with the analytic estimate of Eq. (5.19).

5.4 Inflationary Reheating Dependence

We now consider the more realistic situation in which there is a smooth transition region between the dS and RD phases. When inflation ends, there is typically a period of coherent oscillations ($a_e < a < a_{\rm rh}$) during which the equation of state is close to zero (see e.g. [92, 93, 94]). During that period, the expansion rate behaves as $H \propto a^{-3/2}$ and not a^{-2} as during RD. This difference will lead to an effective dilution of the dark matter particles by the time RD is reached. More precisely, the fermion number density will be diluted as $1/a^3$ as long as the fermion plus anti-fermion number is approximately conserved. As we shall see below, the integrated dilution is typically a function of the reheating temperature during inflation.

Accounting for the dilution, in this section we estimate the relic abundance of fermionic particles (fermions plus anti-fermions).[6] The dilution consideration breaks up naturally into two cases:

[6]This requires the fermion self-annihilation cross section rate to be smaller than the expansion rate throughout its history. Such weak interactions generically can be achieved for sufficiently large particle masses [19], which are allowed as long as the inflationary scale is sufficiently large.

$a_{\rm rh} > a(t_m)$ and $a_{\rm rh} < a(t_m)$. The former case corresponds to the situation in which the dominant particle production occurs during the reheating period, while the latter case corresponds to the complementary situation, which we shall see below is unlikely to be physically important.

Let us begin with the case of $a_{\rm rh} > a_m$, which corresponds to

$$H_e \gg m > H_{\rm rh} \sim \frac{\sqrt{g_*}}{3}\frac{T_{\rm rh}^2}{M_p} = \left(\frac{T_{\rm rh}}{10^9\,{\rm GeV}}\right)^2 \left(\frac{g_*}{100}\right)^{1/2} {\rm GeV}, \tag{5.28}$$

where $H_{\rm rh}$ is the expansion rate at the time radiation domination is achieved. In this case, we have

$$\rho_\Psi(t_{\rm eq}) \sim 0.03 m^4 \left(\frac{H_{\rm rh}}{m}\right)^2 \left(\frac{a_{\rm rh}}{a_{\rm eq}}\right)^3, \tag{5.29}$$

in which we have used the fact that $H \propto a^{-3/2}$ during reheating. We thus find the relic abundance today of fermionic particles to be

$$\Omega_\psi h^2 \sim 3\left(\frac{m}{10^{11}\,{\rm GeV}}\right)^2 \left(\frac{T_{\rm rh}}{10^9\,{\rm GeV}}\right). \tag{5.30}$$

This matches Eq. (44) of [74] (up to a factor of order of a few, part of which is expected from counting fermionic degrees of freedom), which was computed in the context of conformally coupled scalar fields. The match is interesting because the analog of Eq. (A.57) for the conformally coupled scalar field case has a different k/a dependence that converts into an effective m dependence due to the conformal invariance of the fermionic kinetic term. Eq. (5.30) also agrees with the model dependent numerical results of [55] up to a factor of 10. The related ratio of the fermion energy density to the radiation energy density at matter radiation equality, $\rho_\Psi(t_{\rm eq})/\rho_R(t_{\rm eq})$, is the same as Eq. (5.30) up to a factor of 10.

For the case with $a_{\rm rh} < a_m$, we have

$$\rho_\Psi(t_{\rm eq}) \quad \sim \quad 0.03 m^4 \left(\frac{a_{\rm m}}{a_{\rm eq}}\right)^3, \tag{5.31}$$

which leads to

$$\frac{\rho_\Psi(t_{\rm eq})}{\rho_R(t_{\rm eq})} \sim \left(\frac{m}{10^8\,{\rm GeV}}\right)^{5/2} \left(\frac{g_*(t_m)}{100}\right)^{-1/4} \tag{5.32}$$

which up to an order of magnitude is Ω_ψ. However, since this applies only for

$$m < \left(\frac{T_{\rm rh}}{10^9\,{\rm GeV}}\right)^2 \left(\frac{g_*}{100}\right)^{1/2} {\rm GeV}, \tag{5.33}$$

the relic abundance is negligible in this case. For example, a $m \sim 1\,{\rm GeV}$ benchmark point will render $\Omega_\Psi \sim 10^{-20}$.

Chapter 6

Isocurvature 2-Point Correlator

In Chapter 5, we have seen massive fermion particles can also be produced gravitationally during inflation and serve as dark matter candidates. The produced fermions will not necessarily be homogeneous. We want to know how much density perturbation will be produced in this way. Since fermion is not coupled to inflaton directly, this density perturbation will be of the isocurvature type.

The phenomenology of multifield inflationary models has been extensively studied (see e.g. [95]). Our models differs from the usual curvaton model in that fermion does not come from the curvaton decay, rather the light scalar field's inhomogeneous VEV modulate the fermion's non-perturbative production.

Through the study of a Yukawa theory, we have found the following results

- Without Yukawa interaction, the correlation between number density perturbation is undetectable at CMB scale.
- With Yukawa interaction and a light scalar field, the scalar field's perturbation will modulate the fermion production via changing the fermion effective mass.

These result can be understood intuitively as this. At leading order, the scaling dimension of ψ dictate its Green's function decays as $a(t)^{-3}$ at late time and large separation during inflation. In the Yukawa model, the fermions density will preserves the information of the scalar field's fluctuation at the ending time ($m_\psi = H(\eta_*)$), just as the CMB photons captures the density fluctuation of the recombination time.

This Chapter is organized as follows. In Section 6.1, we setup the formalism for the computation. In Section 6.2, we present the leading order result, i.e. diagram (a) in Fig (6.1). In Section 6.3, we present the next leading order result. Finally, in Section 6.5, we discuss the fermion loop correction to the scalar propagator during the de Sitter era. In Appendix C.1, we give the late time large separation limit for the fermion correlator. In Appendix C.2, we demonstrate how to find the dominant Feynman diagram by comparing the scaling behavior of the propagators. In Appendix C.3, we use Bogoliubov transformation to compute the particle production contribution to a loop diagram.

6.1 Setup

In this section, we first define a quantum operator that corresponds to the isocurvature variable in the classical cosmological perturbation theory. Then we introduce the relevant counter-terms to renormalize this composite operator. Finally, we expand its two-point function using the in-in formalism.

The full action is given by Eq. (2.1). In this section, we first approximate by ignore the inflaton sector and ignore the gravitational interactions. The fermion field ψ and scalar field σ are evolving on a quasi de Sitter background. This is assuming that during inflation, ψ and σ are energetically unimportant. The effect of the inflaton fluctuation and the gravitational coupling is considered in the next Chapter.

In classical cosmological perturbation theory, one can define the following gauge-invariant object (see, e.g. Ch 5 of [96])

$$\zeta_O(\vec{q},t) = \frac{A(\vec{q},t)}{2} - H\frac{\delta O(\vec{q},t)}{\dot{\bar{O}}} \tag{6.1}$$

where the scalar metric perturbation is parameterized as

$$\delta g^{(S)}_{\mu\nu} = \begin{pmatrix} -E & aF_{,i} \\ aF_{,i} & a^2[A\delta_{ij} + B_{,ij}] \end{pmatrix} \tag{6.2}$$

and $O = \bar{O} + \delta O$ is a diffeomorphism scalar field. Then under a gauge transformation parametrized by $\epsilon^\mu = (\epsilon^0, a^{-2}\partial_i(\epsilon^S))$, we have

$$\Delta A = -2H\epsilon^0, \Delta B = -\frac{2}{a^2}\epsilon^S \tag{6.3}$$

$$\Delta E = -2\dot{\epsilon}^0, \Delta F = \frac{1}{a}(\epsilon^0 - \dot{\epsilon}^S + 2H\epsilon^S). \tag{6.4}$$

$$\Delta\delta O = -\epsilon^0\dot{\bar{O}} \tag{6.5}$$

which can be obtained from $\Delta(\delta g^{(S)}_{\mu\nu}) = -\mathcal{L}_{\epsilon^\mu \partial_\mu} \bar{g}_{\mu\nu}$. Thus, one can easily check

$$\Delta\zeta_O(\vec{q}, t) = 0 \tag{6.6}$$

Let ρ_ψ represent the energy density of the fermion field, with $\rho_\psi = \bar{\rho}_\psi + \delta\rho_\psi$ separation. Then we can define

$$\begin{aligned} \delta_S &= 3(\zeta_\psi - \zeta_\phi) && (6.7)\\ \zeta_\psi &\equiv \frac{A}{2} - \frac{H}{\dot{\bar{\rho}}_\psi}\delta\rho_\psi && (6.8)\\ \zeta_\phi &\equiv \frac{A}{2} - \frac{H}{\dot{\bar{\phi}}}\delta\phi && (6.9) \end{aligned}$$

Since ζ_ψ and ζ_ϕ are first order gauge invariant, δ_S is first order gauge invariant. In comoving gauge where $\delta\phi = 0$, we have

$$\delta_S = -3\frac{H}{\dot{\bar{\rho}}_\psi}\delta\rho_\psi \tag{6.10}$$

To promote δ_S ,we need to define ρ_ψ as an operator and $\bar{\rho}_\psi(t)$ as a c-number function of time, then $\delta\rho_\psi = \rho_\psi - \bar{\rho}_\psi$ will be naturally defined. From the physics intuition that ψ particles will be produced gravitationally during inflation and behaves as cold dark matter (CDM) after $m > H(t)$, we can expect that at late time $\langle\rho_\psi\rangle \gg \langle p_\psi\rangle$. Thus we make the following approximation,

$$\rho_\psi \approx \rho_\psi + 3p_\psi = T^\mu_\mu = m\bar{\psi}\psi \tag{6.11}$$

where we used the on-shell fermion stress tensor (in vierbein indices)

$$T_{ab} = \frac{i}{2}\bar{\psi}(\gamma_{(a}\nabla_{e_b)} - \gamma_{(a}\overleftarrow{\nabla}_{e_b)})\psi \tag{6.12}$$

and the EOM for ψ. The function $\bar{\rho}_\psi(t)$ can be defined as

$$\bar{\rho}_\psi(t) = \langle in|\rho_\psi|in\rangle \approx m\langle in|\bar{\psi}\psi|in\rangle \tag{6.13}$$

We shall further use the approximation that

$$\frac{d}{dt}\bar{\rho}_\psi(t) = -3H(\bar{\rho}_\psi + \bar{p}_\psi) \approx -3H\bar{\rho}_\psi \tag{6.14}$$

Plug in ρ and $\bar{\rho}$ into Eq. (6.10), we get $\delta_S \approx \frac{\bar{\psi}\psi - \langle\bar{\psi}\psi\rangle}{\langle\bar{\psi}\psi\rangle}$. The subtraction of $\langle\bar{\psi}\psi\rangle$ in the numerator removes the self-contraction diagrams, thus one get only connected diagrams about $\bar{\psi}\psi$. In practice, if we compute the spatial Fourier transform of correlators with δ_S insertion, the $\langle\bar{\psi}\psi\rangle$ in the numerator does not contribute at non-zero momentum. Thus, we may take

$$\delta_S \approx \frac{\bar{\psi}\psi}{\langle\bar{\psi}\psi\rangle} \tag{6.15}$$

in the following computation.

Since $\bar{\psi}\psi$ is a composite operator, we need to renormalize it by mixing it with operators of dimension 3 and lower. The renormalized field operator $(\bar{\psi}\psi)_{x,r}$ is

$$\begin{aligned}(\bar{\psi}\psi)_{x,r} &= (\bar{\psi}_x)_r(\psi_x)_r(1+\delta Z_1) + \delta Z_2(\sigma_{x,r})^3 + \delta Z_3(\sigma_{x,r})^2 \\ &\quad +\delta Z_4\sigma_{x,r} + \delta Z_5 + \delta Z_6\Box\sigma_{x,r} + \delta Z_7 R + \delta Z_8 R\sigma_{x,r} \end{aligned} \tag{6.16}$$

Thus we may define the renormalized partition function,[1]

$$Z_r[K_r] = \int D\sigma D\psi D\bar{\psi} \exp\left\{\frac{i}{\hbar}\int_{CTP}(dx)\left(\mathcal{L}_r + \mathcal{L}_{c.t.} + K_{x,r}(\bar{\psi}\psi)_{x,r}\right)\right\} \tag{6.17}$$

and the renormalized generating functional $W_r[K_r]$

$$\exp[\frac{i}{\hbar}W_r[K_r]] = Z_r[K_r]. \tag{6.18}$$

Our goal in the remaining part of this section is to evaluate

$$\langle(\bar{\psi}\psi)_{x,r}(\bar{\psi}\psi)_{y,r}\rangle_c = \left(\frac{\delta}{\frac{i}{\hbar}\sqrt{g_x}\delta K_{x,r}}\frac{\delta}{\frac{i}{\hbar}\sqrt{g_y}\delta K_{y,r}}\frac{i}{\hbar}W_r[K_r]\right)\Bigg|_{K_r=0} \tag{6.19}$$

for $x^0 = y^0 = t_x$ at a sufficiently late time when particle production has ended. The spatial separation $|\vec{x}-\vec{y}|$ is large enough such that their common past history $I^-(x)\cap I^-(y)$ lives deep within the inflationary era.

The relevant diagrams are given in Fig. (6.1), where diagram (a) is the LO contribution which will be considered in Section 6.2, and diagrams (b-e) will be considered in Section 6.3.

6.2 Leading Order Result

In this section, we evaluate the diagram (a) in fig. (6.1).

$$\langle n_{\psi,x}n_{\psi,y}\rangle_{(a)} = -\mathrm{Tr}\left[\langle\psi_x\bar{\psi}_y\rangle\langle\psi_y\bar{\psi}_x\rangle\right] = \sum_{i,j}\bar{V}_{i,x}U_{j,x}\bar{U}_{j,y}V_{i,y} \tag{6.20}$$

Using contour integral, we can evaluate the mode-sum analytically. The details are in given Appendix B. The result is

$$\langle n_{\psi,x}n_{\psi,y}\rangle_{(a)} = \begin{cases}\frac{1}{\pi^4 a_x^6|\vec{x}-\vec{y}|^6}\left(1+O\left[\left(\frac{m}{H}\right)^2\right]\right) & (m \ll H_{inf}) \\ \frac{1}{\pi^4 a_x^6|\vec{x}-\vec{y}|^6}(4\pi)\left(\frac{m}{H}\right)^3\exp(-2\pi\frac{m}{H}) & (m \gg H_{inf})\end{cases} \tag{6.21}$$

[1] In principle, there should be local terms involving more $K_{x,r}$ in the action, such as $\delta Z\, K_{x,r}^2$ or $\partial K\cdot\partial K$. However we are only interested in $\frac{\delta}{\delta K_x}\frac{\delta}{\delta K_y}W$ $(x\neq y)$, and these terms do not contribute. Thus we can omit them in the action. We note this is only true if we are working in the coordinate space Feynman diagram, as soon as we do Fourier transformation, we shall encounter the $x=y$ case, and $O(K^2)$ terms are needed.

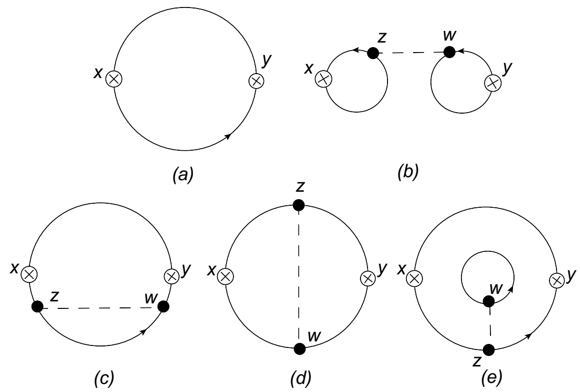

Figure 6.1: The LO and NLO diagrams for $\langle (\bar{\psi}\psi)_{x,r}(\bar{\psi}\psi)_{y,r}\rangle_c$, where the cross-dot vertices is $\bar{\psi}\psi$ insertion.

We can understand this result by backtracking the two points x, y to the time when they were deep inside the horizon, and see what happened when they grow apart.

1. In the heavy mass case ($m \gg H_{inf}$), the Compton radius m^{-1} is smaller than the horizon radius H^{-1}. The physical separation r_{phys} will first grow to the Compton wavelength, and trigger the exponential suppression factor ($\exp(-2mr_{phys})$) in the correlator.

$$\langle \bar{\psi}\psi_x\bar{\psi}\psi_y\rangle_{flat,r_{phys}<m} \quad \sim \quad \frac{m^3}{4\pi^3 r_{phys}^3}\exp(-2mr_{phys}) \tag{6.22}$$

   As the physical separation r_{phys} grows further to exceed the horizon radius H^{-1}, the correlator would freeze and start falling as $(a_r/a_\eta)^6$, where $a_r = 1/(Hr)$ denote the scale factor at the horizon crossing. Then we recover the heavy mass formula, with $a_r = \frac{1}{Hr}, r_{phys} = H^{-1}$

$$\left(\frac{a_r}{a_\eta}\right)^6 \frac{m^3}{4\pi^3 r_{phys}^3}\exp(-2mr_{phys}) \sim \frac{1}{a_x^6 r^6}\left(\frac{m}{H}\right)^3 \exp(-2\frac{m}{H}) \tag{6.23}$$

   Thus we recover the heavy mass result.

2. In the light mass case ($m \ll H_{inf}$), the physical distance will cross the horizon radius first, without the exponential suppression of $\exp(-2mr_{phys})$. From the flat space UV limit result $\frac{1}{r_{phys}^6}$,

$$\langle \bar{\psi}\psi_x\bar{\psi}\psi_y\rangle_{flat,r_{phys}<m} \sim \frac{1}{r_{phys}^6} \tag{6.24}$$

   we get ($a_r = \frac{1}{Hr}, r_{phys} = H^{-1}$)

$$\left(\frac{a_r}{a_\eta}\right)^6 \frac{1}{r_{phys}^6} \sim \frac{1}{a_x^6 r^6} \tag{6.25}$$

   Thus we recover the light mass result.

Unfortunately, the fractional relic density fluctuation at CMB scale[2] is too small

$$\frac{\langle \delta\rho_x \delta\rho_y \rangle}{\langle \rho_\psi \rangle^2} \sim \frac{m^2/(\pi^4 a^6 r_{CMB}^6)}{m^2 m^6 (a_*^6/a^6)} \sim \left(\frac{1}{a_* m r_{CMB}}\right)^6. \tag{6.26}$$

where r_{CMB} is the comoving distance for typical CMB observation scale and the subscript $*$ denotes the time when fermion production ends. Let a_{CMB} denotes the scale factor when CMB scale exits the horizon then we have

$$r_{CMB}^{-1} \sim a_{CMB} H_{inf} \tag{6.27}$$

Assuming the fermion production ends during reheating when $m_\psi = H(t)$, and $H \propto a^{-\alpha}$ during reheating, then we have

$$\frac{a_e H_{inf}}{a_* m_\psi} \sim \frac{a_e H_e}{a_* H_*} \sim \left(\frac{a_e}{a_*}\right)^{1-\alpha} \sim \left(\frac{H_e}{H_*}\right)^{1-\frac{1}{\alpha}} \tag{6.28}$$

Assuming that inflation ends 60 efolds after the CMB scale exits horizon and MD-like reheating i.e. $\alpha = 3/2$, then we have

$$\frac{\langle \delta\rho_x \delta\rho_y \rangle}{\langle \rho_\psi \rangle^2} \sim \left(\frac{a_{CMB} H_{inf}}{a_* m_\psi}\right)^6 \sim \left(\frac{a_{CMB}}{a_e} \frac{a_e H_{inf}}{a_* m_\psi}\right)^6 \sim e^{-300} \left(\frac{H_e}{m_\psi}\right)^2 \tag{6.29}$$

Using the fermion relic abundance formula $\Omega_\psi \sim (m_\psi/10^{10}\text{GeV})^2$, for example, we take $T_{RH} = 10^9\text{GeV}$ and $g_* = 100$, then we get

$$\frac{\langle \delta\rho_x \delta\rho_y \rangle}{\rho_{tot}^2} \sim \Omega_\psi^2 \frac{\langle \delta\rho_x \delta\rho_y \rangle}{\langle \rho_\psi \rangle^2} \sim e^{-300} \left(\frac{H_e}{10^{10}\text{GeV}}\right)^2 \tag{6.30}$$

We see that generically the pure fermion isocurvature perturbations are small on CMB scale.

6.3 Next Leading Order Result

We consider the diagrams (b)-(e) in fig.(6.1). They represent the effect of the Yukawa interaction to the fermion production. Here we only consider the case of light fermion $m_\psi < H_{inf}$.

First, we want to estimate which diagram gives the largest contribution when x, y have large spatial separation. From Appendix C.1 that equal-time correlator $\langle \sigma_x \sigma_y \rangle$ scales as $r^{2\nu-3}$ and $\langle \psi_x \bar{\psi}_y \rangle$ scales as r^{-3}, we can expect that diagrams that has fewer fermion lies stretched between x, y decreases slower when $r \to \infty$. Thus we shall take diagram (b) as the dominating diagrams.

[2] Since the $\langle \delta\rho\delta\rho \rangle$ are frozen as long as the two points are outside of horizon, we can extrapolate this large spatial separation result obtained at the end of inflation to the recombination time.

For diagram (b), we have can expand it using commutators

$$
\begin{aligned}
I_b(x,y) &= \langle(\bar\psi\psi)_{x,r}(\bar\psi\psi)_{y,r}\rangle_{c,diag(b)} && (6.31)\\
&= 4(i\lambda)^2\int^x(dz)\int^y(dw)\langle\bar\psi\psi_{[x}\bar\psi\psi_{z]}\rangle\langle\bar\psi\psi_{[y}\bar\psi\psi_{w]}\rangle\langle\sigma_{\{z}\sigma_{w\}}\rangle && (6.32)\\
&\quad+4(i\lambda)^2\int^x(dz)\int^y(dw)\langle\bar\psi\psi_{\{x}\bar\psi\psi_{z\}}\rangle\langle\bar\psi\psi_{[y}\bar\psi\psi_{w]}\rangle\langle\sigma_{[w}\sigma_{z]}\rangle\Theta(w^0-z^0) && (6.33)\\
&\quad+4(i\lambda)^2\int^x(dz)\int^y(dw)\langle\bar\psi\psi_{[x}\bar\psi\psi_{z]}\rangle\langle\bar\psi\psi_{\{y}\bar\psi\psi_{w\}}\rangle\langle\sigma_{[z}\sigma_{w]}\rangle\Theta(z^0-w^0) && (6.34)\\
&\approx (i\lambda)^2\int^x(dz)\int^y(dw)\langle[\bar\psi\psi_x,\bar\psi\psi_z]\rangle\langle[\bar\psi\psi_y,\bar\psi\psi_w]\rangle\langle\sigma_{\{z}\sigma_{w\}}\rangle && (6.35)
\end{aligned}
$$

where $[\cdots]$ means anti-symmetrization and $\{\cdots\}$ means symmetrization. As derived in Appendix C.2, $\langle[\sigma_{x_1},\sigma_{x_2}]\rangle$ is suppressed by $a^{-2\nu}$ relative to $\langle\{\sigma_{x_1},\sigma_{x_2}\}\rangle$, whereas $\langle[\bar\psi\psi_{x_1},\bar\psi\psi_{x_2}]\rangle$ is suppressed by a^{-1} relative to $\langle\{\bar\psi\psi_{x_1},\bar\psi\psi_{x_2}\}\rangle$. The last line is obtained by keeping only the dominating contribution.

Since the fermion particle production ends at t_* and the previously produced particles are mostly diluted away, we expect the z,w integrals to peak around the time t_*. For late time and large spatial separation, the scalar correlator $\langle\sigma_{\{z}\sigma_{w\}}\rangle$ will be slowly varying. Thus we may approximately take $\langle\sigma_{\{z}\sigma_{w\}}\rangle=\langle\sigma_{\{z_0}\sigma_{w_0\}}\rangle$, where $z_0=(\vec{x},t_*)$ and $w_0=(\vec{y},t_*)$, and factor it outside of the z,w integral.

The remaining fermion integral can be evaluated using the following identity

$$
-i\int^x(dw)\langle[\bar\psi\psi_x,\bar\psi\psi_z]\rangle=\partial_m\langle\bar\psi\psi_x\rangle=\partial_m n_\psi(x) \qquad (6.36)
$$

An explicit check of this integral using Bogoliubov coefficients is given in Appendix C.3. Thus we get

$$
\langle n_x n_y\rangle_{NLO}\approx I_b(x,y)\approx\lambda^2[\partial_m n_\psi(x)][\partial_m n_\psi(y)]\langle\sigma_{\{z_0}\sigma_{w_0\}}\rangle \qquad (6.37)
$$

We can understand that the number density correlator $\langle n_x n_y\rangle$ is generated from the slowly varying scalar corrector $\langle\sigma_{\{z}\sigma_{w\}}\rangle$ via the correction of the fermion mass.

Therefore, we can generalize this result to

$$
\begin{aligned}
\langle n_x n_y\rangle_{NLO} &\approx \langle n_\psi(m_\psi+\lambda\sigma(x))n_\psi(m_\psi+\lambda\sigma(y))\rangle_\sigma\\
&\approx \lambda^2\left(\frac{\partial}{\partial m}n_\psi\right)^2\langle\sigma_{\{(\vec{x},t_*)}\sigma_{(\vec{y},t_*)\}}\rangle\\
&\quad+\frac{1}{4}\lambda^4\left(\frac{\partial^2}{\partial m^2}n_\psi\right)^2\langle\sigma^2_{\{(\vec{x},t_*)}\sigma^2_{(\vec{y},t_*)\}}\rangle+O(\lambda^6) \qquad (6.38)
\end{aligned}
$$

Here note that we expanded it using the perturbative parameter λ assuming this expansion perturbatively under control. However, the assumption may breaks down and the expansion does not hold when the other factors overtake the λ suppression. i.e., $\lambda H_{inf}/m \ll 1$. We shall mainly focus on the parameter region where the perturbation holds in this paper and leave the other region for the future investigation although the region may have interesting features such as large non-Gaussianities.

6.4 Isocurvature Power Spectrum

In the long wavelength limit, the temperature fluctuation is [97]

$$\frac{\Delta T}{T} = -\frac{1}{5}\zeta - \frac{2}{5}S \tag{6.39}$$

where

$$S = \frac{\delta\rho_\psi}{\rho_{DM}} = \frac{\Omega_\psi}{\Omega_{DM}}\delta_S \equiv \omega_\psi \delta_S. \tag{6.40}$$

and $\omega_\psi = \Omega_\psi/\Omega_{DM}$ indicate the relative abundance of the fermion dark matter compared with the total dark matter.

If we define the power spectrum of observable O as

$$\mathcal{P}_O(k) \equiv \frac{k^3}{2\pi^2}\int d^3x \, \langle O_{\vec{0}} O_{\vec{x}} \rangle \, e^{-i\vec{k}\cdot\vec{x}}, \tag{6.41}$$

then we find the power spectrum of the temperature fluctuation

$$\mathcal{P}_{\frac{\Delta T}{T}} = \frac{1}{25}\mathcal{P}_\zeta + \frac{4}{25}\mathcal{P}_S, \tag{6.42}$$

where the cross-correlation term has been dropped. From Eq. (6.38), we have

$$\mathcal{P}_S(k) = \omega_\psi^2 \lambda^2 \left(\frac{\partial_m n_\psi(m_\psi)}{n_\psi}\right)^2 \mathcal{P}_\sigma(k) + O(\lambda^4). \tag{6.43}$$

The power spectrum for a light scalar field can be obtained as

$$\mathcal{P}_\sigma(k) = \frac{H^2(t_k)}{4\pi^2}\left(\frac{k}{a(t_m)H(t_k)}\right)^{-(2/3)(m_\sigma^2/H^2)} \tag{6.44}$$

where t_k is the time when k exits horizon. The slow decay of the σ modes suppresses the modes that exit horizon early, therefore the isocurvature power spectrum is slightly blue-tilted relative to ζ power spectrum. In the region $m_\sigma^2/H^2 \ll 1/60$, we can ignore the slow decay factor in P_σ and get

$$\mathcal{P}_S(k) \approx \omega_\psi^2 \lambda^2 \left(\frac{\partial_m n_\psi(m_\psi)}{n_\psi}\right)^2 \frac{H^2(t_k)}{4\pi^2} \tag{6.45}$$

For example, if we consider MD-like reheating, then from Eq. (5.19) and Eq. (5.30), we get

$$\alpha_S \equiv \frac{\mathcal{P}_S}{\mathcal{P}_\zeta + \mathcal{P}_S} \sim \lambda^2 \left(\frac{m_\psi}{10^3\text{GeV}}\right)^2 \left(\frac{T_{RH}}{10^9\text{GeV}}\right)^2 \left(\frac{H}{10^{13}\text{GeV}}\right)^2 \tag{6.46}$$

The current observational bound[98, 99, 100, 101] of the isocurvature for the uncorrelated case, i.e. $\langle \zeta S \rangle = 0$, is $\alpha_S < 0.07$, which can be used to constrain λ and m_ψ.

6.5 Corrections to Scalar Field Propagator

In this section, we want to consider the effect of the produced fermions on the σ correlator. In general, σ field would acquire a plasma mass when interacting with an ensemble of particles. Here we want to check that whether this effect is so large that we can no longer treat the scalar field as a light field during inflation. Here, to be consistent with our renormalization condition, we will use the WKB mode function to define fermion particles during inflation, i.e. the vacuum choice is the adiabatic vacuum.

The fermion number density is about m_ψ^3 before particle production ends, since the Bogoliubov coefficient $|\beta_{k,\eta}|^2$ 1/2 for $k/a < m_\psi$. Thus the number of fermions per horizon is about m_ψ^3/H^3, which is much less than one. We expect the fermions to have no effect on the scalar mode when it is sub-horizon. After the scalar mode exits horizon, the fermions exert a tiny drag on σ, and shift the zero-point of σ's potential. Apart from this effect, we see the equation of motion is also modified (see [29]),

$$0 \quad = \quad (\Box - m_\sigma^2)\sigma_x - \lambda\langle\bar{\psi}\psi_x\rangle + Y + i\lambda^2 \int^x (dz)\langle[\bar{\psi}\psi_x, \bar{\psi}\psi_z]\rangle\sigma_z + O(\lambda^4) \tag{6.47}$$

Thus, we redefine $\sigma_x = \tilde{\sigma}_x + \frac{-\lambda\langle\bar{\psi}\psi_x\rangle + Y}{m_\sigma^2}$, where $Y \sim O(\lambda)$ absorbs the divergent part in $\lambda\langle\bar{\psi}\psi_x\rangle$. During inflation, we expect $\langle\bar{\psi}\psi_x\rangle$ to be spacetime independent, thus $\Box\langle\bar{\psi}\psi_x\rangle = 0$. Thus we have,

$$\begin{aligned} 0 \quad &= \quad (\Box - m_\sigma^2)\tilde{\sigma}_x + i\lambda^2 \int^x (dz)\langle[\bar{\psi}\psi_x, \bar{\psi}\psi_z]\rangle\tilde{\sigma}_z + \delta m_\sigma^2(\Lambda, m_\psi)\sigma_x + O(\lambda^3) && (6.48) \\ &\approx \quad (\Box - m_\sigma^2)\tilde{\sigma}_x - \lambda^2(\frac{\partial n_\psi}{\partial m_\psi})\tilde{\sigma}_x + O(\lambda^3) + c.t. && (6.49) \\ &\approx \quad (\Box - [m_\sigma^2 + \lambda^2\frac{\partial n_\psi}{\partial m_\psi}])\tilde{\sigma}_x + O(\lambda^3) + c.t. && (6.50) \end{aligned}$$

where in the derivation we used the slow modes approximation for $\tilde{\sigma}_z$, which is valid as long as $k_\sigma \ll a_x m_\psi$. The divergent part in $i\lambda^2 \int^x (dz)\langle[\bar{\psi}\psi_x, \bar{\psi}\psi_z]\rangle\tilde{\sigma}_z$ is canceled by the counter-term $\delta m_\sigma^2(\Lambda, m_\psi)\sigma_x$.

If we estimate $n_\psi = C_1 m_\psi^3$ during inflation, then the mass correction would be $\lambda^2 \frac{\partial n_\psi}{\partial m_\psi} \sim C_1 \lambda^2 m_\psi^2$. Thus, as long as

$$C_1 \lambda^2 m_\psi^2 + m_\sigma^2 \ll H^2 \tag{6.51}$$

we can approximately take σ field to be light during inflation.

Chapter 7

Isocurvature Cross Correlator

In the last chapter, we computed the two point function of the isocurvature operator. This encodes the strength of the energy density fluctuation of the fermion field at various scales at the initial time. In this chapter, we compute the cross correlation of the isocurvature operator and curvature operator. $\langle \delta_S \zeta \rangle$ This encodes how aligned these two types of fluctuations are at the initial time. Since both types of perturbations will lead to temperature fluctuation $\frac{\Delta T}{T}$ of the CMB photon, it is necessary to know how they interfere with each other.

To extract the particle production's contribution to the loop diagram, we used the Bogoliubov subtraction prescription. We find that the cross correlator is small. The intuition is as follows

- The ζ field enjoys a shift symmetry. Recall that ζ reflects the local fluctuation of the scale factor $a(t)$, thus a spacetime constant shift of ζ is equivalent to multiply the scale factor by a constant. However such a change does not affect the background expansion history, which is encoded in $H(t) = d \ln a/dt$. Since any physical observable should be independent of the normalization of $a(t)$, a constant shift of ζ has no effect on the physical observables.

- Fermions are produced continously during inflation and early era of the reheating. Due to the dilution of the early produced quanta, the majority of the remaining fermions are produced latest. Around this production time, let's consider the ζ fluctuation with a comoving wavenumber $p_{\rm CMB}$ that corresponds to the CMB observation scale. Such a fluctuation is nearly constant in space and time, with fractional variation on the order of $\left(\frac{p}{aH}\right)^2 \ e^{-60\times 2}$. This nearly constant ζ will have little effect on the fermions production at late time.

The argument for the cross correlator only depends on the $a^{-3/2}$ scaling behavior of the fermion field mode function. It also applies for heavy scalar field with $m > H_{inf}$. Hence, in the following discussion we do not use the explicit form of the fermion stress tensor $T_{\mu\nu}$. However, we do give explicit check for fermion case in the Appendix.

This chapter is organized as follows. In Section 7.1, we give formal formula for the cross correlator. In Section 7.2, we show that the cross correlator is suppressed in the long wavelength limit due to the derivative coupling of ζ and matter. In Section 7.3, we show that computation in comoving gauge and uniform curvature gauge are equivalent.

7.1 Formal Expression of Cross Correlator

First, we recall the gauge invariant definition for isocurvature curvature and curvature perturbation in Eq. (6.7),

$$\delta_S = 3(\zeta_\psi - \zeta), \quad \zeta_\psi = \frac{A}{2} - \frac{H}{\dot{O}}\delta O \tag{7.1}$$

where we used operator O for the fermion energy density operator and to keep the discussion general. Also we recall the general form of matter-gravity interaction (see e.g. Eq. (2.32, 2.36))

$$S_{int} = \int (dx)\frac{1}{2}T^{\mu\nu}(x)\delta g_{\mu\nu}(x;\zeta) \tag{7.2}$$

the in-in cross correlator to the 1-loop order is given by

$$\langle \delta_S^+(x)\zeta^+(y)\rangle_{1-loop} = i\int_M (dz)\langle \delta_S^+(x)\zeta_\phi^+(y)\frac{1}{2}T^{\mu\nu+}(z;\psi)\delta g_{\mu\nu}^+(z;\zeta)\rangle \tag{7.3}$$

$$-i\int_M (dz)\langle \delta_S^+(x)\zeta_\phi^+(y)\frac{1}{2}T^{\mu\nu-}(z;\psi)\delta g_{\mu\nu}^-(z;\zeta)\rangle \tag{7.4}$$

In the inflationary background, ζ can be quantized as

$$\zeta(\vec{x},t) = \int d^3k[a_{\vec{k}}\frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{\frac{3}{2}}}u_k^{(\zeta)}(t) + a_{\vec{k}}^\dagger\frac{e^{-i\vec{k}\cdot\vec{x}}}{(2\pi)^{\frac{3}{2}}}u_k^{(\zeta)*}(t)] \tag{7.5}$$

$$u_k^{(\zeta)}(t) \approx \frac{1}{\sqrt{4\epsilon(t_k)}}\frac{H(t_k)}{k^{\frac{3}{2}}}e^{i\frac{k}{aH}}(1 - i\frac{k}{aH}) \tag{7.6}$$

where t_k is the time when mode k exit the horizon. In the limit that k is outside of horizon $k/aH \ll 1$, we have the following expansion

$$u_k^{(\zeta)}(t) = u_k^{(\zeta),o}[1 + \frac{1}{2}\left(\frac{k}{aH}\right)^2 + \frac{i}{3}\left(\frac{k}{aH}\right)^3 + \cdots] \tag{7.7}$$

$$u_k^{(\zeta),o} = \frac{1}{\sqrt{4\epsilon(t_k)}}\frac{H(t_k)}{k^{\frac{3}{2}}} \tag{7.8}$$

where $u_k^{(\zeta),o}$ represent the asymptotic value for the mode function when it is outside of horizon.

To proceed further, we choose a gauge. It is not obvious that computation in uniform curvature gauge would give the same result as in comoving gauge, due ot the difference in the form of δ_S and in the interaction action. However, we show that this is indeed the case in Section 7.3. Thus, without loss of generality, we use comoving gauge.

In comoving gauge, the Fourier transform of cross correlator (same Fourier convention as in Chapter 6):

$$
\begin{aligned}
& \langle \delta_S^{(C)}(\vec{k},t)\zeta(\vec{p},t)\rangle_{1-loop} \\
= & -3\frac{H}{\dot{\bar{O}}} i \int^{t_f} (dz) \langle O^+(\vec{k},t)\zeta^+(\vec{p},t)(\mathcal{L}_{int}^{(C)+} - \mathcal{L}_{int}^{(C)-})\rangle \\
= & -3\frac{H}{\dot{\bar{O}}} i \int^{t} (dz) \left\{ \langle \frac{1}{2}\{\zeta(\vec{p},t), \delta g_{\mu\nu}^{(C)}(z;\zeta)\}\rangle \langle [O(\vec{k},t), \frac{1}{2}T^{\mu\nu}(z;\sigma)]\rangle \right. \\
& \left. + \langle \frac{1}{2}\{O(\vec{k},t), \frac{1}{2}T^{\mu\nu}(z;\sigma)\}\rangle \langle [\zeta(\vec{p},t), \delta g_{\mu\nu}^{(C)}(z;\zeta)]\rangle \right\} \qquad (7.9) \\
= & (2\pi)^3 \delta^3(\vec{k}+\vec{p})[I_{\zeta\to\delta_S}(k) + I_{\delta_S\to\zeta}(k)] \qquad (7.10)
\end{aligned}
$$

The first term contains an anti-commutator on $\zeta\zeta$ internal line, with commutator on the matter loop. This represents the effect of ζ's vacuum fluctuation on isocurvature. The second term contains an commutator on $\zeta\zeta$ internal line, with anti-commutator on the matter loop. This represents the effect of isocurvature's vacuum fluctuation on curvature.

7.2 Suppression from Derivative Coupling

To estimate the cross correlator, we first introduce the long-wavelength approximation for ζ. From the fermion relic abundance calculation, we know that the majority of fermions are produced at late time after inflation ends. Around that time, the wavelength for ζ mode is far outside of horizon. The physical length scale of ζ perturbation $\left(\frac{p}{a(t_x)}\right)^{-1}$ is much longer than the gravitational particle production length scale $H(t_x)^{-1}$, thus we may approximately take the long-wavelength approximation for ζ:

$$
\begin{aligned}
\langle \{\zeta(\vec{p},t), \zeta_z\}\rangle &\approx e^{-i\vec{p}\cdot\vec{z}} |u_p^{\zeta,o}|^2\, 2 \qquad (7.11) \\
\langle \{\zeta(\vec{p},t), (-\frac{\zeta}{H} + \epsilon \frac{a^2}{\nabla^2}\dot{\zeta})_{,i}\}\rangle &\approx e^{-i\vec{p}\cdot\vec{z}} |u_p^{\zeta,o}|^2\, 2(-ip_i)(\frac{-1+2\epsilon}{H})_z \qquad (7.12) \\
\langle \{\zeta(\vec{p},t), \frac{\dot{\zeta}_z}{H_z}\}\rangle &\approx e^{-i\vec{p}\cdot\vec{z}} |u_p^{\zeta,o}|^2\, (-4) \left(\frac{p}{aH}\right)^2 \qquad (7.13)
\end{aligned}
$$

and

$$\langle[\zeta(\vec{p},t),\zeta_z]\rangle \approx e^{-i\vec{p}\cdot\vec{z}}|u_p^{\zeta,o}|^2\,\frac{2i}{3}\left(\frac{p}{aH}\right)^3 \tag{7.14}$$

$$\langle[\zeta(\vec{p},t),(-\frac{\zeta}{H}+\epsilon\frac{a^2}{\nabla^2}\dot{\zeta})_{,i}]\rangle \approx e^{-i\vec{p}\cdot\vec{z}}|u_p^{\zeta,o}|^2\left(\frac{p}{aH}\right)^2 a\hat{p}_i\epsilon \tag{7.15}$$

$$\langle[\zeta(\vec{p},t),\frac{\dot{\zeta}_z}{H_z}]\rangle \approx e^{-i\vec{p}\cdot\vec{z}}|u_p^{\zeta,o}|^2\,(-2i)\left(\frac{p}{aH}\right)^3 \tag{7.16}$$

To further simplify this expression, we shall also approximately assume $\bar{O}$ has the following time dependence $\bar{O}(t) \propto a^{-3}(t)$ at late time, which is from the intuition that $\bar{O}$ is proportional to the cold dark matter's number density at late time. From the above approximations on ζ and $\bar{O}(t)$, we get

$$I^{ij}_{\zeta\to\delta_S}(p) \approx \frac{|u_p^{\zeta,o}|^2}{\bar{O}}\,i\int_{t_p}^{t}(dz)e^{-i\vec{p}\cdot\vec{z}}\langle[O(\vec{0},t),T^{ij}(z)]\rangle\left(a^2\delta_{ij}\right) \tag{7.17}$$

$$I^{0i}_{\zeta\to\delta_S}(p) \approx \frac{|u_p^{\zeta,o}|^2}{\bar{O}}\,i\int_{t_p}^{t}(dz)e^{-i\vec{p}\cdot\vec{z}}\langle[O(\vec{0},t),\,T^{0i}(z)]\rangle\left(\frac{ip_i}{H}\right) \tag{7.18}$$

$$I^{00}_{\zeta\to\delta_S}(p) \approx \frac{|u_p^{\zeta,o}|^2}{\bar{O}}\,i\int_{t_p}^{t}(dz)e^{-i\vec{p}\cdot\vec{z}}\langle[O(\vec{0},t),T^{00}(z)]\rangle 2\left(\frac{p}{aH}\right)^2 \tag{7.19}$$

and

$$I^{ij}_{\delta_S\to\zeta}(p) \approx \frac{|u_p^{\zeta,o}|^2}{\bar{O}}\,i\int_{t_p}^{t}(dz)e^{-i\vec{p}\cdot\vec{z}}\langle\frac{1}{2}\{O(\vec{0},t),T^{ij}(z)\}\rangle \times\left(a^2\delta_{ij}\frac{2i}{3}\left(\frac{p}{aH}\right)^3\right) \tag{7.20}$$

$$I^{0i}_{\delta_S\to\zeta}(p) \approx \frac{|u_p^{\zeta,o}|^2}{\bar{O}}\,i\int_{t_p}^{t}(dz)e^{-i\vec{p}\cdot\vec{z}}\langle\frac{1}{2}\{O(\vec{0},t),\,T^{0i}(z)\}\rangle \times\left(a\hat{p}_i\epsilon(t_z)\left(\frac{p}{aH}\right)^2\right) \tag{7.21}$$

$$I^{00}_{\delta_S\to\zeta}(p) \approx \frac{|u_p^{\zeta,o}|^2}{\bar{O}}\,i\int_{t_p}^{t}(dz)e^{-i\vec{p}\cdot\vec{z}}\langle\frac{1}{2}\{O(\vec{0},t),T^{00}(z)\}\rangle \times\left(2i\left(\frac{p}{aH}\right)^3\right) \tag{7.22}$$

Next, we shall show that $I_{\zeta\to\delta_S}(p)$ and $I_{\delta_S\to\zeta}(p)$ are suppressed by $\left(\frac{p}{a(t_*)H}\right)^2$, where t_* is the time for gravitational production (for continous production, it is the time for the end of production). The suppression in $I_{\zeta\to\delta_S}(p)$ is due to that ζ and σ's gravitational couplings are actually derivative coupling, and at late time ζ is nearly constant in space and time. One only need to show

$$\int_{t_p}^{t}(dz)e^{-i\vec{p}\cdot\vec{z}}\langle[O(\vec{0},t),a_z^2\delta_{ij}T^{ij}(z)]\rangle \sim O(p^2) \tag{7.23}$$

$$\int_{t_p}^{t}(dz)e^{-i\vec{p}\cdot\vec{z}}\langle[O(\vec{0},t),\,T^{0i}(z)]\rangle \sim O(p) \tag{7.24}$$

$$\int_{t_p}^{t}(dz)e^{-i\vec{p}\cdot\vec{z}}\langle[O(\vec{0},t),T^{00}(z)]\rangle a_z^{-2} \sim O(1) \tag{7.25}$$

Since there is no inverse power of p coming from the matter-loop, it is obvious Eq. (7.25) holds.

To see Eq. (7.24) holds, we may set $p=0$ on the LHS. By isotropy, the LHS is zero. Thus Eq. (7.24) start from p linear term.

To see Eq. (7.23), we may set $p=0$ on the LHS again. We claim that

$$i\int^{t} dt_z d^3z\, a_z^3\langle[O(\vec{0},t),T_z^{ij}a_z^2\delta_{ij}]\rangle = 0 \tag{7.26}$$

The reason is that $\langle in|O(\vec{0},t)|in\rangle$ is invariant under a spatial dilation flow centered at $(\vec{0},t)$. More precisely, we define a local dilation flow $X^\mu(x)=W(x)(0,x^1,x^2,x^3)$, where $W(x)$ is a window function that is one inside the past lightcone of point $(\vec{0},t)$ and smoothly goes to zero at spatial infinity. Then, we get

$$\nabla_i X_j = a^2\delta_{ij} + a^2(x^j\partial_i W). \tag{7.27}$$

We can replace $T_z^{ij}a_z^2\delta_{ij}$ by $T_z^{ij}\nabla_i X_j - T_z^{ij}a^2(x^j\partial_i W)$. However the W dependent part only has support outside of the light-cone while the commutator only has support inside the light-cone, thus the W dependent part does not contribute. We get

$$i\int^{t} dt_z d^3z\, a_z^3\langle in|[\sigma^2(\vec{0},t),T_z^{ij}a_z^2\delta_{ij}]|in\rangle = i\int^{t} dt_z d^3z\, a_z^3\langle in|[\sigma^2(\vec{0},t),T_z^{ij}\nabla_i X_j]|in\rangle \tag{7.28}$$

Next, since

$$\nabla_0 X_0 = 0, \quad \nabla_i X_0 + \nabla_0 X_i = 0 \tag{7.29}$$

we have

$$T_z^{ij}(\nabla_i X_j) = T^{\mu\nu}\nabla_\nu X_\mu \tag{7.30}$$

Plug this back into the above equation, we get

$$i\int^{t} dt_z d^3z\, a_z^3\langle in|[O(\vec{0},t),T^{\mu\nu}\nabla_\nu X_\mu]|in\rangle \tag{7.31}$$

$$= -X^\mu(\vec{0},t)\partial_\mu|_{x=(\vec{0},t)}\langle in|O(x)|in\rangle \tag{7.32}$$

$$= 0 \tag{7.33}$$

where the last step used $X^\mu = 0$ at the origin. Thus, we have conclude the proof of the claim Eq. (7.26). If we assume analyticity of LHS of Eq. (7.23) in p^2 and use isotropy of the background, we see the next leading term is of order p^2.

Thus, we have shown all three terms in $I_{\zeta\to\delta_S}$ are p^2 suppressed. Combined with the assumption that the time integral is peaked near t_*, the suppression factor should be $\left(\frac{p}{a(t_*)H}\right)^2$.

It is also clear that all three terms in $I_{\delta_S\to\zeta}$ are p^2 suppressed, since the matter field mode function has no inverse p dependence. Thus we conclude the cross correlator is suppressed in the long wavelength limit.

7.3 Gauge Independence of $\langle \delta_S \zeta \rangle$

In uniform curvature gauge, we have

$$\delta_S^{(U)} = -3\frac{H}{\dot{\bar{O}}}(O - \bar{O}) - 3\zeta. \tag{7.34}$$

The full correlator up to 1-loop level can be expanded using in-in formalism as

$$\begin{aligned} &\langle \delta_S^{(U)}(\vec{k},t)\zeta(\vec{p},t)\rangle_{1-loop} \\ = \quad & -3\langle \zeta(\vec{k},t)\zeta(\vec{p},t)\rangle - 3\frac{H}{\dot{\bar{O}}} i\int^{t_f}(dz)\langle O^+(\vec{k},t)\zeta^+(\vec{p},t)(\mathcal{L}_{int}^{(U)+} - \mathcal{L}_{int}^{(U)-})\rangle. \end{aligned} \tag{7.35}$$

The cross correlators at 1-loop level in comoving gauge and in uniform curvature gauge have different expressions, see Eq. (7.9) and Eq. (7.35). It seems that in uniform curvature gauge, one would get a strong correlation due to the $\langle\zeta\zeta\rangle$ term in Eq. (7.35). However, the paradox can be resolved by computing everything to the same $\hbar$ order. In Eq. (7.35), the second term (one-loop diagram) has the same order of $\hbar$ as the first term (tree diagram), since the denominator $\bar{O}$ of the second term also contains one-loop. In the remaining part of this subsection, we show that this is indeed the case.

First, we take the difference of Eq. (7.9) and Eq. (7.35):

$$\begin{aligned} \Delta\langle\delta_S\zeta\rangle \quad &\equiv \quad \langle \delta_S^{(U)}(\vec{k},t)\zeta(\vec{p},t)\rangle_{1-loop} - \langle \delta_S^{(C)}(\vec{k},t)\zeta(\vec{p},t)\rangle_{1-loop} \\ &= \quad -3\langle\zeta(\vec{k},t)\zeta(\vec{p},t)\rangle \\ &\quad\quad -3\frac{H}{\dot{\bar{O}}} i\int^{t_f}(dz)\langle O^+(\vec{k},t)\zeta^+(\vec{p},t)(\Delta\mathcal{L}_{int}^+ - \Delta\mathcal{L}_{int}^-)\rangle \end{aligned} \tag{7.36}$$

where $\Delta\mathcal{L}_{int}^{\pm} = \mathcal{L}_{int}^{\pm(U)} - \mathcal{L}_{int}^{\pm(C)}$.

Next, we compute $\Delta\mathcal{L}_{int}$. From Eq. (2.30) and Eq.(2.35), we get

$$\Delta g_{\mu\nu} = \delta g^{(U)}_{\mu\nu} - \delta g^{(C)}_{\mu\nu} = \begin{pmatrix} 2\frac{d}{dt}(\frac{\zeta}{H}) & (-\frac{\zeta}{H})_{,i} \\ (-\frac{\zeta}{H})_{,i} & -a^2\delta_{ij}2\zeta \end{pmatrix} = -[\mathcal{L}_{\xi^\mu\partial_\mu}\bar{g}]_{\mu\nu}, \tag{7.37}$$

where

$$\xi^0 = -\frac{\zeta}{H}, \quad \xi^i = 0. \tag{7.38}$$

Their interaction actions differ by

$$\begin{aligned} \Delta S_{int} = S^{(U)}_{int} - S^{(C)}_{int} &= \int^{t_f} dt d^3x a_x^3 \frac{1}{2} T^{\mu\nu}(\bar{g},\sigma)[-\mathcal{L}_{\xi^\mu\partial_\mu}g]_{\mu\nu} \\ &= \int^{t_f} dt d^3x a_x^3 \, \nabla_\mu T^{\mu\nu}(\bar{g},\sigma)\xi_\nu \end{aligned} \tag{7.39}$$

In the second line, we have dropped a total derivative term

$$\int^{t_f} dt d^3x a_x^3 \, \nabla_\mu [T^{\mu\nu}(\bar{g},\sigma)\xi_\nu]. \tag{7.40}$$

If t_f were taken to $+\infty$ we assume that this term would be negligible. Thus, we get

$$\Delta\mathcal{L}_{int} = \nabla_\mu T^{\mu\nu}(\bar{g},\sigma)\xi_\nu(\zeta). \tag{7.41}$$

Plug $\Delta\mathcal{L}_{int}$ back into Eq. (7.36), and use the Ward Identity Eq(4.20,4.21), we get

$$\begin{aligned} \Delta\langle\delta_S\zeta\rangle &= -3\langle\zeta(\vec{k},t)\zeta(\vec{p},t)\rangle \\ &\quad -3\frac{H}{\dot{O}} i \int^{t_f} (dz)\langle O^+(\vec{k},t)\nabla_\mu T^{\mu\nu}(z;\bar{g},\sigma^+)\rangle\langle\zeta^+(\vec{p},t)\xi_\nu(z;\zeta^+)\rangle \\ &= -3\langle\zeta(\vec{k},t)\zeta(\vec{p},t)\rangle \\ &\quad -3\frac{H}{\dot{O}}\int d^3x\, e^{-i\vec{k}\cdot\vec{x}}\partial^\nu\langle O^+(\vec{x},t)\rangle\langle\zeta^+(\vec{p},t)\xi_\nu((\vec{x},t);\zeta^+)\rangle \\ &= -3\langle\zeta(\vec{k},t)\zeta(\vec{p},t)\rangle - 3\frac{H}{\dot{O}}\frac{d}{dt}\langle O(t)\rangle\langle\zeta^+(\vec{p},t)(-\frac{\zeta^+(\vec{k},t)}{H})\rangle \\ &= 0 \end{aligned} \tag{7.42}$$

In short, we have shown that the cross correlator in the uniform curvature gauge is the same and in the comoving gauge at 1-loop level. For any other gauge fixing condition with resultant metric perturbation $\delta g_{\mu\nu}$, as long as $\delta g_{\mu\nu} - \delta g^{(C)}_{\mu\nu} = \mathcal{L}_\xi\bar{g}$ for some ξ, we can use the above procedure to show that the cross-correlator agrees with that in comoving gauge at 1-loop level.

We may also check $\langle\delta_S\delta_S\rangle$ is the same in uniform curvature gauge and in the comoving gauge, as long as we systematically expand everything to order $O(\hbar)$. Thus, it is necessary to consider

Feynman diagrams involving gravitational interactions to ensure gauge invariance. In comoving gauge, the gravitational interaction diagram gives suppressed contribution due to derivative coupling, thus we can neglect their contribution to $\langle \delta_S \delta_S \rangle$. In uniform curvature gauge, the gravitational interaction diagrams are important to cancel out the ζ dependence in δ_S as in Eq. (7.34).

Chapter 8

Discussions

8.1 Other Interactions

In general, fermions can couple to gauge field as well. Here we shall show that gauge coupling would not create large density perturbation. The main reason is the late time large separation scaling behavior of A_μ 2-point function decreases as $1/r^2$, unlike the massless minimally cooupled scalar field correlator which does not decrease in the large r limit. Physically, since the early produced fermions would dilute away during the inflation and do not contribute much to the density perturbation, we shall focus on those fermions that are produced at the last production surface, namely equal-time hypersurface with $H(t) = m_\psi$.

From the conformal symmetry of the classical action,

$$S(g_{\mu\nu}, A_\mu) = \int d^4x \sqrt{g}(-\frac{1}{4})g^{\alpha\beta}g^{\mu\nu}F_{\mu\alpha}F_{\nu\beta} \tag{8.1}$$

$$= \int d^4x \sqrt{e^{8\sigma}g}(-\frac{1}{4})e^{-4\sigma}g^{\alpha\beta}g^{\mu\nu}F_{\mu\alpha}F_{\nu\beta} \tag{8.2}$$

$$= S(e^{2\sigma}g_{\mu\nu}, A_\mu) \tag{8.3}$$

we know an on-shell configuration $A_\mu(x)$ in flat spacetime maps to an on-shell configuration in FRW spacetime. Hence the two-point equal-time correlator $\langle A_\mu(\vec{0}, \eta)A_\nu(\vec{r}, \eta)\rangle$ scales as $1/r^2$, as in flat spacetime, where we used conformal time η. This implies in vierbein indices,

$$\langle A_a(\vec{0}, \eta)A_b(\vec{r}, \eta)\rangle \propto 1/(a(t)r)^2 \tag{8.4}$$

$$\langle F_{ab}(\vec{0}, \eta)F_{cd}(\vec{r}, \eta)\rangle \propto 1/(a(t)r)^4. \tag{8.5}$$

. We see the field strength correlators are suppressed at large separation, unlike the massless minimally coupled scalar. Thus any effect of the gauge field on the fermion production, such as QED Schwinger effect or the polarization of the produced pairs are suppressed by $(p_{CMB}/(aH))^4$, thus negligible.

8.2 Parameter bounds

First, we estimate the parameters in this model. We leave H_{inf} and T_{RH} as external parameters, not part of the model. For the mass of the scalar field, we want it to be small enough such that we can treat it almost as massless during inflation. This enable us to use the estimation that $\delta\sigma \sim H_{inf}$, and the mass fluctuation $\delta m_\psi \sim \lambda\delta\sigma \sim \lambda H_{inf}$. Furthermore, since after inflation the scalar field will start to oscillate when $m_\sigma > H(t)$, we want this to happen after the fermions are produced but before RD-MD equality time. Once these criterias are met, we do not need to know the precise value of m_σ and we are left with two parameter, namely λ and m_ψ.

There are two observational bounds, the relic abundance of dark matter and the fluctuation of the relic dark matter.

$$\Omega_\psi < 1 \tag{8.6}$$

$$\Omega_\psi \frac{\delta m}{m_{eff}} < \zeta \sim 10^{-5} \tag{8.7}$$

For small mass correction case, i.e. $\lambda H_{inf} < m_\psi$, we have

$$\Omega_\psi \sim \left(\frac{m_\psi}{10^{10}\text{GeV}}\right)^2 \left(\frac{T_{RH}}{10^9\text{GeV}}\right) \left(\frac{g_*}{100}\right)^{1/4} \tag{8.8}$$

$$\frac{\delta m}{m_{eff}} \sim \frac{\lambda H_{inf}}{m_\psi}. \tag{8.9}$$

For large mass correction case, i.e. $\lambda H_{inf} > m_\psi$, we have

$$\Omega_\psi \sim \frac{m_\psi}{\lambda H_{inf}} \left(\frac{\lambda H_{inf}}{10^{10}\text{GeV}}\right)^2 \left(\frac{T_{RH}}{10^9\text{GeV}}\right) \left(\frac{g_*}{100}\right)^{1/4} \tag{8.10}$$

$$\frac{\delta m}{m_{eff}} \sim 1. \tag{8.11}$$

Combining the above consideration, we have the parameter plot shown in fig. (8.1).

The large mass correction case maybe is the most interesting case, since n_ψ depends on $|m_\psi + \lambda\sigma|$ and it may happen that $m_\psi + \lambda\sigma$ become negative at certain patches of the universe. However, the perturbative calculation of n_ψ is invalid for this case. We leave the analysis of the large mass correction case for future work.

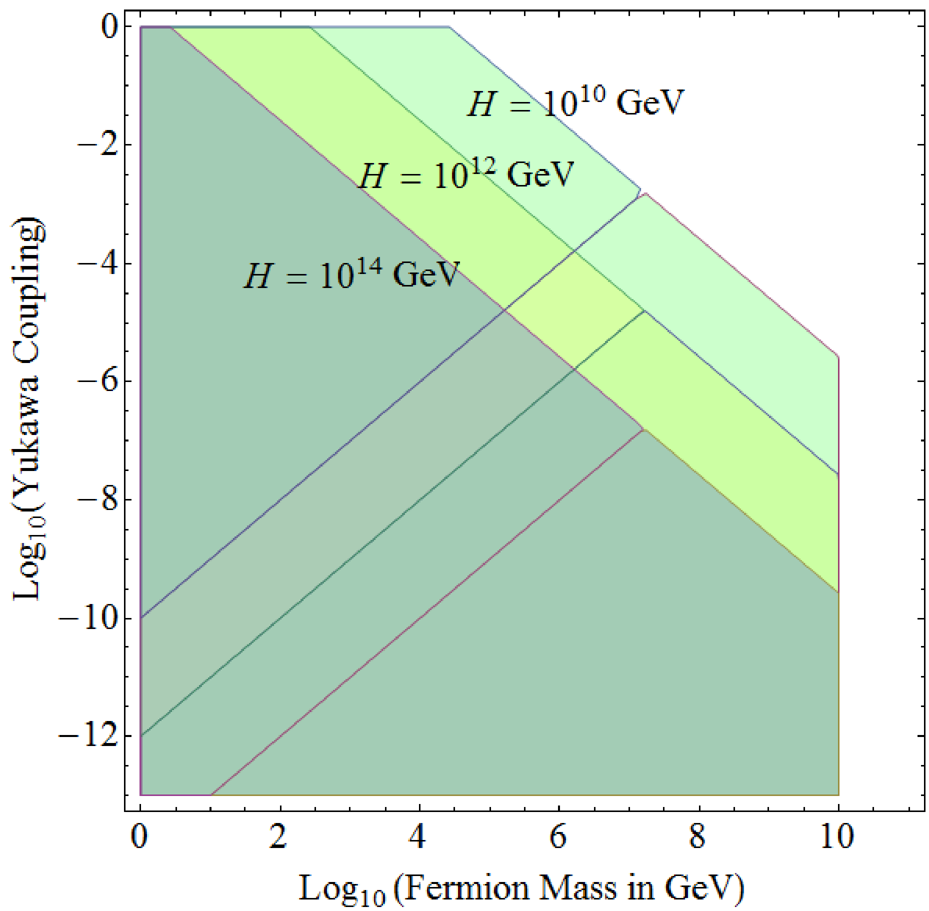

Figure 8.1: Bounds on the fermion mass and Yukawa coupling for various inflationary Hubble scales. The vertical and diagonal bounds corresponds to total relic density constraint and density fluctuation constraint. The splitting lines in each region separates the small mass and large mass correction regime.

8.3 Non-Gaussianities

Next, we give an estimation of the non-Gaussianities of this model. We compute the bi-spectrum $B_S(\vec{p}_1, \vec{p}_2, \vec{p}_3)$ defined by

$$(2\pi)^3 \delta^{(3)}(\sum_i \vec{p}_i) B_S(\vec{p}_1, \vec{p}_2, \vec{p}_3) = \int d^3x_1 d^3x_2 d^3x_3 e^{-i\sum_i \vec{p}_i \cdot \vec{x}_i} \left\langle S_{\vec{x}_1} S_{\vec{x}_2} S_{\vec{x}_3} \right\rangle . \tag{8.12}$$

The fermion density fluctuation is intrinsically non-Gaussian, since n_ψ is the non-linear function of σ. If we assume the mass fluctuation is small, we can expand the number density as

$$n_\psi\left(m_\psi + \lambda\sigma_*(\vec{x})\right) = n_\psi\left(m_\psi\right) + \lambda\left(\partial_m n_\psi(m_\psi)\right)\sigma(\vec{x}) + \frac{1}{2}\lambda^2\left(\partial_m^2 n_\psi(m_\psi)\right)\sigma^2(\vec{x}) + O(\lambda^3). \tag{8.13}$$

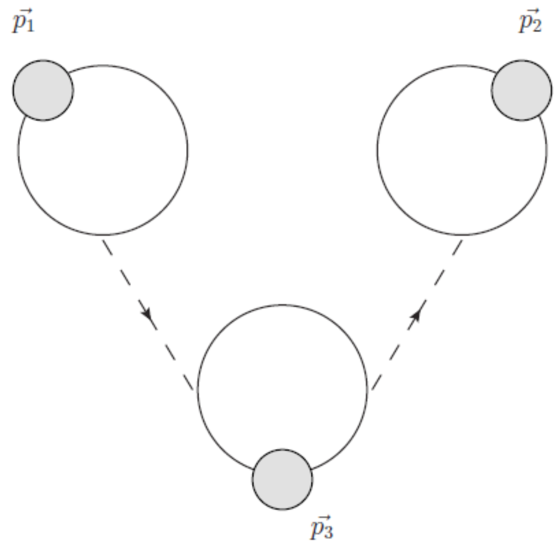

Figure 8.2: The two leading order diagrams to 3-point function $\langle SSS \rangle$. The shaded blob indicates $\bar{\psi}\psi$ insertion.

Also note that no cross-correlation term arises unlike the other multi-field inflationary model. Then the bi-spectrum is written diagrammatically as Fig.(8.2), and we have

$$B_S(\vec{p}_1, \vec{p}_2, \vec{p}_3) \quad = \quad \lambda^4 \omega_\psi^3 \frac{(\partial_m n_\psi)^2 (\partial_m^2 n_\psi)}{n_\psi^3} [\mathcal{P}_\sigma(p_1)\mathcal{P}_\sigma(p_2) + 2 \text{ perms}] + O(\lambda^6) \tag{8.14}$$

Now we compare it with the observational non-Gaussianities using the conventional parameter f_{NL} defined by

$$B_\zeta(\vec{p}_1, \vec{p}_2, \vec{p}_3) \equiv \frac{6}{5} f_{NL} \left[\mathcal{P}_\zeta(p_1)\mathcal{P}_\zeta(p_2) + 2 \text{ perms}\right]. \tag{8.15}$$

Identifying B_ζ as the bi-spectrum of the temperature fluctuation using Eq.(6.39) and compare it with B_S, we find

$$f_{NL}^S = \frac{8\, B_S}{B_\zeta|_{f_{NL}=1}} = 8 \times \frac{5}{6} \lambda^4 \omega_\psi^3 \frac{(\partial_m n_\psi)^2 (\partial_m^2 n_\psi)}{n_\psi^3} \frac{\mathcal{P}_\sigma(p_1)\mathcal{P}_\sigma(p_2) + 2 \text{ perms.}}{\mathcal{P}_\zeta(p_1)\mathcal{P}_\zeta(p_2) + 2 \text{ perms.}} \tag{8.16}$$

Note that the factor 8 arises due to the fact that the isocurvature contribution is twice larger than the curvature perturbation at the long wavelength limit as in Eq.(6.39).Although the isocurvature non-Gaussianities parameter f_{NL}^S should not be compared directly with f_{NL} defined by the curvature perturbation[102], this can be corrected easily by $O(1)$ factor[25, 103, 104].

Next, we specialize to the case of MD-like reheating scenarios. During the early stage of the reheating when the inflaton field oscillates coherently, the equation of state of the inflaton is

$$w(t) = \frac{p(t)}{\rho(t)} = \frac{\frac{1}{2}\dot{\bar{\varphi}}^2 - V(\bar{\varphi})}{\frac{1}{2}\dot{\bar{\varphi}}^2 + V(\bar{\varphi})}, \tag{8.17}$$

which oscillates between ± 1. If the coherent oscillation's time scale is much less than the Hubble expansion rate H^{-1}, then during the period when $H(t)$ drops below m (the duration of this period is about $H/\dot{H} \sim H^{-1}$), fermions would experience the time averaged effect of the oscillation. Thus, after the time average, $w_{eff} = 0$ and we are in a MD-like era.

After approximating the early stage of the reheating to a MD-like era, we get (see Eq. (5.19))

$$n_\psi(t) \sim \frac{m^3}{3\pi^2}\left(\frac{a(t_m)}{a_t}\right)^3 \sim mH_e^2\left(\frac{a_e}{a_t}\right)^3 \tag{8.18}$$

However, this leading order result gives $\partial_m^2 n_\psi = 0$ which renders $f_{NL}^S = 0$ via Eq.(8.16).

To find the non-zero result of f_{NL}^S, we need to study the mass dependence of n_ψ in more detail, which in turn requires the knowledge of $|\beta_k(t;m)|^2$. To this point, we have approximated our spectrum by $|\beta_k(t;m)|^2 \sim 1/2\Theta(k_* - k)$, where $k_* = a(t_*)m$ and $t_*$is the time when $m = H$. However, in general the spectrum should contain more than one characteristic scale, such as $k_e = a(t_e)H_e$ where t_e marks the end of inflation. Thus, in general, the number density should contain a fudge factor $f(\frac{m}{H_e})$ i.e.

$$n_\psi \sim mH_e^2\left(\frac{a_e}{a_t}\right)^3 f(\frac{m}{H_e}) \tag{8.19}$$

and $f(0) = 1$. This higher order correction to n_ψ would render $\partial_m^2 n_\psi \neq 0$ for MD-like reheating scenario.

For simplicity, if we assume that $f(x) = 1 + a_1 x + a_2 x^2 + \cdots$, then in the limit where $\mathcal{P}_\sigma, \mathcal{P}_\zeta, \mathcal{P}_S$ are scale invariant

$$f_{NL}^S \quad \sim \quad a_1 O(10)\Omega_{DM}\alpha_S^2\left(\frac{10^6 GeV}{m_\psi}\right)\left(\frac{10^9 GeV}{T_{RH}}\right)\left(\frac{10^{14} GeV}{H_e}\right) \tag{8.20}$$

However, the justification of the Taylor expansion for $f(x)$ and the estimation of the coefficient a_i will be left for future work.

Chapter 9

Conclusions

In this thesis, we studied the cosmological effects of gravitationally produced fermions. Since most of the current isocurvature models depends on fluctuations of bosonic condensates, it is interesting to know the effects of fermionic fluctuations. Furthermore, many particle physics models for inflation contains massive fermions, it is useful to have cosmological probes for these models. Here we used Bogoliubov transformation to compute the relic abundance of fermionic dark matter, and we used the in-in formalism to compute the correlator and the cross correlator of the fermionic isocurvature perturbations.

First, we revisited the gravitational fermion productions, to clarity its analytic structures. For light fermions with mass smaller than the inflationary Hubble expansion rate, they are continuously produced until the Hubble rate drops below the mass during reheating. For modes that are non-relativistic when the particle production ends, their occupation numbers are about $1/2$. Unlike the bosonic case, fermion statistics forbid the occupation number to be larger than 1. Heavy fermions with mass larger than Hubble expansion rate are produced predominantly at the end of inflation. The occupation number per state is exponentially suppressed by the factor $\exp(-c_1 m/H)$ where $c_1 \sim O(1)$. The analytic estimates are confirmed by the numerical computation in a toy model.

The fermion isocurvature two point correlator, which is proportional to $\langle(\bar{\psi}\psi)(x)(\bar{\psi}\psi)(y)\rangle$, is computed at the leading order (one-loop level) and the next leading order (two-loop level). The leading order contribution comes from the fermions that are generated when the CMB scale exits the horizon. Due to the $1/a^3$ scaling behavior of the fermion equal-time correlator, coming from the fermion operator's conformal dimension, the leading order contribution is undetectable. The next leading order contribution comes from the scalar field σ's interaction. Since a light scalar field's fluctuations are long-lived, its fluctuations at CMB scale would effectively change the fermion mass and modulate the fermion relic number density.

Next, we computed the curvature and isocurvature cross correlator. We used gravitational Ward identities to show that the cross correlation contains a suppression factor of $[p_{CMB}/(a(t_*)H(t_*))]^2 < \exp(-120)$, where p_{CMB} is the comoving scale relevant for CMB observation and t_* is the time at the end of fermion production. Thus, the fermionic isocurvatures are of the uncorrelated type.

As a self-consistency check, we used the Ward identities to show the gauge invariance of our results. The gauge-invariance of the correlation function regarding isocurvature is only true if we include all the diagram with gravitational coupling at the same $\hbar$ order. If δ_S is a composite operator, as is in our case, one need to consider the tree-level diagrams and the one-loop diagrams together to get gauge-invariance.

Our results regarding the gravitational fermion production can be applied to any model with heavy stable fermions. The fermion mass is constrained from the dark matter relic density and the Yukawa coupling constant with light scalar field is constrained from the CMB isocurvature bound. Our results about the smallness of the cross correlator can be generalized to other field types, as long as the fields carrying the isocurvature perturbations are generated at late time. Our proof for the gauge-invariance of isocurvature correlators at one-loop level should pave the road for future loop computations.

In the future, it would be interesting to consider the non-Gaussianities signal from this model more closely. If the mass correction from the scalar field to the fermion field is large, one need to give up the perturbative calculation and use the stochastic approach. It would be interesting to know how the resulting bispectrum behaves. It would also be useful to prove gauge-invariance beyond one-loop level and study the construction of gauge-invariant quantum operators relevant for cosmology.

Appendix A

QFT in curved spacetime

A.1 Quantization for Scalar field

In this section, we quantize the free scalar field in a fixed curved background and work out the case for FRW background. We follow the approach in [36].

From the classical action

$$S_\sigma = \int (dx)(-\frac{1}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - \frac{1}{2}m^2\sigma^2) \tag{A.1}$$

we can extremize with respect to σ to get the equation of motion

$$\nabla_\mu(g^{\mu\nu}\partial_\nu\sigma) - m^2\sigma = 0. \tag{A.2}$$

The set of solutions to the equation of motion forms a complex vector space $\mathcal{N}_\sigma$. We can define an inner product on the solution space as

$$(U_1, \Psi_2) = \int d\Sigma^\mu e_a^\mu \bar{\Psi}_1\gamma^a\Psi_2 \tag{A.3}$$

where $d\Sigma^\nu = n^\nu d\Sigma$, with n^ν a future-directed unit vector orthorgonal to the spacelike hypersurface Σ and $d\Sigma$ is the volume element in Σ. Use Stoke's theorem, and the current conservation we can show that the inner product is independent of the choice of Σ.

There exists a complete and orthonormal set of mode solutions $u_i(x)$ of (A.2), such that

$$(u_i, u_j) = \delta_{ij}, \qquad (u_i, u_j^*) = 0 \tag{A.4}$$

where label i schematically represetns the set of quantities necessary to label the modes. Then we can expand the field operator $\phi(x)$ as

$$\phi(x) = \sum_i a_i u_i(x) + a_i^\dagger u_i^*(x) \tag{A.5}$$

where we impose the commutation relations

$$[a_i, a_j^\dagger] = \delta_{ij}, \qquad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0 \tag{A.6}$$

For unperturbed FRW spacetime, we use the 3-momentum $\vec{k}$ to label the modes. The mode function can be written as

$$u_{\vec{k}}(\vec{x}, t) = \frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{\frac{3}{2}}} u_k(t) \tag{A.7}$$

where the time-dependent factor of the mode solution $u_k(t)$ obeys the equation from Eq.(A.2)

$$\ddot{u}_k(t) + 3H(t)\dot{u}_k(t) + (\frac{k^2}{a^2} + m^2)u_k(t) = 0 \tag{A.8}$$

and satisfies the normalization condition from Eq. (A.4)

$$u_k \partial_t u_k^* - u_k^* \partial_t u_k = \frac{i}{a^3(t)} \tag{A.9}$$

Thus for FRW metric, the mode decomposition can be written as

$$\phi(\vec{x}, t) = \int d^3k \, [a_{\vec{k}} \frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{\frac{3}{2}}} u_k(t) + a_{\vec{k}}^\dagger \frac{e^{-i\vec{k}\cdot\vec{x}}}{(2\pi)^{\frac{3}{2}}} u_k^*(t)] \tag{A.10}$$

A.2 Quantization for Spinor field

In this section, we quantize the spinor field on a fixed curved background.

From the classical action

$$S_\psi = \int (dx) \bar{\psi}(i\gamma^a \nabla_{e^a} - m)\psi \tag{A.11}$$

we can extremize with respect to $\bar{\psi}, \psi$ to get the equation of motions

$$(i\gamma^\alpha \nabla_{e_\alpha} - m)\psi = 0, \qquad \nabla_{e_a}\bar{\psi}(-i\gamma^a) - \bar{\psi}m = 0. \tag{A.12}$$

The set of solutions to the equation of motion forms a complex vector space $\mathcal{N}_\psi$. We can define an inner product on the solution space as

$$(\Psi_1, \Psi_2) = \int d\Sigma^\mu e_{a\mu} \bar{\Psi}_1 \gamma^a \Psi_2 \tag{A.13}$$

where the notation is the same as in the scalar field case. Use Stoke's theorem, we can show that the inner product is independent of the choice of Σ.

Analogous the scalar field case where $u(x) \mapsto u^*(x)$ defines pairing in the solution space $\mathcal{N}_\sigma$, in the spinor case $\Psi \mapsto -i\gamma^2\Psi^*$ defines a pairing in the solution space $\mathcal{N}_\psi$.

There exists a complete and orthonormal set of mode solutions $U_i(x)$ of (A.12), such that

$$(U_i, U_j) = \delta_{ij}, \qquad (V_i, V_j) = \delta_{ij}, \qquad (U_i, V_j) = 0 \tag{A.14}$$

where $V_i \equiv -i\gamma^2 U_i^*$ and label i schematically represents the set of quantities necessary to label the modes. Then we can expand the field operator $\Psi(x)$ as

$$\Psi(x) \quad = \quad \sum_i a_i U_i + b_i^\dagger V_i, \tag{A.15}$$

where we impose the commutation relations

$$\{a_i, a_j^\dagger\} = \delta_{ij}, \qquad \{b_i, b_j^\dagger\} = \delta_{ij}. \tag{A.16}$$

For unperturbed FRW spacetime, we use the 3-momentum $\vec{k}$ and the helicity $r = \pm 1$ to label the modes. The mode function can be written as

$$U_{\vec{k},r}(\vec{x}, t) \quad = \quad \frac{e^{i\vec{k}\cdot\vec{x}}}{a(t)^{3/2}(2\pi)^{3/2}} \begin{pmatrix} u_{A,k}(t) \\ r\, u_{B,k}(t) \end{pmatrix} \otimes h_{\hat{k},r} \tag{A.17}$$

$$V_{\vec{k},r}(\vec{x}, t) \quad = \quad \frac{e^{-i\vec{k}\cdot\vec{x}}}{a(t)^{3/2}(2\pi)^{3/2}} \begin{pmatrix} r\, u^*_{B,k}(t) \\ -u^*_{A,k}(t) \end{pmatrix} \otimes (-i\sigma^2) h^*_{\hat{k},r} \tag{A.18}$$

where $h_{\hat{k},r}$ is eigenvector of $\hat{k}\cdot\vec{\sigma}$, that satisfies

$$\hat{k}\cdot\vec{\sigma} h_{\hat{k},r} \quad = \quad r h_{\hat{k},r}, \; r = \pm 1. \tag{A.19}$$

$$h^\dagger_{\hat{k},r} h_{\hat{k},s} \quad = \quad \delta_{rs} \tag{A.20}$$

More concretely, if $\hat{k} = (\theta, \phi)$ in spherical coordinates, then the normalization factor can be chosen such that

$$h_{\hat{k},+1} \equiv \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\phi} \\ \sin\frac{\theta}{2} \end{pmatrix}, h_{\hat{k},-1} \equiv \begin{pmatrix} \sin\frac{\theta}{2} e^{-i\phi} \\ -\cos\frac{\theta}{2} \end{pmatrix}. \tag{A.21}$$

One can easily check that due to this phase convention

$$-i\sigma^2 (h_{\hat{k},r})^* = -r e^{-ir\phi} h_{\hat{k},-r}, \qquad h_{-\hat{k},r} = -h_{\hat{k},-r}. \tag{A.22}$$

The time-dependent factor of the mode solution $(u_{k,A}(t), u_{k,B}(t))$ obeys the equation

$$i\partial_t \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} m & \frac{k}{a} \\ \frac{k}{a} & -m \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} \tag{A.23}$$

and satisfies the normalization condition from Eq. (A.14)

$$|u_A|^2 + |u_B|^2 = 1. \tag{A.24}$$

Thus for FRW metric, the mode decomposition can be written as

$$\Psi(\vec{x}, t) = \int d^3k \sum_r [a_{\vec{k},r} \frac{e^{i\vec{k}\cdot\vec{x}}}{a^{\frac{3}{2}}(2\pi)^{\frac{3}{2}}} \begin{pmatrix} u_{A,k}(t) \\ r\, u_{B,k}(t) \end{pmatrix} \otimes h_{\hat{k},r} \tag{A.25}$$

$$+ b^{\dagger}_{\vec{k}} \frac{e^{i\vec{k}\cdot\vec{x}}}{a^{\frac{3}{2}}(2\pi)^{\frac{3}{2}}} \begin{pmatrix} r\, u^*_{B,k}(t) \\ -u^*_{A,k}(t) \end{pmatrix} \otimes (-i\sigma^2) h^*_{\hat{k},r}] \tag{A.26}$$

A.3 Bogoliubov Transformation

In the previous two sections, we have quantized the scalar field and the spinor field on a curved spacetime background. And we have used the mode decomposition to express the field operators as creation and annihilation operators. However, there is ambiguity in such a mode decomposition.

For scalar field $\phi(x)$, consider another set of complete orthonormal basis $\{\tilde{u}_i, \tilde{u}^*_i\}$, which lead to another mode decomposition

$$\phi(x) = \sum_i \tilde{a}_i \tilde{u}_i + \tilde{a}^*_i \tilde{u}^*_i. \tag{A.27}$$

The new basis is related to the old basis as

$$\tilde{u}_j = \sum_i (\alpha_{ji} u_i + \beta_{ji} u^*_i). \tag{A.28}$$

Conversely

$$u_i = \sum_j (\tilde{u}_j \alpha^*_{ji} - \tilde{u}^*_j \beta_{ji}) \tag{A.29}$$

These relations are known as Bogoliubov transformations.

We can also write down the transformation for the creation and annihilation operators

$$a_i = \sum_j (\tilde{a}_j \alpha_{ji} + \tilde{a}^{\dagger}_j \beta^*_{ji}) \tag{A.30}$$

$$\tilde{a}_j = \sum_i (\alpha^*_{ji} a_i - \beta^*_{ji} a^{\dagger}_i) \tag{A.31}$$

If we define the matrix $[\alpha]_{ij} = \alpha_{ij}, [\beta]_{ij} = \beta_{ij}$, then the Bogoliubov coefficients satisfies the relation

$$\begin{pmatrix} [\alpha] & [\beta] \\ [\beta]^* & [\alpha]^* \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} [\alpha]^\dagger & [\beta]^T \\ [\beta]^\dagger & [\alpha]^T \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{A.32}$$

Vacuum states defined using a_i and $\tilde{a}_i$ are different. For example, if we define $|0\rangle$ by $a_i|0\rangle = 0$ and $|\tilde{0}\rangle$ by $\tilde{a}_i|\tilde{0}\rangle = 0$, then we have

$$a_i|\tilde{0}\rangle = \sum_j \tilde{a}_j^\dagger \beta_{ji}^* |\tilde{0}\rangle \neq 0 \tag{A.33}$$

It then follows that

$$\langle\tilde{0}|a_i^\dagger a_i|\tilde{0}\rangle = \sum_j |\beta_{ji}|^2 \tag{A.34}$$

i.e. the vacuum defined using $\tilde{a}_i$ is not empty of particles defined bya_i.

In the case of FRW spacetime, due to momentum conservation, the only non-zero β_{ij} components are $\beta_{\vec{k},-\vec{k}}$. In this case, we can factor out $\hat{k}$ dependence and introduce α_k, β_k by

$$\tilde{u}_k(t) = \alpha_k u_k(t) + \beta_k u_k^*(t) \tag{A.35}$$

The constraints on α and β are now

$$|\alpha_k|^2 - |\beta_k|^2 = 1 \tag{A.36}$$

Similarly, we can analyze the spinor field $\psi(x)$. Consider another set of complete orthonormal basis $\{\tilde{U}_i, \tilde{V}_i\}$, which lead to another mode decomposition

$$\tilde{U}_i = \sum_j \alpha_{ij} U_j + \beta_{ij} V_j \tag{A.37}$$

Apply $-i\gamma^2(.)^*$on both sides, we get

$$\tilde{V}_i = \sum_j \alpha_{ij}^* V_j + \beta_{ij}^* U_j \tag{A.38}$$

By requiring the new modes satisfies the orthonormal conditions, we get these constraints on the Bogolubov coefficients:

$$\sum_m \alpha_{im}^* \alpha_{jm} + \beta_{im}^* \beta_{jm} = \delta_{ij} \tag{A.39}$$

$$\sum_m \alpha_{im} \beta_{jm} + \beta_{im} \alpha_{jm} = 0 \tag{A.40}$$

or equivalently in matrix notation

$$\begin{pmatrix} [\alpha] & [\beta] \\ [\beta]^* & [\alpha]^* \end{pmatrix} \begin{pmatrix} [\alpha]^\dagger & [\beta]^T \\ [\beta]^\dagger & [\alpha]^T \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{A.41}$$

In the case of FRW spacetime, due to momentum conservation, the only non-zero β_{ij} components are $\beta_{(\vec{k},r),(-\vec{k},r)}$. In this case, we can factor out $\hat{k}, r$ dependence and introduce α_k, β_k by

$$\begin{pmatrix} \tilde{u}_A \\ \tilde{u}_B \end{pmatrix}_{k,t} = \alpha_k \begin{pmatrix} u_A \\ u_B \end{pmatrix}_{k,t} + \beta_k \begin{pmatrix} u_B^* \\ -u_A^* \end{pmatrix}_{k,t} \tag{A.42}$$

The constraints on α and β are now

$$|\alpha_k|^2 + |\beta_k|^2 = 1 \tag{A.43}$$

The Bogoliubov coefficients are extracted using the scalar product of the mode functions evaluated at time η as follows:

$$\alpha_{(\vec{k},s)(\vec{k},s)} = u^*_{A,k,\eta}\tilde{u}_{A,k,\eta} + u^*_{B,k,\eta}\tilde{u}_{B,k,\eta} \tag{A.44}$$

$$\beta_{(\vec{k},s)(-\vec{k},s)} = e^{-is\phi(\hat{k})}(u_{A,k,\eta}\tilde{u}_{B,k,\eta} - u_{B,k,\eta}\tilde{u}_{A,k,\eta}) \tag{A.45}$$

Since we shall only consider $|\beta_k|^2$ in this work, we can drop the $e^{-is\phi(\hat{k})}$ factor in the β_k definition without loss of generality. Here one of the bases (corresponding to the Heisenberg state of the universe) is specified by asymptotic conditions such as the Bunch-Davies boundary condition as the in-vacuum.) Similarly, the other basis is the observable operator basis as specified by asymptotic conditions at late times, which is referred to as the out-vacuum.

Finally, we introduce the time dependent Bogoliubov coefficients.First, we define the adiabatic vacuum at time t. Recall that on flat spacetime, the fermion equation of motion is

$$i\partial_t \begin{pmatrix} u_A \\ u_B \end{pmatrix}_{k,t} = \begin{pmatrix} m & k \\ k & -m \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} \tag{A.46}$$

and the solution with positive frequency ($e^{-i\omega t}$ time dependence) takes the form of

$$\begin{pmatrix} u_A \\ u_B \end{pmatrix}_{k,t} = \begin{pmatrix} \sqrt{\frac{\omega+m}{2\omega}} \\ \sqrt{\frac{\omega-m}{2\omega}} \end{pmatrix} e^{-i\omega t} \tag{A.47}$$

where $\omega^2 = m^2 + k^2$. In the case that the matrix $\begin{pmatrix} m(t) & k(t) \\ k(t) & -m(t) \end{pmatrix}$ is time-dependent but varying very slowly, compared with the scale of $\omega(t)$, we may still use Eq. (A.46) as an approximate solution. More precisely, the approximate solution is

$$\begin{pmatrix} u_A \\ u_B \end{pmatrix}^{WKB}_{k,t} = \begin{pmatrix} \sqrt{\frac{\omega(t)+m(t)}{2\omega(t)}} \\ \sqrt{\frac{\omega(t)-m(t)}{2\omega(t)}} \end{pmatrix} e^{-i\int^t \omega(t')dt'}. \tag{A.48}$$

The exact solution can be decomposed as

$$\begin{pmatrix} u_A \\ u_B \end{pmatrix}_{k,t} = \alpha_{k,t} \begin{pmatrix} u_A \\ u_B \end{pmatrix}^{WKB}_{k,t} + \beta_{k,t} \begin{pmatrix} u_B^* \\ -u_A^* \end{pmatrix}^{WKB}_{k,t} \tag{A.49}$$

where α and β are now time dependent. The field operator can be decomposed as

$$\psi(x) = \sum_i a_i^{WKB}(t) U_i^{WKB}(x) + b_i^{WKB,\dagger}(t) V_i^{WKB}(x) \tag{A.50}$$

and the vacuum $|0\rangle_t$ annihilated by $a_i^{WKB}(t)$ and $b_i^{WKB}(t)$ for a fixed time t is called the adiabatic vacuum centered at time t.

If at very early time $t \to -\infty$, the WKB solution approaches an exact solution, then we can use Eq.(A.49) with $\alpha = 1, \beta = 0$ at $t = -\infty$ to define the positive frequency mode function. This will corresponds to the in-vacuum. Similarly, if at very late time, the WKB solution approaches an exact solution, then we can use $\alpha = 1, \beta = 0$ at $t = +\infty$ to define the out-vacuum.

Next, we derive the evolution equation obeyed by $\alpha_{k,t}, \beta_{k,t}$. In the case that $k(t) = \frac{k}{a(t)} \equiv k_p$ and $m(t) = m$ constant, we get[1]

$$\partial_t \alpha = \frac{m k_p H}{2\omega^2} e^{2i \int^t \omega} \beta \tag{A.53}$$

$$\partial_t \beta = -\frac{m k_p H}{2\omega^2} e^{-2i \int^t \omega} \alpha. \tag{A.54}$$

We may define $\epsilon_{\text{non-ad}} = \frac{m k_p H}{2\omega^3}$ as the non-adiabatic parameter. It is easy to see that at very early time and very late time, $\epsilon_{\text{non-ad}}$ is suppressed, thus we are indeed in the adiabatic regime.

To get the Bogoliubov coefficients between the in-mode and out-mode, defined as

$$\begin{pmatrix} u_A \\ u_B \end{pmatrix}^{in}_{k,t} = \alpha_k^{in-out} \begin{pmatrix} u_A \\ u_B \end{pmatrix}^{out}_{k,t} + \beta_k^{in-out} \begin{pmatrix} u_B^* \\ -u_A^* \end{pmatrix}^{out}_{k,t} \tag{A.55}$$

one only need to consider Eq. (A.49) at late time. Thus

$$\beta_k^{in-out} = \lim_{t\to\infty} \beta_{k,t}. \tag{A.56}$$

Since for fermion field, $\alpha \sim O(1)$, we may formally integrate Eq. (A.54) to get

$$\beta_k \approx -\int_{-\infty}^{+\infty} dt \frac{m \frac{k}{a(t)} H(t)}{2(m^2 + \frac{k^2}{a^2})} e^{-2i \int^t d\tilde{t} \omega(t)} \tag{A.57}$$

[1] More generally, if we set $k \mapsto k(t)$ and $m \mapsto m(t)$ in Eq. (A.46) and plug the ansatz Eq. (A.49) into it, we get

$$\partial_t \alpha = -\frac{m \partial_t k - k \partial_t m}{2\omega^2} e^{2i \int^t \omega} \beta \tag{A.51}$$

$$\partial_t \beta = \frac{m \partial_t k - k \partial_t m}{2\omega^2} e^{-2i \int^t \omega} \alpha \tag{A.52}$$

A.4 Stress Tensor for Dirac Spinor

We use $(-+++)$ for Lorentzian metric signature. We use vielbein $\{e_a\}_{a=0,\cdots,3}$ and its dual $\{\theta^a\}_a$, with the following decomposition

$$
\begin{aligned}
e_a &= e_a^{\mu}\partial_{\mu} && \text{(A.58)}\\
\theta^a &= \theta^a_{\mu} dx^{\mu}. && \text{(A.59)}
\end{aligned}
$$

The spin connection one-form is given by

$$\omega = \omega_{\mu}^{A}\, T^A \otimes dx^{\mu} \tag{A.60}$$

where T^A are generators of $\mathbf{so}(3,1)$. In its fundamental representation, T^A act on e_a by

$$T^A(e_a) = e_b[T^A]^b_{\ a} \tag{A.61}$$

then we define

$$[\omega_{\mu}]^b_{\ a} \equiv \omega_{\mu}^{A}[T^A]^b_{\ a}. \tag{A.62}$$

The curvature and torsion 2-forms are defined by $T = D\theta, R = D\omega$which in component form reads

$$
\begin{aligned}
T^a &= d\theta^a + \omega^a_{\ b}\wedge\theta^b && \text{(A.63)}\\
R^a_{\ b} &= d\omega^a_{\ b} + \omega^a_{\ c}\wedge\omega^c_{\ b} && \text{(A.64)}
\end{aligned}
$$

We shall impose $T = 0$ constraint as we vary the vielbein.

If we take the vielbein and connection 1-form as the fundamental object and define the metric and connection from it, then we may consider a perturbation of the metric caused by the perturbation of the vielbein. Consider the following infinitesimal transformations

$$
\begin{aligned}
e_a &\mapsto e_a + \delta e_a && \text{(A.65)}\\
\theta^a &\mapsto \theta^a + \delta\theta^a && \text{(A.66)}\\
[\omega_{\mu}]^a_{\ b} &\mapsto [\omega_{\mu}]^a_{\ b} + [\delta\omega_{\mu}]^a_{\ b} && \text{(A.67)}
\end{aligned}
$$

We may parametrize the perturbation as

$$\delta e_a = e_b\epsilon^b_{\ a} = e^b(\epsilon^A_{ba} + \epsilon^S_{ba}) \tag{A.68}$$

where superscript A, S denote the symmetric and anti-symmetric part of the matrix ϵ_{ba}. At linear perturbation level, ϵ^A_{ba} and ϵ^S_{ba} can be considered independently. ϵ^A_{ba} corresponds to a rotation (in

$SO(3,1)$ sense) of the orthonormal basis, under which the action is invariant. ϵ^S_{ba} causes metric perturbation, and we may define the stress-tensor T^{ab} in vielbein indices as

$$\delta S = -\int_M \epsilon^S_{ba} T^{ab} \tag{A.69}$$

which is symmetric. Since ϵ^A does not do anything, we will set it to zero and treat ϵ_{ab} as a symmetric matrix.

The $\delta\theta^a$ and $\delta\omega$ are determined by the linearization of constraints

$$\theta^a(e_b) = \delta^a_b \tag{A.70}$$

$$T = 0 \tag{A.71}$$

i.e.

$$\delta\theta^a(e_b) + \theta^a(\delta e_b) = 0 \tag{A.72}$$

$$d(\delta\theta^a) + \omega^a_b \wedge (\delta\theta^b) + \delta(\omega^a_b) \wedge \theta^b = 0 \tag{A.73}$$

The first one gives

$$\delta\theta^a = -\epsilon^a_b \theta^b. \tag{A.74}$$

The second one gives, skematically

$$\delta\omega \wedge \theta = -D(\delta\theta) \tag{A.75}$$

$$= D(\epsilon\theta) \tag{A.76}$$

$$= d(\epsilon\theta) + \omega \wedge (\epsilon\theta) \tag{A.77}$$

$$= (d\epsilon) \wedge \theta + \epsilon(d\theta) + \omega \wedge (\epsilon\theta) \tag{A.78}$$

$$= (d\epsilon) \wedge \theta + \epsilon(-\omega \wedge \theta) + \omega \wedge (\epsilon\theta) \tag{A.79}$$

$$= (d\epsilon + [\omega, \epsilon]) \wedge \theta \tag{A.80}$$

If we write $\delta(\omega^a_b) = \delta\Gamma^a_{cb}\theta^c$,$(d\epsilon + [\omega,\epsilon])^a_b = M^a_{cb}\theta^c$ and lower the indices by $\delta\Gamma_{c,ab} = \eta_{ad}\delta\Gamma^d_{cb}$, $M_{c,ab} = \eta_{ab}M^d_{cb}$, then

$$\delta\Gamma_{c,ab}\theta^c \wedge \theta^b = M_{c,ab}\theta^c \wedge \theta^b \tag{A.81}$$

$$\Leftrightarrow \delta\Gamma_{c,ab} - \delta\Gamma_{b,ac} = M_{c,ab} - M_{b,ac} \tag{A.82}$$

Since $\delta\Gamma_{c,ab}$ is anti-symmetric in ab indices, we may permute the indices to get

$$\delta\Gamma_{c,ba} - \delta\Gamma_{a,bc} = M_{c,ba} - M_{a,bc} \tag{A.83}$$

$$\delta\Gamma_{b,ca} - \delta\Gamma_{a,cb} = M_{b,ca} - M_{a,cb} \tag{A.84}$$

If we do Eq.(A.82)-Eq.(A.83)-Eq.(A.84) and divide by 2, then we get

$$\delta\Gamma_{c,ab} = M^A_{c,ab} + M^S_{a,cb} - M^S_{b,ca} \tag{A.85}$$

where $M^A_{c,ab}$ is defined using ϵ^A and is anti-symmetric in ab indices, etc. Thus

$$\begin{aligned}
\delta[\omega_\mu]_{ab} &= \theta^c_\mu \delta\Gamma_{c,ab} && \text{(A.86)}\\
&= \partial_\mu(\epsilon^A_{ab}) + ([\omega_\mu, \epsilon^A])_{ab} && \text{(A.87)}\\
&\quad +\theta^c_\mu e^\nu_a[\partial_\nu \epsilon^S_{bc} + ([\omega_\nu, \epsilon^S])_{bc}] && \text{(A.88)}\\
&\quad -\theta^c_\mu e^\nu_b[\partial_\nu \epsilon^S_{ac} + ([\omega_\nu, \epsilon^S])_{ac}] && \text{(A.89)}\\
&= \theta^c_\mu[\nabla_c(\epsilon^A_{ab}) + \nabla_a(\epsilon^S_{bc}) - \nabla_b(\epsilon^S_{ac})] && \text{(A.90)}
\end{aligned}$$

where

$$(\nabla_b \epsilon)_{cd} = e^\nu_b[\partial_\nu \epsilon_{cd} + ([\omega_\nu, \epsilon])_{cd}. \tag{A.91}$$

In FRW metric, using conformal time, we have

$$ds^2 = a^2(\eta)(-d\eta^2 + d\vec{x}^2) \tag{A.92}$$

The vielbein is simply the rescale of the ∂_μ,

$$e_0 = \frac{1}{a}\frac{\partial}{\partial\eta}, e_i = \frac{1}{a}\frac{\partial}{\partial x^i} \tag{A.93}$$

$$\theta^0 = a d\eta, \theta^i = a dx^i \tag{A.94}$$

The connection one-form in the vierbein indices are $\omega^a_b = \theta^c(\omega_c)^a_b$. Define $\omega_{c,ab} = \eta_{ad}(\omega_c)^d_b$, then the only non-zero components are

$$\omega_{i,i0} = -\omega_{i,0i} = H(\eta). \tag{A.95}$$

Consider the following action for Dirac spinor on a curved background

$$S = \int_M \bar{\psi}(i\gamma^a \nabla_a - m)\psi \tag{A.96}$$

The volume form $d^4x\sqrt{-g} = d^4x\det(\theta^a_\mu)$ is implied in $\int_M$. The Lagrangian density is not real, we may extract its real part and imaginary part

$$\mathcal{L}_R = \frac{1}{2}(\bar\psi i\gamma^a\nabla_a\psi - \nabla_a(\bar\psi)i\gamma^a\psi) - m\bar\psi\psi \quad \text{(A.97)}$$

$$\mathcal{L}_I = \frac{1}{2}(\bar\psi i\gamma^a\nabla_a\psi + \nabla_a(\bar\psi)i\gamma^a\psi) \quad \text{(A.98)}$$

$$= \frac{1}{2}\nabla_a(\bar\psi i\gamma^a\psi) \quad \text{(A.99)}$$

We see $\mathcal{L}_I$ is a total derivative, and does not affect the e^{iS} in the path integral, thus we may drop it safely.

Under the vielbein perturbation $\delta e_b = e_a\epsilon^a_b$, where ϵ_{ab} is taken as a symmetric matrix, the action changes as

$$\delta S = \int_M \left\{\delta[\ln\det(\theta^a_\mu)]\mathcal{L} + \bar\psi i\gamma^a\delta(e^\mu_a)\nabla_\mu\psi + \bar\psi i\gamma^a e^\mu_a\frac{1}{2}\delta[\omega_\mu]_{bc}\Sigma^{bc}\psi\right\} \quad \text{(A.100)}$$

$$= \int_M \left\{-\epsilon_{ab}\eta^{ab}\mathcal{L} + \epsilon_{ab}\bar\psi i\gamma^a\nabla^b\psi + \bar\psi i\gamma^a(\nabla_b\epsilon^S)_{ca}\Sigma^{bc}\psi\right\} \quad \text{(A.101)}$$

Note that $(\nabla_b\epsilon^S)_{ca}$ is still a real symmetric matrix, thus for any N_{ac} symmetric, we get

$$N_{ac}\gamma^a\Sigma^{bc} = -\frac{1}{4}N_{ac}\gamma^a(\gamma^b\gamma^c - \gamma^c\gamma^b) \quad \text{(A.102)}$$

$$= -\frac{1}{4}N_{ac}\gamma^a(-2\eta^{bc} - 2\gamma^c\gamma^b) \quad \text{(A.103)}$$

$$= \frac{1}{2}N_{ac}(\gamma^a\eta^{bc} - \gamma^b\eta^{ac}). \quad \text{(A.104)}$$

We may work on the last term in δS

$$\int_M(\nabla_b\epsilon^S)_{ca}\bar\psi i\gamma^a\Sigma^{bc}\psi \quad \text{(A.105)}$$

$$= -\int_M(\epsilon^S)_{ca}\nabla_b[\bar\psi i\gamma^a\Sigma^{bc}\psi] \quad \text{(A.106)}$$

$$= -\frac{1}{2}\int_M(\epsilon^S)_{ca}\nabla_b[\bar\psi i(\gamma^a\eta^{bc} - \gamma^b\eta^{ac})\psi] \quad \text{(A.107)}$$

$$= \frac{1}{2}\int_M -(\epsilon^S)_{ca}\nabla^c(\bar\psi i\gamma^a\psi) + \epsilon_{ac}\eta^{ac}\nabla_b(\bar\psi i\gamma^b\psi) \quad \text{(A.108)}$$

Plug back in, we get

$$\delta S = \int_M\left\{-\epsilon_{ab}\eta^{ab}\mathcal{L}_R + \epsilon_{ab}\frac{1}{2}[\bar\psi i\gamma^a\nabla^b\psi - \nabla^b(\bar\psi)i\gamma^a\psi]\right\} \quad \text{(A.109)}$$

Thus, we get the stress tensor for fermion field in vierbein indices (indicated by (v))

$$T^{ab(v)} = -\frac{i}{2}[\bar\psi\gamma^{(a}\nabla^{b)}\psi - \nabla^{(b}(\bar\psi)\gamma^{a)}\psi] + \eta^{ab}\mathcal{L}_R \quad \text{(A.110)}$$

which is real and symmetric. Note in the above derivation, we did not use the equation of motion for ψ.

Appendix B

Details for Relic Abundance Computation

B.1 Demonstration that $|\beta_k|^2 \sim \frac{1}{2}$ for Small k

We begin with the determination of β_k from Eq. (A.45) evaluated at very late times when the out-modes can be directly replaced by their asymptotic values. In the limit in which $am/k \to \infty$, we see that we then only need to find the asymptotic values of the in-modes:

$$|\beta_k| = |u^{out}_{A,k,\eta} u^{in}_{B,k,\eta} - u^{out}_{B,k,\eta} u^{in}_{A,k,\eta}| \tag{B.1}$$

$$= |\sqrt{\frac{\omega + am}{2\omega}} u^{in}_{B,k,\eta} - \sqrt{\frac{\omega - am}{2\omega}} u^{in}_{A,k,\eta}| \tag{B.2}$$

$$= \lim_{\eta \to \infty} |u^{in}_{B,k,\eta}|. \tag{B.3}$$

Let us consider the evolution equations as given in Eq. (A.23) with boundary conditions as given in Eq. (5.22). For concreteness, we choose a time η_i that is early enough such that $u_A(\eta_i) \approx u_B(\eta_i) \approx \frac{1}{\sqrt{2}}$. The system can be formally solved to obtain

$$\begin{pmatrix} u_A \\ u_B \end{pmatrix}_f = T \exp\left\{-i \int d\Phi \sigma(\theta)\right\} \begin{pmatrix} u_A \\ u_B \end{pmatrix}_i \tag{B.4}$$

in which $\omega \cos\theta = k$, $\omega \sin\theta = am$, $\omega d\eta = d\Phi$, and $\sigma(\theta) = \sigma_1 \cos\theta + \sigma_3 \sin\theta$ ($0 \leq \theta \leq \pi/2$). The time evolution is thus expressed as a series of infinitesimal $SU(2)$ rotations that act successively on the complex vector $u \equiv (u_A\, u_B)$.

For fixed θ, the evolution corresponds to precession about the axis defined by $\sigma(\theta)$. However, throughout the evolution of the universe, $\sigma(\theta)$ evolves from its initial direction along σ_1 ($am \ll k$) to its final direction along σ_3 ($am \gg k$). If the switching of the axis is much faster than the precession time scale, u remains in the xy-plane and rotates around the new axis σ_3, while if the switching is much slower compared with the precession time scale, u adheres closely to the rotation axis and thus ends up in the σ_3 direction. The time scale of the axis switching is given by the Hubble expansion rate, since the universe needs to expand several e-folds for am to overtake k, while the time scale of the precession is given by the physical frequency ω/a, which is on the order of m during the transition. Hence, fast transitions occur when $m \ll H$, for which $|u_B|^2$ stabilizes at $\frac{1}{2}$ and $|\beta_k|^2 = \frac{1}{2}$. After $H(\eta)$ drops below m, only slow transitions occur and $|\beta_k|^2$ is small.

B.2 Heavy mass case ($m > H_e$)

As we expect the particle production spectrum $|\beta_k|^2$ to be exponentially suppressed by m/H, we can adopt a similar approach as the heavy mass scalar case [73] to look for a one-pole approximation to the time integral that determines β_k. We shall consider the time-dependent Bogoliubov coefficients between the in-modes and the zeroth adiabatic modes with boundary conditions such that

$$\begin{pmatrix} u_A \\ u_B \end{pmatrix}^{(\eta_1)}_{k,\eta=\eta_1} = \begin{pmatrix} \sqrt{\frac{\omega+am}{2\omega}} \\ \sqrt{\frac{\omega-am}{2\omega}} \end{pmatrix}. \tag{B.5}$$

In the above, the superscript (η_1) indicates the time that the boundary conditions are imposed. The in-modes can be decomposed into the zeroth adiabatic mode basis as follows:

$$\begin{pmatrix} u_A \\ u_B \end{pmatrix}^{in}_{k,\eta_1} = \alpha_k^{in-(\eta_1)} \begin{pmatrix} u_A \\ u_B \end{pmatrix}^{(\eta_1)}_{k,\eta_1} + \beta_k^{in-(\eta_1)} \begin{pmatrix} -u_B^* \\ u_A^* \end{pmatrix}^{(\eta_1)}_{k,\eta_1}. \tag{B.6}$$

For $\eta_1 \to \infty$, the instantaneous-modes will coincide with the out-modes up to an overall phase, and hence

$$|\beta_k| = \lim_{\eta_1 \to \infty} |\beta_k^{in-(\eta_1)}|. \tag{B.7}$$

Inserting this decomposition into Eq. (A.23) (and writing $\alpha_k^{in-(\eta_1)}$ as $\alpha_k(\eta_1)$, etc. for notational simplicity) results in

$$\begin{aligned} \dot{\alpha}_k(\eta_1) &= -\frac{mk}{2\omega^2}\dot{a}e^{2i\int^{\eta_1} d\eta\,\omega(\eta)}\beta_k(\eta_1) && \text{(B.8)} \\ \dot{\beta}_k(\eta_1) &= \frac{mk}{2\omega^2}\dot{a}e^{-2i\int^{\eta_1} d\eta\,\omega(\eta)}\alpha_k(\eta_1), && \text{(B.9)} \end{aligned}$$

with the initial conditions $\alpha_k(\eta_i) = 1$, $\beta_k(\eta_i) = 0$ for the time η_i early enough that the mode is inside the dS event horizon. Since we expect $|\beta_k| \ll 1$ and $a_k \approx 1$, we can replace $\alpha = 1$ in Eq. (B.9) and formally write the solution as

$$\beta_k(\eta_f) = \int_{\eta_i}^{\eta_f} d\tau \frac{mk}{2\omega^2} \dot{a}(\tau) e^{-2i\int^{\tau} d\eta \omega(\eta)}. \tag{B.10}$$

The steepest descent method can be applied to evaluate this integral in a similar fashion as was done for the scalar case in [73]. Despite the different k dependence in Eq. (B.10), the result is the same as Eq. (41) of [73]:

$$|\beta_k|^2 \approx \exp\left\{-4\left[\frac{[k/a(r)]^2}{m\sqrt{H^2(r)+R(r)/6}} + \frac{m}{\sqrt{H^2(r)+R(r)/6}}\right]\right\}, \tag{B.11}$$

in which r is the real part of the complexified conformal time $\tilde{\eta}$ at which $\omega(\tilde{\eta}) = 0$ and R is the Ricci scalar. This is approximately due to the fact that the branch point occurs when $\omega = 0$, such that the dominant contribution occurs when $|k/a| \sim m$. Eq. (B.11) leads to the particle number density (fermion plus anti-fermion) as

$$\rho_\psi(t) \approx \frac{1}{2\pi^{3/2}} \left(\frac{a(r)}{a(t)}\right)^3 m \left[\frac{m}{4}\sqrt{H^2(r)+R(r)/6}\right]^{3/2} \exp\left(\frac{-4m}{\sqrt{H^2(r)+R(r)/6}}\right). \tag{B.12}$$

To estimate the relic abundance from this equation, one can use the formula

$$\Omega_\psi h^2 \sim 100 \left(\frac{T_{\text{rh}}}{10^9\text{GeV}}\right)\left(\frac{H(t_e)}{10^{13}\text{GeV}}\right)^{-2} \frac{\rho_\psi(t_e)}{(10^{12}\text{GeV})^4}, \tag{B.13}$$

where one is only formally evaluating $\rho_\psi(t_e)$ at the end of inflation time t_e even though the particle densities are well defined at times far later than time. Unlike the formulae presented in the body of the text, the exponential sensitivity and the approximations made in obtaining the saddle-point does not allow one to guarantee an order of magnitude numerical accuracy, especially for large $m/H(r)$ [73]. However, the spectral and mass cutoffs can be well estimated by Eqs. (B.11) and (B.12).

Appendix C

Details for Isocurvature Correlator Computation

C.1 Asymptotic behavior of $\langle \psi_x \bar{\psi}_y \rangle$ at large r

In this section we want to derive the result about leading order contribution to $\langle n_{\psi,x} n_{\psi,y} \rangle$, i.e. Eq.(6.21). By Wick contraction, this reduces to computing the field correlator $\langle \psi_x \bar{\psi}_y \rangle$. The standard way to compute the correlator is to plug in the mode decomposition Eq.(A.15) and compute the mode functions $\{U_i, V_i\}$. The difficulties lies in how to obtain the mode functions on a curved spacetime. For inflationary background spacetime, one can use the de Sitter spacetime as an approximation and obtain exact analytic solutions. However, it is unclear how do these mode solutions evolve after inflation ends. Here we give an approach that answers this question.

First, we plug in the mode decomposition to the equal-time correlator:

$$\langle \psi_x \bar{\psi}_y \rangle \tag{C.1}$$

$$= \sum_i U_i(x) \bar{U}_i(y) \tag{C.2}$$

$$= \int d^3k \sum_{r=\pm 1} \frac{1}{a_x^3} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{(2\pi)^3} \begin{pmatrix} |u_{A,k,x^0}|^2 & -r\, u_{A,k,x^0} u^*_{B,k,x^0} \\ r\, u_{B,k,x^0} u^*_{A,k,x^0} & -|u_{B,k,x^0}|^2 \end{pmatrix} \otimes h_{\hat{k},r} h^\dagger_{\hat{k},r} \tag{C.3}$$

$$= \int d^3k \frac{1}{a_x^3} \frac{e^{i\vec{k}\cdot\vec{r}}}{(2\pi)^3} \begin{pmatrix} |u_{A,k,x^0}|^2 \otimes I_2 & -u_{A,k,x^0} u^*_{B,k,x^0} \otimes (\hat{k}\cdot\vec{\sigma}) \\ u_{B,k,x^0} u^*_{A,k,x^0} \otimes (\hat{k}\cdot\vec{\sigma}) & -|u_{B,k,x^0}|^2 \otimes I_2 \end{pmatrix} \tag{C.4}$$

where we performed the spin-sum in the last step. Since

$$\int d^3k \frac{e^{i\vec{k}\cdot\vec{r}}}{(2\pi)^3}|u_{A,k,x^0}|^2 = \int d^3k \frac{e^{i\vec{k}\cdot\vec{r}}}{(2\pi)^3}(1-|u_{B,k,x^0}|^2) \tag{C.5}$$

$$= \delta^3(\vec{r}) - \int d^3k \frac{e^{i\vec{k}\cdot\vec{r}}}{(2\pi)^3}|u_{B,k,x^0}|^2 \tag{C.6}$$

and $\vec{r} \neq 0$, we see the diagonal elements are the same. Then we perform the angular integral $d^2\hat{k}$. Recall that

$$\int d^3k\, e^{i\vec{k}\cdot\vec{r}} f(k) = \int 4\pi k^2 dk\, \frac{\sin(kr)}{kr} f(k) \tag{C.7}$$

$$\int d^3k\, e^{i\vec{k}\cdot\vec{r}} \hat{k}_i f(k) = \int d^3k\, e^{i\vec{k}\cdot\vec{r}} k_i \frac{f(k)}{k} \tag{C.8}$$

$$= (-i\partial_{r_i}) \int d^3k\, e^{i\vec{k}\cdot\vec{r}} \frac{f(k)}{k} \tag{C.9}$$

$$= (-i\hat{r}_i \partial_r) \int 4\pi k^2 dk\, \frac{\sin(kr)}{kr}\frac{f(k)}{k} \tag{C.10}$$

After the angular integral, we have

$$\langle \psi_x \bar{\psi}_y \rangle = \int \frac{4\pi k^2 dk}{(2\pi)^3} \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \tag{C.11}$$

$$A = |u_{A,k,\eta}|^2 \cdot \frac{\sin(kr)}{kr} \tag{C.12}$$

$$B = (i\hat{r}\cdot\vec{\sigma}) u_{A,k,\eta} u^*_{B,k,\eta} \cdot \partial_r \frac{\sin(kr)}{kr}\frac{1}{k} \tag{C.13}$$

$$C = -|u_{B,k,\eta}|^2 \cdot \frac{\sin(kr)}{kr} \tag{C.14}$$

It is sufficient to study these two integrals for the diagonal and off-diagonal elements.

$$I_{11} = I_{22} = \int_0^\infty \frac{4\pi k^2 dk}{(2\pi)^3} |u_{A,k,\eta}|^2 \cdot \frac{\sin(kr)}{kr} \tag{C.15}$$

$$I_{12} = I^*_{21} = \partial_r \int_0^\infty \frac{4\pi k^2 dk}{(2\pi)^3} u_{A,k,\eta} u^*_{B,k,\eta} \frac{\sin(kr)}{kr}\frac{1}{k} \tag{C.16}$$

Now, we only need to find the mode function u_A, u_B, and perform the mode sum.

Let's consider the mode functions first. Since we are interested in evaluting the fermion field correlator at a time when the fermion production has ended, i.e. when $m \gg H(x^0)$ and in the limit $r \to \infty$, we can make the following approximations about the mode functions $\{u_{A,k,x^0}, u_{B,k,x^0}\}$. First, since the particle production has stopped, the non-adiabatic parameter is suppressed by $\frac{H(t)}{m}$, thus we can approximately replace the Bogoliubov coefficients by their late time asymptotic values, i.e.

$$\alpha_{k,x^0} \approx \alpha_k, \beta_{k,x^0} \approx \beta_k. \tag{C.17}$$

Second, since we want to capture the particle production effect on the correlator and the produced particles are non-relativistic at the time of production, by the time x^0 which is sufficiently long after the production has ended, we may approximate the produced modes all have $k \ll a(x^0)m$. Thus, the WKB modes can be approximated by

$$\left(\begin{array}{c} u_A \\ u_B \end{array}\right)^{WKB}_{k,\eta,IR} = \left(\begin{array}{c} \sqrt{\frac{\omega+am}{2\omega}} \\ \sqrt{\frac{\omega-am}{2\omega}} \end{array}\right) e^{-i\int^{\eta}\omega d\eta'} \to \left(\begin{array}{c} \frac{1}{\sqrt{2}} \\ 0 \end{array}\right) e^{-i\int^{\eta}\omega d\eta'}. \tag{C.18}$$

Combining these two approximations, we have

$$\left(\begin{array}{c} u_A \\ u_B \end{array}\right)^{in}_{k,\eta,IR} \approx \left(\begin{array}{c} \alpha_k \frac{1}{\sqrt{2}} e^{-i\int^{\eta}\omega d\eta'} \\ -\beta_k \frac{1}{\sqrt{2}} e^{i\int^{\eta}\omega d\eta'} \end{array}\right) \tag{C.19}$$

Thus we can easily evaluate I_{11}, I_{12}:

$$2\pi^2 I_{11,IR} = \int_0^{\infty} k^2 dk\, \frac{1}{2}|\alpha_k|^2 \cdot \frac{\sin(kr)}{kr} \tag{C.20}$$

$$= \int_0^{\infty} k^2 dk\, \frac{1}{2}[1-n(k)] \cdot \frac{\sin(kr)}{kr} \tag{C.21}$$

$$= \frac{1}{r}\mathrm{Im}\int_0^{\infty} kdk\, \frac{1}{2}[1-n(k)] \cdot e^{ikr} \tag{C.22}$$

We note that for the contribution from 1 vanishes

$$\frac{1}{r}\mathrm{Im}\int_0^{\infty} kdk\,[1] \cdot e^{ikr} = \frac{1}{r}\mathrm{Im}\int_0^{\infty} (is)ids\,[1] \cdot e^{-sr} = 0 \tag{C.23}$$

For the contribution from $n(k)$, we may assume it to be a real analytic function on $\mathbb{R}^+$and can be analytically continuated to upper-right quadrant of the complex k plane. The location of singularity of $n(k)$ determines contour of k. For example, we may consider the $n(k)$ for heavy fermion case ($m > H_{inf}$):

$$n(k)_{heavy} = \exp\left[-\frac{4(k/a_{nad})^2}{mH} - \frac{4m}{H}\right] \tag{C.24}$$

where a_{nad} is at the non-adiabatic time point. In this case, the non-adiabatic time is the transition from de Sitter era to the reheating era, i.e. $a_{nad} = a_e$. One can apply steepest descent to find that

$$
\begin{aligned}
& 2\pi^2 I_{11,heavy,IR} && \text{(C.25)}\\
= & -\frac{1}{r}\exp[-4\frac{m}{H}]\mathrm{Im}\int_0^\infty kdk\,\exp\left[-\frac{4(k/a_e)^2}{mH}+ikr\right] && \text{(C.26)}\\
= & -\frac{1}{r}\exp[-\frac{4m}{H}]\mathrm{Im}\int_0^\infty kdk\,\exp\left[-(\frac{2k/a_e}{\sqrt{mH}}+i\frac{1}{4}\sqrt{mH}r)^2-\frac{1}{16}mHr^2\right] && \text{(C.27)}\\
\approx & -\frac{1}{r}\exp[-\frac{4m}{H}-\frac{1}{16}mHr^2](a_e^2 mH)\mathrm{Im}[-i\frac{1}{4}\sqrt{mH}a_e r\frac{1}{2}\sqrt{\pi}] && \text{(C.28)}\\
= & \frac{1}{8}\sqrt{\pi}a_e^3(mH)^{\frac{3}{2}}\exp[-\frac{4m}{H}-\frac{1}{16}a_e^2 mHr^2] && \text{(C.29)}
\end{aligned}
$$

For light fermion, we may approximate the number density spectrum as

$$
n(k)_{light} = \frac{1}{1+\exp(\frac{k^2}{(a_{nad}m)^2})} \tag{C.30}
$$

where the non-adiabatic point occurs when H drops below m, i.e. $a_{nad} = a(\eta_*) = a_*$. This ansatz is only used to mimick the cut-off of the spectrum at $k \sim a_{nad}m$. The singularity lies at

$$
\frac{k^2}{a_*^2 m^2} = (2n+1)\pi i, \qquad n = 0, 1, 2\cdots \tag{C.31}
$$

or $k_{*,n} = a_* m\sqrt{(2n+1)\pi}e^{\frac{\pi}{4}i}$. Again, one can perform the steepest descent around the $n = 0$ singularity $k_* = a_* m\sqrt{\pi}e^{\frac{\pi}{4}i}$. Let $\delta = (k - k_*)/a_* m$, we have

$$
\begin{aligned}
2\pi^2 I_{11,light,IR} &= -\frac{1}{r}\mathrm{Im}\left(\int_0^\infty kdk\,\frac{1}{1+\exp(\frac{k^2}{(a_*m)^2})}e^{ikr}\right) && \text{(C.32)}\\
&= -\frac{1}{r}\mathrm{Im}\left(e^{ik_*r}k_*a_*m\int_C d\delta\frac{1}{1-\exp 2\frac{k_*}{a_*m}\delta}e^{ia_*mr\delta}\right) && \text{(C.33)}\\
&= -\frac{1}{r}\mathrm{Im}\left(-e^{ik_*r}(a_*m)^2(\pi i)\right) && \text{(C.34)}\\
&= \pi a_*^3\frac{m^2}{a_*r}\exp[-\sqrt{\frac{\pi}{2}}a_*mr]\cos(\sqrt{\frac{\pi}{2}}a_*mr) && \text{(C.35)}
\end{aligned}
$$

For both the heavy and light fermion case, $I_{11} \propto \exp(-a_* Mr)$, where $a_* M$ is the scale that $n(k)$ start to cut-off. This can be viewed as the screening effect of the produced fermions. We should also remind ourself that the UV vacuum contributions also exist, which scales as

$$
I_{11,UV} \propto \exp[-a_\eta mr] \tag{C.36}
$$

due to the singularity at $k = a_\eta m$ in the mode functions u_A^{WKB}, u_B^{WKB}. Thus we have shown that the diagonal element of Eq(C.11) is always exponentially suppressed.

Next, we turn to look at the off diagonal element I_{12}. Unlike the I_{11} case, whose integrand $|u_A|^2$ has constant asymptotic value in the IR region, the I_{12}'s IR contribution

$$u_{A,k,\eta}u^*_{B,k,\eta} = \alpha_k\beta^*_k e^{-2i\int^\eta \omega d\eta'} \tag{C.37}$$

contains e^{-2imt} time dependence. Physically, if we decompose the in-state into WKB vacuum and excitation state

$$|\text{in,vac}\rangle =\sim |\text{WKB,vac}\rangle + \sim |\text{WKB,2-particles}\rangle + \sim |\text{WKB,4-particles}\rangle \tag{C.38}$$

then this term comes from the interference term

$$\langle \text{WKB, vac}|\psi_x\bar{\psi}_y|\text{WKB, 2-particles}\rangle \in \langle \text{in, vac}|\psi_x\bar{\psi}_y|\text{in, vac}\rangle. \tag{C.39}$$

If we care about r large enough, for example corresponding to the CMB observation scale at recombination, we may assume the relevant k scale exit horizon and become non-relativistic during inflation. Thus we may safely use the dS mode function to evaluate $I_{12,IR,CMB}$.

Recall that during dS era, we have Eq. (5.23), where we choose the end of inflation time t_e as the reference point. Thus

$$u_{A,k,\eta}u^*_{B,k,\eta} = \frac{1}{2\pi}e^{-2im(t-t_e)+2i\frac{m}{H}\ln(2k/a_eH)}\Gamma^2(\frac{1}{2}-i\frac{m}{H}) \tag{C.40}$$

Performing the integral using steepest descent, we found the leading contribution comes from $k\sim 0$ singularity in $u_{A,k,\eta}u^*_{B,k,\eta}$. We note that the k dependent phase factor $e^{2i\frac{m}{H}\ln(2k/H)}$ cannot be absorbed by a redefinition of the mode functions $u_{A,k,\eta}, u_{B,k,\eta}$, since this phase factor depends on the relative phase of $u_{A,k,\eta}, u_{B,k,\eta}$ which is fixed by the Bunch-Davies initial condition.

Pluggin in the Eq(C.16), we have

$$
\begin{aligned}
& 2\pi^2 I_{12,IR} && \text{(C.41)}\\
=\ & \frac{1}{2\pi}e^{-2im(t-t_e)}\Gamma^2(\frac{1}{2}-i\frac{m}{H})\partial_r\left[\frac{1}{r}\int_0^\infty dk\, e^{2i\frac{m}{H}\ln(2k/a_eH)}\sin(kr)\right] && \text{(C.42)}\\
=\ & \frac{1}{2\pi}e^{-2im(t-t_e)}\Gamma^2(\frac{1}{2}-i\frac{m}{H})\Gamma(1+2i\frac{m}{H}) && \text{(C.43)}\\
& (2+2i\frac{m}{H})\cosh(\frac{m}{H})\left(\frac{2}{a_eHr}\right)^{2i\frac{m}{H}}\frac{1}{r^3} && \text{(C.44)}\\
=\ & -e^{-2im(t-t(r))+i\phi(\frac{m}{H})}r^{-3}\sqrt{\frac{2\pi\frac{m}{H}}{\sinh(2\pi\frac{m}{H})}\left(1+\left(\frac{m}{H}\right)^2\right)} && \text{(C.45)}
\end{aligned}
$$

where $\phi(\frac{m}{H}) = \text{Arg}(\Gamma(2+ix)\Gamma(\frac{1}{2}-ix))$ and $t(r)$ is the time when $a(t_r)Hr = 4$. We may consider the light mass limit

$$2\pi^2 I_{12,IR,light} \approx -e^{-2im(t-t(r))}r^{-3} \tag{C.46}$$

and the heavy mass limit

$$2\pi^2 I_{12,IR,heavy} \approx -(4\pi)^{\frac{1}{2}} \left(\frac{m}{H}\right)^{\frac{3}{2}} \exp(-\pi\frac{m}{H}) e^{-2im(t-t(r))} r^{-3} \tag{C.47}$$

We may also consider the effect of having an IR cut-off k_{IR}, which is the scale that exit horizon at the beginning of inflation. Such an IR cut-off will introduce a $\exp(-k_{IR}r)$ type of exponential suppression factor. However, for observable universe with comoving radius R_{obs}, as long as $k_{IR}R_{obs} \ll 1$, we may ignore this suppression factor.

After evaluating the matrix element for the fermion correlators, we find that

1. For the light fermion case, i.e. $m \ll H_{inf}$, in the limit $r \to \infty$

$$\langle \psi_x \bar{\psi}_y \rangle \approx \frac{1}{a_x^3} \frac{1}{2\pi^2} \begin{pmatrix} A & B \\ B^* & A \end{pmatrix} \tag{C.48}$$

where

$$A = \frac{1}{2}\pi a_*^3 \frac{m^2}{a_* r} \exp[-\sqrt{\frac{\pi}{2}} a_* mr] \cos(\sqrt{\frac{\pi}{2}} a_* mr) \tag{C.49}$$

$$B = -i\hat{r} \cdot \vec{\sigma} e^{-2im(t-t_r)} r^{-3} \tag{C.50}$$

where a_* in evaluated at η_*.

2. For the heavy fermion case, i.e. $m \gg H_{inf}$, in the limit $r \to \infty$, we find in Eq. (C.48)

$$A = \frac{1}{16}\sqrt{\pi} a_e^3 (mH_e)^{\frac{3}{2}} \exp[-\frac{4m}{H_e} - \frac{1}{16} a_e^2 m H_e r^2] \tag{C.51}$$

$$B = -i\hat{r} \cdot \vec{\sigma} (4\pi)^{\frac{1}{2}} \left(\frac{m}{H_e}\right)^{\frac{3}{2}} \exp(-\pi\frac{m}{H_e}) e^{-2im(t-t(r))} r^{-3} \tag{C.52}$$

and a_e is evaluated at the end of inflation.

Finally, we plug in the field correlator to $\langle n_{\psi,x} n_{\psi,y} \rangle$, and drop the term that are exponentially suppressed when $r \to \infty$, to get Eq. (6.21).

C.2 Relative suppresion of $\langle in|[O_x, O_y]|in\rangle$

In this subsection, we want compare the dependence on the scale factor $a(t)$ between $\langle in|[O_x, O_y]|in\rangle$ and $\langle in|\{O_x, O_y\}|in\rangle$, where O_x is a bosonic hermitian operator and x, y are spacetime points located near the end of inflation. For simplicity, we take H as a constant. In particular, we are interested in the cases where $O = \sigma, \bar{\psi}\psi, \zeta$. We want to show that the commutator of O suffers from additional suppression factor than the anti-commtutator.

In general, the diagonal matrix elements of products of hermitian operator obeys

$$(\langle in|O_x O_y|in\rangle)^* = \langle in|O_y O_x|in\rangle \tag{C.53}$$

therefore

$$\langle in|[O_x, O_y]|in\rangle = 2i\mathrm{Im}\langle in|O_x O_y|in\rangle \tag{C.54}$$

$$\langle in|\{O_x, O_y\}|in\rangle = 2\mathrm{Re}\langle in|O_x O_y|in\rangle \tag{C.55}$$

We can just study $\langle in|O_x O_y|in\rangle$. We may use the mode expansion of the field operator to evaluate such an expression, and focus on modes that are outside of horizon at both times η_x, η_y.

We shall first take $O = \sigma$. We assume that the scalar is light, i.e. $m_\sigma < \frac{3}{2}H$, such that ν is real.

$$\langle in|\sigma_x\sigma_y|in\rangle = \sum_i u_i(x)u_i^*(y) \tag{C.56}$$

$$= \int d^3k \frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}a(\eta_x)} f_k(\eta_x) \frac{e^{-i\vec{k}\cdot\vec{y}}}{(2\pi)^{3/2}a(\eta_y)} f_k^*(\eta_y) \tag{C.57}$$

$$= \int 4\pi k^2 dk \frac{[\int d^2\hat{k} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}]}{(2\pi)^3 a_x^{3/2} a_y^{3/2}} \frac{1}{H}\frac{\pi}{4}[J_x J_y + Y_x Y_y + i(Y_x J_y - J_x Y_y)] \tag{C.58}$$

where $J_x = J_\nu(\frac{k}{a_x H}), Y_x = Y_\nu(\frac{k}{a_x H})$ are the first and second kinds of Bessel functions with real values. The $d^2\hat{k}$ is the angular integral with normalization $\int d^2\hat{k} = 1$, and $\int d^2\hat{k} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} = \sin(kr)/kr$ is real. If we focus on the k modes that are outside of horizon, i.e. $k/aH \ll 1$, we may use the small argument expansion of the Bessel function, i.e. when ($0 < z < \sqrt{1+\nu}$)

$$J_\nu(z) \approx \frac{1}{\Gamma(\alpha+1)}\left(\frac{z}{2}\right)^\nu \tag{C.59}$$

$$Y_\nu(z) \approx -\frac{\Gamma(\alpha)}{\pi}\left(\frac{2}{z}\right)^\nu. \tag{C.60}$$

Then, under the common scaling of $a_x \to \lambda a_x, a_y \to \lambda a_y$, with λ increasing, we see the various term in the correlator scales as

$$a_x^{-3/2} a_y^{-3/2} J_x J_y \propto \lambda^{-2\nu-3} \tag{C.61}$$

$$a_x^{-3/2} a_y^{-3/2} Y_x Y_y \propto \lambda^{2\nu-3} \tag{C.62}$$

$$a_x^{-3/2} a_y^{-3/2}(Y_x J_y - J_x Y_y) \propto \lambda^{-3} \tag{C.63}$$

Thus, we see under this common scaling, the IR contribution to the two point functions are

$$\langle in|\{\sigma_x,\sigma_y\}|in\rangle_{IR} = 2\int_{IR} 4\pi k^2 dk \frac{[\int d^2\hat{k} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}]}{(2\pi)^3 a_x^{3/2} a_y^{3/2}} \frac{1}{H}\frac{\pi}{4}(J_x J_y + Y_x Y_y) \propto \lambda^{2\nu-3} \quad (C.64)$$

$$\langle in|[\sigma_x,\sigma_y]|in\rangle_{IR} = 2i\int_{IR} 4\pi k^2 dk \frac{[\int d^2\hat{k} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}]}{(2\pi)^3 a_x^{3/2} a_y^{3/2}} \frac{1}{H}\frac{\pi}{4}(Y_x J_y - J_x Y_y) \propto \lambda^{-3} \quad (C.65)$$

Thus, we have shown under the scaling $a \to \lambda a$, the commutator of σ is suppressed by $\lambda^{-2\nu}$ factor relative to its anti-commutator. For small mass scalar, $\lambda^{-2\nu} \approx \lambda^{-3+\frac{2m^2}{3H^2}}$.

For the case of $O = \zeta$, we have similar statements as the scalar case with $\nu = \frac{3}{2}$, i.e. $\langle[\zeta_x,\zeta_y]\rangle_{IR}$ is suppressed by λ^{-3} relative to $\langle\{\zeta_x,\zeta_y\}\rangle_{IR}$ under the scaling of $a \to \lambda a$.

Next, we consider the case of $O = \bar{\psi}\psi$. Using the mode decomposition Eq.(A.15) and mode functions Eq. (A.17,A.18), we have

$$\langle\bar{\psi}\psi_x\bar{\psi}\psi_y\rangle \quad (C.66)$$

$$= \sum_{i,j} \bar{V}_{i,x} U_{j,x} \bar{U}_{j,y} V_{i,y} \quad (C.67)$$

$$= \sum_{i,j} \frac{1}{a_x^3 a_y^3} \frac{e^{i(\vec{k}_i+\vec{k}_j)\cdot(\vec{x}-\vec{y})}}{(2\pi)^6} [h_i^T(i\sigma_2)h_j][h_j^\dagger(-i\sigma_2)h_i^*] F_{ij,x} F_{ij,y}^* \quad (C.68)$$

where

$$F_{ij,x} = r_i u_{B,i,x} u_{A,j,x} + (i \leftrightarrow j) \quad (C.69)$$

$$F_{ij,x}F_{ij,y}^* = 2[r_i u_{B,i,x} u_{A,j,x} + (i \leftrightarrow j)](r_i u_{B,i,y}^* u_{A,j,y}^*) \quad (C.70)$$

$$= 2[u_{B,i,x} u_{A,j,x} u_{B,i,y}^* u_{A,j,y}^* + r_i r_j u_{B,i,x} u_{A,j,x} u_{B,j,y}^* u_{A,i,y}^*]. \quad (C.71)$$

We note that in Eq.(C.68), the factor $e^{i(\vec{k}_i+\vec{k}_j)\cdot(\vec{x}-\vec{y})}$ after angular average is real, and the factor $[h_i^T(i\sigma_2)h_j][h_j^\dagger(-i\sigma_2)h_i^*] = |[h_i^T(i\sigma_2)h_j]|^2$ is also real, thus the imaginary and real part of $F_{ij,x}F_{ij,y}^*$ correspond to the commutator and anti-commutator respectively.

Next, we consider the two terms in Eq. (C.71) one by one, using explicit expression of fermion de Sitter mode function to get

$$u_{B,i,x}u_{A,j,x}u^*_{B,i,y}u^*_{A,j,y} \tag{C.72}$$

$$= \sqrt{\frac{\pi}{4}\frac{k_i}{a_xH}}e^{i\frac{\pi}{2}(1+i\frac{m}{H})})H^{(1)}_{\frac{1}{2}+i\frac{m}{H}}(\frac{k_i}{a_xH})\sqrt{\frac{\pi}{4}\frac{k_j}{a_xH}}e^{i\frac{\pi}{2}(1-i\frac{m}{H})})H^{(1)}_{\frac{1}{2}-i\frac{m}{H}}(\frac{k_j}{a_xH}) \tag{C.73}$$

$$\sqrt{\frac{\pi}{4}\frac{k_i}{a_yH}}e^{-i\frac{\pi}{2}(1-i\frac{m}{H})})H^{(2)}_{\frac{1}{2}-i\frac{m}{H}}(\frac{k_i}{a_yH})\sqrt{\frac{\pi}{4}\frac{k_j}{a_yH}}e^{-i\frac{\pi}{2}(1+i\frac{m}{H})})H^{(2)}_{\frac{1}{2}+i\frac{m}{H}}(\frac{k_j}{a_yH}) \tag{C.74}$$

$$= \sqrt{\frac{\pi}{4}\frac{k_i}{a_xH}}\sqrt{\frac{\pi}{4}\frac{k_j}{a_xH}}\sqrt{\frac{\pi}{4}\frac{k_i}{a_yH}}\sqrt{\frac{\pi}{4}\frac{k_j}{a_yH}} \tag{C.75}$$

$$(J_{+,i,x}+iY_{+,i,x})(J_{-,j,x}+iY_{-,j,x})(J_{-,i,y}-iY_{-,i,y})(J_{+,j,y}-iY_{+,j,y}) \tag{C.76}$$

where

$$J_{\pm,i,x} = J_{\frac{1}{2}\pm i\frac{m}{H}}(\frac{k_i}{a_xH}), \quad Y_{\pm,i,x} = Y_{\frac{1}{2}\pm i\frac{m}{H}}(\frac{k_i}{a_xH}). \tag{C.77}$$

Using the small z expansion of Bessel function again, where Re $(\nu) = \frac{1}{2}$ in all the cases, we can extract its scaling behavior under $a \to \lambda a$,

$$(J_{+,i,x}+iY_{+,i,x})(J_{-,j,x}+iY_{-,j,x})(J_{-,i,y}-iY_{-,i,y})(J_{+,j,y}-iY_{+,j,y}) \tag{C.78}$$

$$= Y_{+,i,x}Y_{-,j,x}Y_{-,i,y}Y_{+,j,y}\cdots\cdots \propto \lambda^2, \text{real} \tag{C.79}$$

$$-iJ_{+,i,x}Y_{-,j,x}Y_{-,i,y}Y_{+,j,y} - iY_{+,i,x}J_{-,j,x}Y_{-,i,y}Y_{+,j,y}\cdots\cdots \propto \lambda^1, \text{imaginary} \tag{C.80}$$

$$+iY_{+,i,x}Y_{-,j,x}J_{-,i,y}Y_{+,j,y} + iY_{+,i,x}Y_{-,j,x}Y_{-,i,y}J_{+,j,y}\cdots\cdots \propto \lambda^1, \text{imaginary} \tag{C.81}$$

$$+\text{terms subdominant in } \lambda \text{expansion}. \tag{C.82}$$

Thus the imaginary part is suppresed by λ^{-1} relative to the real part. We can do similar analysis to the second part $r_ir_ju_{B,i,x}u_{A,j,x}u^*_{B,j,y}u^*_{A,i,y}$ in Eq. (C.71) and found the same behavior. Thus, for $\bar{\psi}\psi$ operator, we have the following scaling law

$$\langle\{\bar{\psi}\psi_x, \bar{\psi}\psi_y\}\rangle_{IR} \propto \lambda^{-6} \tag{C.83}$$

$$\langle[\bar{\psi}\psi_x, \bar{\psi}\psi_y]\rangle_{IR} \propto \lambda^{-7}. \tag{C.84}$$

Thus, we see the commutator for $\bar{\psi}\psi$ gives additional suppression of a^{-1} factor compared with the anti-commutator, whereas the commutator for σ and ζ gives additional suppression of a^{-3} factor.

C.3 Explict check of the mass term insertion formula

Expressing both side of Eq.(6.36) using the mode sum, we see the left hand side is

$$-i\int^{y}(dw)\langle[\bar{\psi}\psi_x,\bar{\psi}\psi_w]\rangle=\frac{16}{a_x^3}\int^{y^0}dw^0\,a_w\int\frac{d^3k}{(2\pi)^3}\text{Im}[(u_{A,k}u_{B,k})_x(u_{A,k}u_{B,k})^*_w] \tag{C.85}$$

and the right hand side is

$$\partial_m\langle\bar{\psi}\psi_x\rangle=\frac{2}{a_x^3}\int\frac{d^3k}{(2\pi)^3}\partial_m(|u_B|^2-|u_A|^2) \tag{C.86}$$

Thus, we only need to check for each given k, the following equation is right

$$\partial_m(|u_B|^2-|u_A|^2)=8\int^{y^0}dw^0\,a_w\text{Im}[(u_{A,k}u_{B,k})_x(u_{A,k}u_{B,k})^*_w] \tag{C.87}$$

From the left hand side, we have

$$\partial_m(|u_B|^2-|u_A|^2)=-2\text{Re}\left[\begin{pmatrix}u_A^* & u_B^*\end{pmatrix}\sigma_3\frac{\partial}{\partial m}\begin{pmatrix}u_A\\ u_B\end{pmatrix}_{k,x}\right] \tag{C.88}$$

and upon expressing mode function at time x^0 in term of evolution operator acting on the initial value, we have

$$\frac{\partial}{\partial m}\begin{pmatrix}u_A\\ u_B\end{pmatrix}_{k,x} = \frac{\partial}{\partial m}\left\{T\exp\left\{-i\int_{\eta_i}^{x^0}d\eta\begin{pmatrix}am & k\\ k & -am\end{pmatrix}\right\}\begin{pmatrix}u_A\\ u_B\end{pmatrix}_{k,\eta_i}\right\} \tag{C.89}$$

$$= -i\int_{\eta_i}^{x}dz^0\,U(x^0\leftarrow z^0)\,\frac{\partial}{\partial m}\begin{pmatrix}am & k\\ k & -am\end{pmatrix} \tag{C.90}$$

$$\times U(z^0\leftarrow\eta_i)\begin{pmatrix}u_A\\ u_B\end{pmatrix}_{k,i} \tag{C.91}$$

Combining these two expression, we can obtain the desired result after some algebra.

However, the remaining d^3k integrals in Eq. (C.85) and Eq. (C.86) are UV divergent. To make them finite, we express both side in terms of Bogoliubov coefficients and dropped the pure vacuum contribution to get

$$-i\int^{x^0}(dw)\langle[\bar{\psi}\psi_x,\bar{\psi}\psi_w]\rangle \tag{C.92}$$

$$\approx\ 16\int\frac{d^3k}{(2\pi a_x)^3}(\frac{am}{\omega_k})_x\int^{x}d\eta_w\,a_w(\frac{am}{\omega})_w\text{Im}[(\alpha\beta)_x(\alpha\beta)^*_w] \tag{C.93}$$

and

$$\partial_m \langle \bar{\psi}\psi_x \rangle \tag{C.94}$$

$$\approx \frac{2}{a_x^3} \int \frac{d^3k}{(2\pi)^3} \partial_m [2|\beta_{k,x}|^2 \frac{a_x m}{\omega_{k,x}}] \approx \frac{4}{a_x^3} \int \frac{d^3k}{(2\pi)^3} (\frac{a_x m}{\omega_{k,x}}) \partial_m |\beta_{k,x}|^2 \tag{C.95}$$

Now, we only need to check

$$\partial_m |\beta_{k,x}|^2 = 4 \int^x d\eta_w \, a_w (\frac{am}{\omega})_w \mathrm{Im}[(\alpha\beta)_x (\alpha\beta)_w^*] \tag{C.96}$$

Suppose, x^0 is late enough such that $\beta_{k,x}$ is stablized and equals to its value at asymptotic future β_k, then we get

$$\partial_m |\beta_k|^2 \tag{C.97}$$

$$= 2\mathrm{Re} \left\{ \beta_k^* \partial_m \left[(0, -1) T \exp \left\{ -i \int_{\eta_i}^{x_f^0} d\eta \begin{pmatrix} am & k \\ k & -am \end{pmatrix} \right\} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right] \right\} \tag{C.98}$$

$$= 2\mathrm{Re}\{ \left[(-i) \int_{\eta_i}^{x^0} dz^0 a_z \, |\beta_k^*|^2 (|u_A|^2 - |u_B|^2)_{k,x}^{in} + \alpha_k^* \beta_k^* (2u_A u_B)_{k,z}^{in} \right] \} \tag{C.99}$$

$$= 4 \int_{\eta_i}^{x^0} dz^0 a_z \frac{am}{\omega} \mathrm{Im}(\alpha_k \beta_k)_x (\alpha\beta)_z^* \tag{C.100}$$

Thus, Eq. (6.36) is compatible with the Bogoliubov projection.

Appendix D

Example: Cross Correlator for Fermion

In this appendix, we explicitly check the claim that the non-derivative coupling term does not cause large isocurvature and curvature cross correlator in the fermion case. In other word, we check Eq. (7.26) with operator O as $\bar{\psi}\psi$ and $T_{\mu\nu}$ as in Eq. (A.110). Upon integration by part, we only need to show

$$i\int_{-\infty}^{t}(dz)\langle[\bar{\psi}\psi_x, 3\mathcal{L}_{\psi,0}(z) - (\bar{\psi}i\gamma^i a^{-1}\partial_i\psi)_z]\rangle = 0 \tag{D.1}$$

We may plug in the mode decomposition for ψ as in Eq. (A.15), then sum over the modes running in the loop. We may perform the spacetime integral first, then we shall find the mode sum is UV divergent. We assume such a UV divergence can be regulated via a covariant regulator, such as Pauli-Villars regulator or the Schwinger proper time regulator, and the regulator dependence can be removed by some counter-terms in the operator mixing of $\bar{\psi}\psi_x$. Here, we use the adiabatic subtraction method to extract the particle production contribution. Let

$$A_1 = i\int_{-\infty}^{t}(dz)\langle[\bar{\psi}\psi_x, 3\mathcal{L}_{\psi,0}(z)]\rangle \tag{D.2}$$

$$A_2 = i\int_{-\infty}^{t}(dz)\langle[\bar{\psi}\psi_x, (\bar{\psi}i\gamma^i a^{-1}\partial_i\psi)_z]\rangle \tag{D.3}$$

we only need to show that $A_1 = A_2$. To evaluate A_1, we may consider varying $\hbar$ in the closed-time-path path integral with free field Lagrangian

$$\langle in|\bar{\psi}\psi_x|in\rangle = \int_{CTP} D\psi D\bar{\psi} e^{\frac{i}{\hbar}\int\mathcal{L}}\bar{\psi}\psi_x \tag{D.4}$$

to get

$$\frac{\partial}{\partial \ln \hbar}\langle in|\bar{\psi}\psi_x|in\rangle = \int_{CTP} D\psi D\bar{\psi} e^{\frac{i}{\hbar}\int_{CTP}\mathcal{L}}\bar{\psi}\psi_x(-)\frac{i}{\hbar}\int_{CTP}\mathcal{L} \tag{D.5}$$

$$= (-\frac{i}{\hbar})\int_{-\infty}^{t}(dz)\langle[\bar{\psi}\psi_x, \mathcal{L}_{\psi,0}(z)]\rangle \tag{D.6}$$

Since $\langle in|\bar{\psi}\psi_x|in\rangle$ is a one-loop calculation, we get

$$\frac{\partial}{\partial \ln \hbar}\langle in|\bar{\psi}\psi_x|in\rangle = \langle in|\bar{\psi}\psi_x|in\rangle. \tag{D.7}$$

After setting $\hbar$ back to 1, we get

$$A_1 = -3\langle in|\bar{\psi}\psi_x|in\rangle. \tag{D.8}$$

Then we apply Bogoliubov transformation to obtain the particle production's contribution to A_1:

$$A_1 = -3\frac{2}{a_x^3}\int \frac{d^3k}{(2\pi)^3}(|u_{B,k}(t)|^2 - |u_{A,k}(t)|^2) \tag{D.9}$$

$$= -3\frac{2}{a_x^3}\int \frac{d^3k}{(2\pi)^3}[(|\alpha_{k,t}|^2 - |\beta_{k,t}|^2)(|\bar{u}_{B,k,t}|^2 - |\bar{u}_{A,k,t}|^2) \tag{D.10}$$

$$-8\mathrm{Re}(\alpha_{k,t}\beta_{k,t}\bar{u}^*_{A,k,t}\bar{u}_{B,k,t})]. \tag{D.11}$$

Here we repeat that (u_A, u_B) are the mode function with Bunch-Davies initial condition, while $(\bar{u}_A, \bar{u}_B)$ are WKB mode functions. They are related by time-dependent Bogoliubov coefficients as in Eq. (A.49). If this expression is evaluated when the particle production has ended, then α, β are constant over time. Plug in the explicit expression for WKB modes, we get

$$A_1 = -3\frac{2}{a_x^3}\int \frac{d^3k}{(2\pi)^3}[(1-2|\beta_{k,t}|^2)(\frac{-m}{\omega}) - 8\mathrm{Re}(\alpha_{k,t}\beta_{k,t}\frac{a_x^{-1}k}{2\omega}e^{2i\int \omega dt})] \tag{D.12}$$

where $\omega = \sqrt{a_x^{-2}k^2 + m^2}$. The first term in A_1 is non-oscillatory, and the second term is oscillatory with frequency 2ω. The Bogoliubov subtraction involves subtract the same expression of A_1 with $\alpha = 1, \beta = 0$. And we further simplify by assuming that the time t_x is late enough such that the produced particle are all non-relativistic, i.e. $ka_x^{-1}\omega^{-1} \ll 1$. This approximation enables us to drop the second term and take $m/\omega \approx 1$ in the first term. After these simplification, we get

$$A_{1,IR} = -3\frac{4}{a_x^3}\int \frac{d^3k}{(2\pi)^3}|\beta_{k,t}|^2 \tag{D.13}$$

where subscript IR denote the above procedure in extracting the particle production contribution.

Next we compute A_2. Plug in the mode expansion, we get

$$A_2 = -4\int^x dt_z \int \frac{d^3k}{(2\pi a_x)^3}\frac{k}{a_z}\,\mathrm{Im}(2u_A u_B)_x(u_A^2 - u_B^2)^*_z. \tag{D.14}$$

We may perform the Bogoliubov transformation and drop the oscillatory part to get

$$A_{2,IR} = 16 \int^x dt_z \int \frac{d^3k}{(2\pi a_x)^3} \left(\frac{k}{a_z}\right)^2 \frac{1}{\omega_{k,z}} \frac{m}{\omega_{k,x}} \mathrm{Im}(\alpha_{k,x}\beta_{k,x})(\alpha_{k,z}\beta_{k,z})^* \tag{D.15}$$

After switching order of the time integral and the momentum integral, we find the time integral to be

$$\int^x dt_z \left(\frac{k}{a_z}\right)^2 \frac{1}{\omega_{k,z}} \mathrm{Im}(\alpha_{k,x}\beta_{k,x}\alpha^*_{k,z}\beta^*_{k,z}). \tag{D.16}$$

It is interesting to note that this time integral is peaked around the time when mode k is being excited. Since before k mode is excited, the integrand is suppressed by $\beta_{k,z}$; after k mode is excited, the integrand is suppressed due to $\mathrm{Im}(\alpha_{k,x}\beta_{k,x}\alpha^*_{k,z}\beta^*_{k,z}) \approx \mathrm{Im}(|\alpha_k\beta_k|^2) = 0$. Due to the peak of the time integral, we can approximately take $\alpha_{k,x}\beta_{k,x} \approx \alpha_k\beta_k$ and set the time integral's upper bound to $+\infty$.

We claim that the integral I is related to $\frac{\partial}{\partial \ln k}|\beta_k|^2$. Recall that β_k is the time-independent Bogoliubov coefficients between the in-vacuum and out-vacuum, which can be found from Eq. (A.42):

$$\beta_k = (u^{out}_{B,k,t}, -u^{out}_{A,k,t}) \begin{pmatrix} u^{in}_{A,k,t} \\ u^{in}_{B,k,t} \end{pmatrix} \tag{D.17}$$

$$\alpha_k = (u^{out*}_{A,k,t}, u^{out*}_{B,k,t}) \begin{pmatrix} u^{in}_{A,k,t} \\ u^{in}_{B,k,t} \end{pmatrix} \tag{D.18}$$

where the matching time t can be arbitrarily taken. If we take t to be $+\infty$, then the out-mode is approximately the WKB modes,

$$t \to +\infty : \begin{pmatrix} u^{out}_{A,k,t} \\ u^{out}_{B,k,t} \end{pmatrix} \to \begin{pmatrix} \sqrt{\frac{\omega+m}{2\omega}} \\ \sqrt{\frac{\omega-m}{2\omega}} \end{pmatrix} e^{-i\int\omega} \approx \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\int\omega} \tag{D.19}$$

where the last step is due to $\frac{k}{a} \ll m$ in the asymptotic future. Thus, we can have

$$\beta_k \approx (0, -1) \begin{pmatrix} u^{in}_{A,k,\infty} \\ u^{in}_{B,k,\infty} \end{pmatrix} \tag{D.20}$$

$$\approx (0,-1) T \exp\left\{-i \int_{t_i}^{t_f} dt \begin{pmatrix} m & k/a \\ k/a & -m \end{pmatrix}\right\} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \tag{D.21}$$

where we have used the evolution equation Eq. (A.23) to formally express the in-mode mode functions. The initial and final time t_i and t_f, can be chosen arbitrarily, as long as they are in the asymptotically early and late region. Such arbitrariness in t_i and t_f only affect β_k by a constant phase factor. Next, we consider

$$\begin{aligned} & \frac{\partial}{\partial \ln k}|\beta_k|^2 && \text{(D.22)} \\ = \quad & 2\mathrm{Re}[\beta_k^* \frac{\partial}{\partial \ln k}\beta_k] && \text{(D.23)} \\ = \quad & 2\mathrm{Re}[\beta_k^*(0,-1)\frac{\partial}{\partial \ln k} T\exp\left\{-i\int_{t_i}^{t_f} dt \begin{pmatrix} m & k/a \\ k/a & -m \end{pmatrix}\right\}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}] && \text{(D.24)} \\ = \quad & 2\mathrm{Re}[\beta_k^*(0,-1)[-i\int_{t_i}^{t_f} dt\, \frac{k}{a}\, U(t_f,t)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u^{in}_{A,k,t} \\ u^{in}_{B,k,t} \end{pmatrix}]. && \text{(D.25)} \end{aligned}$$

Using the identity matrix

$$\mathbb{I} = \begin{pmatrix} u^{in}_{A,k,t} \\ u^{in}_{B,k,t} \end{pmatrix}(u^{in,*}_{A,k,t}, u^{in,*}_{B,k,t}) + \begin{pmatrix} u^{in*}_{B,k,t} \\ -u^{in*}_{A,k,t} \end{pmatrix}(u^{in}_{B,k,t}, -u^{in}_{A,k,t}), \tag{D.26}$$

we get

$$\begin{aligned} & \frac{\partial}{\partial \ln k}|\beta_k|^2 && \text{(D.27)} \\ = \quad & 2\mathrm{Re}[\beta_k^*(0,-1)[-i\int_{t_i}^{t_f} dt\, \frac{k}{a}\, U(t_f,t)\mathbb{I}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u^{in}_{B,k,t} \\ u^{in}_{A,k,t} \end{pmatrix}]] && \text{(D.28)} \\ = \quad & 2\mathrm{Re}[\beta_k^*(0,-1)(-i)[\int_{t_i}^{t_f} dt\, \frac{k}{a}\begin{pmatrix} u^{in}_{A,k,t_f} \\ u^{in}_{B,k,t_f} \end{pmatrix} 2\mathrm{Re}(u^{in}_{B,k,t}u^{in,*}_{A,k,t}) && \text{(D.29)} \\ & + \begin{pmatrix} u^{in*}_{B,k,t} \\ -u^{in*}_{A,k,t} \end{pmatrix}((u^{in}_{B,k,t})^2 - (u^{in}_{A,k,t})^2)]] && \text{(D.30)} \\ = \quad & 2\mathrm{Re}[-i\int_{t_i}^{t_f} dt\, \frac{k}{a}[|\beta_k|^2 2\mathrm{Re}(u^{in}_{B,k,t}u^{in,*}_{A,k,t}) + \alpha_k^*\beta_k^*((u^{in}_{B,k,t})^2 - (u^{in}_{A,k,t})^2)]] && \text{(D.31)} \end{aligned}$$

The first term vanishes after taking the real part, the second term gives

$$\frac{\partial}{\partial \ln k}|\beta_k|^2 = 2\int_{t_i}^{t_f} dt\, \frac{k}{a}\mathrm{Im}[\alpha_k^*\beta_k^*((u^{in}_{B,k,t})^2 - (u^{in}_{A,k,t})^2)] \tag{D.32}$$

We use Bogoliubov decomposition on $u_B^2 - u_A^2$ to get

$$\frac{\partial}{\partial \ln k}|\beta_k|^2 \approx 4\int_{t_i}^{t_f} dt\, \left(\frac{k}{a}\right)^2 \frac{1}{\omega_{k,t}}\mathrm{Im}[\alpha_k\beta_k\alpha^*_{k,t}\beta^*_{k,t}] \tag{D.33}$$

where the approximation is due to the omission of the oscillatory terms.

This above identity enables us to write Eq.(D.15) as

$$A_{2,IR} \approx 4\int \frac{d^3k}{(2\pi a_x)^3}\frac{\partial}{\partial \ln k}|\beta_k|^2 \tag{D.34}$$

$$= -3\times 4\int \frac{d^3k}{(2\pi a_x)^3}\,|\beta_k|^2 \tag{D.35}$$

where in the second step, we used integration by part in $\frac{\partial}{\partial \ln k}$. We see $A_{2,IR} = A_{1,IR}$ indeed.

To summarize, to prove Eq. (D.1), we only need to prove $A_1 = A_2$, where A_1 and A_2 are defined in Eq. (D.2) and Eq. (D.3). We used the approximation that t_x is late enough, such that the particle production has stopped and the produced particles are non-relativistic by the time of t_x. We used the Bogoliubov subtraction prescription to extract the particle production contribution, i.e. $A_{1,IR}$ and $A_{2,IR}$. We are able to show that $A_{1,IR}$ and $A_{2,IR}$ are indeed equal.

Bibliography

[1] J. C. Mather, D. Fixsen, R. Shafer, C. Mosier, and D. Wilkinson, "Calibrator design for the COBE far infrared absolute spectrophotometer (FIRAS)," *Astrophys.J.* **512** (1999) 511–520, arXiv:astro-ph/9810373 [astro-ph].

[2] **Supernova Cosmology Project** Collaboration, S. Perlmutter *et al.*, "Measurements of Omega and Lambda from 42 high redshift supernovae," *Astrophys.J.* **517** (1999) 565–586, arXiv:astro-ph/9812133 [astro-ph].

[3] **Supernova Cosmology Project** Collaboration, S. Perlmutter *et al.*, "Discovery of a supernova explosion at half the age of the Universe and its cosmological implications," *Nature* **391** (1998) 51–54, arXiv:astro-ph/9712212 [astro-ph].

[4] D. Langlois and B. van Tent, "Isocurvature modes in the CMB bispectrum," arXiv:1204.5042 [astro-ph.CO].

[5] A. D. Linde, "Generation of Isothermal Density Perturbations in the Inflationary Universe," *Phys.Lett.* **B158** (1985) 375–380.

[6] J. Silk and M. S. Turner, "Double Inflation," *Phys.Rev.* **D35** (1987) 419.

[7] R. Bean, J. Dunkley, and E. Pierpaoli, "Constraining Isocurvature Initial Conditions with WMAP 3-year data," *Phys.Rev.* **D74** (2006) 063503, arXiv:astro-ph/0606685 [astro-ph].

[8] I. Sollom, A. Challinor, and M. P. Hobson, "Cold Dark Matter Isocurvature Perturbations: Constraints and Model Selection," *Phys.Rev.* **D79** (2009) 123521, arXiv:0903.5257.

[9] A. Mangilli, L. Verde, and M. Beltran, "Isocurvature modes and Baryon Acoustic Oscillations," *JCAP* **1010** (2010) 009, arXiv:1006.3806 [astro-ph.CO].

[10] S. M. Kasanda, C. Zunckel, K. Moodley, B. Bassett, and P. Okouma, "The sensitivity of BAO Dark Energy Constraints to General Isocurvature Perturbations," arXiv:1111.2572 [astro-ph.CO].

[11] H. Li, J. Liu, J.-Q. Xia, and Y.-F. Cai, "Cold Dark Matter Isocurvature Perturbations: Cosmological Constraints and Applications," *Phys.Rev.* **D83** (2011) 123517, arXiv:1012.2511 [astro-ph.CO].

[12] M. Kawasaki, T. Sekiguchi, and T. Takahashi, "Differentiating CDM and Baryon Isocurvature Models with 21 cm Fluctuations," *JCAP* **1110** (2011) 028, arXiv:1104.5591 [astro-ph.CO].

[13] J. Valiviita, M. Savelainen, M. Talvitie, H. Kurki-Suonio, and S. Rusak, "Constraints on scalar and tensor perturbations in phenomenological and two-field inflation models: Bayesian evidences for primordial isocurvature and tensor modes," *Astrophys.J.* **753** (2012) 151, arXiv:1202.2852 [astro-ph.CO].

[14] D. Polarski and A. A. Starobinsky, "Isocurvature perturbations in multiple inflationary models," *Phys.Rev.* **D50** (1994) 6123–6129, arXiv:astro-ph/9404061 [astro-ph].

[15] D. Seckel and M. S. Turner, "Isothermal Density Perturbations in an Axion Dominated Inflationary Universe," *Phys.Rev.* **D32** (1985) 3178.

[16] M. Beltran, J. Garcia-Bellido, and J. Lesgourgues, "Isocurvature bounds on axions revisited," *Phys.Rev.* **D75** (2007) 103507, arXiv:hep-ph/0606107 [hep-ph].

[17] A. D. Linde and V. F. Mukhanov, "Nongaussian isocurvature perturbations from inflation," *Phys.Rev.* **D56** (1997) 535–539, arXiv:astro-ph/9610219 [astro-ph].

[18] D. H. Lyth and D. Wands, "Generating the curvature perturbation without an inflaton," *Phys.Lett.* **B524** (2002) 5–14, arXiv:hep-ph/0110002 [hep-ph].

[19] D. J. Chung, E. W. Kolb, and A. Riotto, "Superheavy dark matter," *Phys.Rev.* **D59** (1999) 023501, arXiv:hep-ph/9802238 [hep-ph]. In *Venice 1999, Neutrino telescopes, vol. 2* 217-237.

[20] D. J. Chung, E. W. Kolb, A. Riotto, and L. Senatore, "Isocurvature constraints on gravitationally produced superheavy dark matter," *Phys.Rev.* **D72** (2005) 023511, arXiv:astro-ph/0411468 [astro-ph].

[21] J. M. Maldacena, "Non-Gaussian features of primordial fluctuations in single field inflationary models," *JHEP* **0305** (2003) 013, arXiv:astro-ph/0210603 [astro-ph].

[22] N. Bartolo, S. Matarrese, and A. Riotto, "Nongaussianity from inflation," *Phys.Rev.* **D65** (2002) 103505, arXiv:hep-ph/0112261 [hep-ph].

[23] M. Kawasaki, K. Nakayama, T. Sekiguchi, T. Suyama, and F. Takahashi, "Non-Gaussianity from isocurvature perturbations," *JCAP* **0811** (2008) 019, arXiv:0808.0009 [astro-ph].

[24] D. Langlois, F. Vernizzi, and D. Wands, "Non-linear isocurvature perturbations and non-Gaussianities," *JCAP* **0812** (2008) 004, arXiv:0809.4646 [astro-ph].

[25] D. J. Chung and H. Yoo, "Isocurvature Perturbations and Non-Gaussianity of Gravitationally Produced Nonthermal Dark Matter," arXiv:1110.5931 [astro-ph.CO].

[26] R. L. Arnowitt, S. Deser, and C. W. Misner, "The Dynamics of general relativity," arXiv:gr-qc/0405109 [gr-qc].

[27] S. Miao, N. Tsamis, and R. Woodard, "The Graviton Propagator in de Donder Gauge on de Sitter Background," *J.Math.Phys.* **52** (2011) 122301, arXiv:1106.0925 [gr-qc].

[28] S. L. Adler, "Einstein Gravity as a Symmetry Breaking Effect in Quantum Field Theory," *Rev.Mod.Phys.* **54** (1982) 729.

[29] E. Calzetta and B. Hu, *Nonequilibrium Quantum Field Theory*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2008.

[30] S. Weinberg, "Quantum contributions to cosmological correlations," *Phys.Rev.* **D72** (2005) 043514, arXiv:hep-th/0506236 [hep-th].

[31] B. DeWitt, *The Global Approach to Quantum Field Theory*. No. v. 1 in International Series of Monographs on Physics. Clarendon Press, 2003.

[32] S. Weinberg, "Ultraviolet Divergences in Cosmological Correlations," *Phys.Rev.* **D83** (2011) 063508, arXiv:1011.1630 [hep-th].

[33] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*. International Series of Monographs on Physics. Clarendon Press, 2002.

[34] B. S. DeWitt, "Quantum Field Theory in Curved Space-Time," *Phys.Rept.* **19** (1975) 295–357.

[35] M. Bruni, S. Matarrese, S. Mollerach, and S. Sonego, "Perturbations of space-time: Gauge transformations and gauge invariance at second order and beyond," *Class.Quant.Grav.* **14** (1997) 2585–2606, arXiv:gr-qc/9609040 [gr-qc].

[36] N. Birrell and P. Davies, *Quantum Fields in Curved Space*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1984.

[37] A. E. Lawrence and E. J. Martinec, "String field theory in curved space-time and the resolution of space - like singularities," *Class.Quant.Grav.* **13** (1996) 63–96, arXiv:hep-th/9509149 [hep-th].

[38] S. S. Gubser, "String production at the level of effective field theory," *Phys.Rev.* **D69** (2004) 123507, arXiv:hep-th/0305099 [hep-th].

[39] N. Turok, M. Perry, and P. J. Steinhardt, "M theory model of a big crunch / big bang transition," *Phys.Rev.* **D70** (2004) 106004, arXiv:hep-th/0408083 [hep-th].

[40] J. J. Friess, S. S. Gubser, and I. Mitra, "String creation in cosmologies with a varying dilaton," *Nucl.Phys.* **B689** (2004) 243–256, arXiv:hep-th/0402156 [hep-th].

[41] A. J. Tolley and D. H. Wesley, "String pair production in a time-dependent gravitational field," *Phys.Rev.* **D72** (2005) 124009, arXiv:hep-th/0509151 [hep-th].

[42] S. Cremonini and S. Watson, "Dilaton dynamics from production of tensionless membranes," *Phys.Rev.* **D73** (2006) 086007, `arXiv:hep-th/0601082 [hep-th]`.

[43] S. R. Das and J. Michelson, "Matrix membrane big bangs and D-brane production," *Phys.Rev.* **D73** (2006) 126006, `arXiv:hep-th/0602099 [hep-th]`.

[44] C.-J. Feng, X. Gao, M. Li, W. Song, and Y. Song, "Reheating and cosmic string production," *Nucl.Phys.* **B800** (2008) 190–203, `arXiv:0707.0908 [hep-th]`.

[45] J. S. Schwinger, "On gauge invariance and vacuum polarization," *Phys.Rev.* **82** (1951) 664–679.

[46] E. Brezin and C. Itzykson, "Pair production in vacuum by an alternating field," *Phys.Rev.* **D2** (1970) 1191–1199.

[47] J. Audretsch and G. Schaefer, "Thermal Particle Production in a Radiation Dominated Robertson-Walker Universe," *J.Phys.A* **A11** (1978) 1583–1602.

[48] S. G. Mamaev and V. M. Mostepanenko, "Particle creation by the gravitational field, and the problem of the cosmological singularity," *Pis ma Astronomicheskii Zhurnal* **4** (June, 1978) 203–206.

[49] S. G. Mamaev, V. M. Mostepanenko, and V. M. Frolov, "Fermion pair creation near the Friedmann singularity," *Soviet Astronomy Letters* **1** (Oct., 1975) 179–+.

[50] L. Parker, "Quantized fields and particle creation in expanding universes. 2.," *Phys.Rev.* **D3** (1971) 346–356.

[51] L. Parker, "Quantized fields and particle creation in expanding universes. 1.," *Phys.Rev.* **183** (1969) 1057–1068.

[52] V. A. Kuzmin and I. I. Tkachev, "Ultrahigh-energy cosmic rays and inflation relics," *Phys.Rept.* **320** (1999) 199–221, `arXiv:hep-ph/9903542 [hep-ph]`.

[53] S. V. Anischenko, S. L. Cherkas, and V. L. Kalashnikov, "Cosmological production of fermions in a flat Friedman universe with linearly growing scale factor: Exactly solvable model," *Nonlin.Phenom.Complex Syst.* **13** (2010) 315–319, `arXiv:0911.0769 [gr-qc]`.

[54] S. Tsujikawa and H. Yajima, "Massive fermion production in nonsingular superstring cosmology," *Phys. Rev. D* **64** (Jun, 2001) 023519.

[55] V. Kuzmin and I. Tkachev, "Matter creation via vacuum fluctuations in the early universe and observed ultrahigh-energy cosmic ray events," *Phys.Rev.* **D59** (1999) 123006, `arXiv:hep-ph/9809547 [hep-ph]`.

[56] J. R. Ellis, J. L. Lopez, and D. V. Nanopoulos, "Confinement of fractional charges yields integer charged relics in string models," *Phys.Lett.* **B247** (1990) 257.

[57] K. Benakli, J. R. Ellis, and D. V. Nanopoulos, "Natural candidates for superheavy dark matter in string and M theory," *Phys.Rev.* **D59** (1999) 047301, `arXiv:hep-ph/9803333 [hep-ph]`.

[58] A. Kusenko and M. E. Shaposhnikov, "Supersymmetric Q balls as dark matter," *Phys.Lett.* **B418** (1998) 46–54, `arXiv:hep-ph/9709492 [hep-ph]`.

[59] T. Han, T. Yanagida, and R.-J. Zhang, "Adjoint messengers and perturbative unification at the string scale," *Phys.Rev.* **D58** (1998) 095011, `arXiv:hep-ph/9804228 [hep-ph]`.

[60] G. Dvali, "Infrared hierarchy, thermal brane inflation and superstrings as superheavy dark matter," *Phys.Lett.* **B459** (1999) 489–496, `arXiv:hep-ph/9905204 [hep-ph]`.

[61] K. Hamaguchi, K. Izawa, Y. Nomura, and T. Yanagida, "Longlived superheavy particles in dynamical supersymmetry breaking models in supergravity," *Phys.Rev.* **D60** (1999) 125009, `arXiv:hep-ph/9903207 [hep-ph]`.

[62] C. Coriano, A. E. Faraggi, and M. Plumacher, "Stable superstring relics and ultrahigh-energy cosmic rays," *Nucl.Phys.* **B614** (2001) 233–253, `arXiv:hep-ph/0107053 [hep-ph]`.

[63] H.-C. Cheng, K. T. Matchev, and M. Schmaltz, "Radiative corrections to Kaluza-Klein masses," *Phys.Rev.* **D66** (2002) 036005, `arXiv:hep-ph/0204342 [hep-ph]`.

[64] G. Shiu and L.-T. Wang, "D matter," *Phys.Rev.* **D69** (2004) 126007, `arXiv:hep-ph/0311228 [hep-ph]`.

[65] V. Berezinsky, M. Kachelriess, and M. Solberg, "Supersymmetric superheavy dark matter," *Phys.Rev.* **D78** (2008) 123535, `arXiv:0810.3012 [hep-ph]`.

[66] T. W. Kephart and Q. Shafi, "Family unification, exotic states and magnetic monopoles," *Phys.Lett.* **B520** (2001) 313–316, `arXiv:hep-ph/0105237 [hep-ph]`.

[67] T. W. Kephart, C.-A. Lee, and Q. Shafi, "Family unification, exotic states and light magnetic monopoles," *JHEP* **0701** (2007) 088, `arXiv:hep-ph/0602055 [hep-ph]`.

[68] C. Barbot and M. Drees, "Detailed analysis of the decay spectrum of a super-heavy X particle," *Astropart. Phys.* **20** (2003) 5–44, `arXiv:hep-ph/0211406`.

[69] I. F. Albuquerque and L. Baudis, "Direct detection constraints on superheavy dark matter," *Phys.Rev.Lett.* **90** (2003) 221301, `arXiv:astro-ph/0301188 [astro-ph]`.

[70] M. Taoso, G. Bertone, and A. Masiero, "Dark Matter Candidates: A Ten-Point Test," *JCAP* **0803** (2008) 022, `arXiv:0711.4996 [astro-ph]`. * Brief entry *.

[71] J. Bovy and G. R. Farrar, "Connection between a possible fifth force and the direct detection of Dark Matter," *Phys.Rev.Lett.* **102** (2009) 101301, `arXiv:0807.3060 [hep-ph]`.

[72] I. F. Albuquerque and C. Perez de los Heros, “Closing the Window on Strongly Interacting Dark Matter with IceCube,” *Phys.Rev.* **D81** (2010) 063510, arXiv:1001.1381 [astro-ph.HE].

[73] D. J. Chung, “Classical inflation field induced creation of superheavy dark matter,” *Phys.Rev.* **D67** (2003) 083514, arXiv:hep-ph/9809489 [hep-ph].

[74] D. J. Chung, P. Crotty, E. W. Kolb, and A. Riotto, “On the gravitational production of superheavy dark matter,” *Phys.Rev.* **D64** (2001) 043503, arXiv:hep-ph/0104100 [hep-ph].

[75] B. Garbrecht, T. Prokopec, and M. G. Schmidt, “Particle number in kinetic theory,” *Eur.Phys.J.* **C38** (2004) 135–143, arXiv:hep-th/0211219 [hep-th].

[76] S. A. Ramsey, B. L. Hu, and A. M. Stylianopoulos, “Nonequilibrium inflaton dynamics and reheating. II. Fermion production, noise, and stochasticity,” *Phys. Rev. D* **57** (May, 1998) 6003–6021.

[77] B. A. Bassett, M. Peloso, L. Sorbo, and S. Tsujikawa, “Fermion production from preheating amplified metric perturbations,” *Nucl.Phys.* **B622** (2002) 393–415, arXiv:hep-ph/0109176 [hep-ph].

[78] M. Peloso and L. Sorbo, “Preheating of massive fermions after inflation: Analytical results,” *JHEP* **0005** (2000) 016, arXiv:hep-ph/0003045 [hep-ph].

[79] J. Garcia-Bellido, S. Mollerach, and E. Roulet, “Fermion production during preheating after hybrid inflation,” *JHEP* **0002** (2000) 034, arXiv:hep-ph/0002076 [hep-ph].

[80] J. Baacke, K. Heitmann, and C. Pätzold, “Nonequilibrium dynamics of fermions in a spatially homogeneous scalar background field,” *Phys. Rev. D* **58** (Nov, 1998) 125013.

[81] P. B. Greene and L. Kofman, “Preheating of fermions,” *Phys.Lett.* **B448** (1999) 6–12, arXiv:hep-ph/9807339 [hep-ph].

[82] A. D. Dolgov and D. P. Kirilova, “Production of particles by a variable scalar field,” *Sov. J. Nucl. Phys.* **51** (1990) 172–177. [Yad.Fiz.51:273-282,1990].

[83] D. J. Chung, E. W. Kolb, A. Riotto, and I. I. Tkachev, “Probing Planckian physics: Resonant production of particles during inflation and features in the primordial power spectrum,” *Phys.Rev.* **D62** (2000) 043508, arXiv:hep-ph/9910437 [hep-ph].

[84] B. Garbrecht and T. Prokopec, “Fermion mass generation in de Sitter space,” *Phys. Rev. D* **73** (Mar, 2006) 064036.

[85] A. L. Maroto and A. Mazumdar, “Production of spin 3/2 particles from vacuum fluctuations,” *Phys. Rev. Lett.* **84** (2000) 1655–1658, arXiv:hep-ph/9904206.

[86] R. Kallosh, L. Kofman, A. D. Linde, and A. Van Proeyen, “Gravitino production after inflation,” *Phys.Rev.* **D61** (2000) 103503, arXiv:hep-th/9907124 [hep-th].

[87] G. Giudice, A. Riotto, and I. Tkachev, "Thermal and nonthermal production of gravitinos in the early universe," *JHEP* **9911** (1999) 036, `arXiv:hep-ph/9911302 [hep-ph]`.

[88] H. P. Nilles, M. Peloso, and L. Sorbo, "Coupled fields in external background with application to nonthermal production of gravitinos," *JHEP* **04** (2001) 004, `arXiv:hep-th/0103202`.

[89] H. P. Nilles, M. Peloso, and L. Sorbo, "Nonthermal production of gravitinos and inflatinos," *Phys. Rev. Lett.* **87** (2001) 051302, `arXiv:hep-ph/0102264`.

[90] M. Kawasaki, F. Takahashi, and T. T. Yanagida, "The gravitino overproduction problem in inflationary universe," *Phys. Rev.* **D74** (2006) 043519, `arXiv:hep-ph/0605297`.

[91] D. J. Chung, A. Notari, and A. Riotto, "Minimal theoretical uncertainties in inflationary predictions," *JCAP* **0310** (2003) 012, `arXiv:hep-ph/0305074 [hep-ph]`.

[92] E. W. Kolb and M. S. Turner, "The Early universe," *Front.Phys.* **69** (1990) 1–547.

[93] D. H. Lyth and A. Riotto, "Particle physics models of inflation and the cosmological density perturbation," *Phys.Rept.* **314** (1999) 1–146, `arXiv:hep-ph/9807278 [hep-ph]`.

[94] A. Mazumdar, "The origin of dark matter, matter-anti-matter asymmetry, and inflation," `arXiv:1106.5408 [hep-ph]`. * Temporary entry *.

[95] A. Mazumdar and J. Rocher, "Particle physics models of inflation and curvaton scenarios," *Phys.Rept.* **497** (2011) 85–215, `arXiv:1001.0993 [hep-ph]`.

[96] S. Weinberg, *Cosmology*. 2008.

[97] D. Langlois, "Correlated adiabatic and isocurvature perturbations from double inflation," *Phys.Rev.* **D59** (1999) 123512, `arXiv:astro-ph/9906080 [astro-ph]`.

[98] E. Komatsu, J. Dunkley, M. R. Nolta, C. L. Bennett, B. Gold, G. Hinshaw, N. Jarosik, D. Larson, M. Limon, L. Page, D. N. Spergel, M. Halpern, R. S. Hill, A. Kogut, S. S. Meyer, G. S. Tucker, J. L. Weiland, E. Wollack, and E. L. Wright, "Five-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation," *arXiv* **astro-ph** (Mar., 2008) .

[99] E. Komatsu, K. M. Smith, J. Dunkley, C. L. Bennett, B. Gold, G. Hinshaw, N. Jarosik, D. Larson, M. R. Nolta, L. Page, D. N. Spergel, M. Halpern, R. S. Hill, A. Kogut, M. Limon, S. S. Meyer, N. Odegard, G. S. Tucker, J. L. Weiland, E. Wollack, and E. L. Wright, "SEVEN-YEAR WILKINSON MICROWAVE ANISOTROPY PROBE (WMAP1) OBSERVATIONS: COSMOLOGICAL INTERPRETATION," *arXiv* **astro-ph.CO** (Jan., 2010) .

[100] D. Larson, J. Dunkley, G. Hinshaw, E. Komatsu, M. Nolta, C. Bennett, B. Gold, M. Halpern, R. Hill, and N. Jarosik, "Seven-year Wilkinson Microwave Anisotropy Probe (WMAP) observations: power spectra and WMAP-derived parameters," *The Astrophysical Journal Supplement Series* **192** (2011) 16.

[101] R. Bean, J. Dunkley, and E. Pierpaoli, "Constraining isocurvature initial conditions with WMAP 3-year data," *Physical Review D* **74** no. 6, (2006) 063503.

[102] E. Komatsu and D. N. Spergel, "Acoustic signatures in the primary microwave background bispectrum," *Phys. Rev.* **D63** (2001) 063002, `arXiv:astro-ph/0005036`.

[103] C. Hikage, K. Koyama, T. Matsubara, T. Takahashi, and M. Yamaguchi, "Limits on Isocurvature Perturbations from Non-Gaussianity in WMAP Temperature Anisotropy," *Mon.Not.Roy.Astron.Soc.* **398** (2009) 2188–2198, `arXiv:0812.3500 [astro-ph]`. * Brief entry *.

[104] T. Takahashi, M. Yamaguchi, and S. Yokoyama, "Primordial Non-Gaussianity in Models with Dark Matter Isocurvature Fluctuations," *Phys.Rev.* **D80** (2009) 063524, `arXiv:0907.3052 [astro-ph.CO]`.